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Modeling and Analyzing Fractal Point Processes

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Abstract

The temporal pattern of the action potential discharges of auditory nerve fibers have been presumed to be random in nature and well described as a renewal process. Recently, we observed that the discharges were characterized by excess variability and low frequency spectra. Simple stationary point process models, point process models with a chaotic intensity, and doubly stochastic point process models with a strongly mixing intensity process do not describe the data. But a fractal point process, defined to be a doubly stochastic point process with a fractal waveform intensity process, described the data more generally. The sample paths of the counting process of a fractal point process are not self-similar; it displays self-similar characteristics only over large time scales.

The Fano factor, rather than the power spectrum, captures these self-similar characteristics. The *fractal dimension* and *fractal time* characterize the Fano factor. The fractal dimension is characteristic of the fractal intensity process and is measured from the asymptotic slope of the Fano factor plot. The fractal time is the time before the onset of the fractal behavior and delineates the short-term characteristics of the

data. Absolute and relative refractory effects, serial dependence, and average rate modify the fractal time.

To generate a fractal point process, fractal intensity processes are derived through memoryless, nonlinear transformations of fractional Gaussian noise. All the transformations considered preserve the self-similarity property of the fractional Gaussian noise. A theorem due to Snyder [32] relates the estimate of the asymptotic pulse number distribution (PND) to the asymptotic distribution of the integrated intensity process of a doubly stochastic Poisson process. The shape of the probability density function of the intensity process is greatly influenced by the choice of the transformation. The intensity density function, in combination with the Fano factor form a criterion to choose the transformation that best describes the data.

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Chapter 1

Background

Point process models are appropriate for a wide variety of physical phenomenon. According to Snyder [32], they describe electron emissions, radioactivity, neural activity, atmospheric noise caused by lightning discharges, photoelectron emissions, astronomy, scintigraphy in nuclear medicine, electron microscopy, dynamic studies in nuclear medicine and seismic events. In modeling discharge patterns from the lower auditory nervous system, we are primarily concerned with the representation and processing of the acoustical environment by the system. The discharge patterns of most single neurons are well described by *point process models* [14]. Johnson and Swami [14] determined that a renewal model described the discharge patterns of auditory-nerve fibers by accurately modeling the refractory effects present in neural firing. Johnson *et al* used a “shifting model” [15] to describe discharges from the Lateral Superior Olive (LSO) that indicated a first order Markovian dependence in the time series of inter-spike intervals; this model described the sustained and transient responses of most LSO units.

In all of these models, the emphasis is on describing short-term response characteristics. A study we conducted five years ago [16] indicated the presence of long-term

variations in the data. More recently, we showed that these long-term characteristics are well described by *fractal point processes* wherein the intensity process of a doubly stochastic Poisson process is a fractal waveform process. The goal of this thesis is to develop tools to measure, analyze, and model fractal activity.

In this thesis we will first demonstrate that the data with the long-term variations cannot be described by a stationary Poisson process. We will then test a point process model with a chaotic intensity function; to this end we will present some results from the theory of chaos. In § 3.2.1 we will illustrate the fact that a doubly stochastic Poisson process with a simple Markovian dependence does not describe the fractal characteristics of the data. Only a doubly stochastic Poisson model with a fractal intensity process describes the variability and low frequency characteristics of the data. The fractal intensity process is derived through nonlinear transformation of fractional Gaussian noise. Fractional Gaussian noise is, by definition, the stationary derivative of fractional Brownian motion. The properties of fractional Brownian motion are presented in § 2.2.1; a method to generate fractional Gaussian noise is also described there. The Fano factor is the primary indicator and measure of fractal activity in point process data; this quantity is characterized by the *fractal dimension* and *fractal time*. The factors that modify these aspects of the Fano factor of a doubly stochastic point process are determined analytically in § 3.2.1. Different nonlinear transformations (full-wave rectifier, half-wave rectifier, square law transformation, shifted version of these transformations, exponentiation, and combinations of these

transformations) are used in deriving the intensity process. The characteristics of the resulting intensity process and the fractal point process generated thereof are studied to determine criteria for selecting a particular transformation for a given fractal dataset. We look at the self-similarity parameter of the fractal intensity process, its relation to the self-similarity parameter of the input to the transformation – fractional Gaussian noise, and the constraints on these parameters. The Fano factor of the resulting fractal point process, the number of fractal exponents present in the Fano factor, and the probability density function of the intensity process are studied for possible selection criteria. The validity of the fractal point process generated in this manner is established. A theorem due to Snyder [32] is stated that relates the asymptotic distributions of the normalized and centered versions of the counting and integrated intensity processes for fractal point processes; this establishes the probability density function of the intensity process as an important selection criterion. We, then, establish the mean-ergodicity of the fractal intensity process and the convergence of rate estimates; the convergence rates are also discussed. The factors that modify the fractal dimension and fractal time of fractal point processes are then determined.

1.1 Point Process Theory and Neural Modeling

Recordings from single neurons in the lower auditory system indicate [14] that the acoustical information is encoded in the timing of the action potential (henceforth called the event) rather than in the amplitude of the waveform of each event (the

waveform of the individual events are identical). The discharge patterns of these events are well modeled by point processes wherein the probability law is assigned to the time of occurrence of the identical events. Let N_t denote the number of events of a point process that have occurred in the time interval $[t_0, t)$ where t_0 is the time at which observation begins. It is usually assumed that $N_{t_0} = 0$. $\{N_t; t \geq t_0\}$ is termed the *counting process* associated with the point process. The increments of this process, $N_{s,t} = N_t - N_s$, $t \geq s \geq t_0$, measure the number of events occurring in the time interval $[s, t)$ and is termed the increment process. Let $\mathbf{w}$ be the vector of times at which these events occur: $\mathbf{w} = [w_1, \dots, w_{N_t}]$ where $t_0 \leq w_1 \leq w_2 \leq \dots \leq w_{N_t} \leq t$. The inter-event interval, τ_n , is given by $w_n - w_{n-1}$.

Define the quantity $\mu(t; N_t; \mathbf{w})$ as

$$\mu(t; N_t; \mathbf{w}) = \lim_{\Delta t \rightarrow 0} \frac{\Pr[N_{t,t+\Delta t} = 1 | N_t, \mathbf{w}]}{\Delta t}.$$

If the limit on the right exists, then $\mu(\cdot)$ is known as the *intensity* of the point process.

For small Δt , we can approximate the conditional probability of one event occurring in $[t, t + \Delta t)$ by

$$\Pr[\text{one event in } [t, t + \Delta t) | N_t, \mathbf{w}] \approx \mu(t; N_t; \mathbf{w}) \Delta t.$$

The intensity completely describes the point process by expressing the dependence of event occurrence on time and on the history of the process: times of occurrence and number of previous events. Johnson *et al* [13] have proposed an efficient scheme to generate point processes having a specific intensity. Figure (1.1), reproduced from

[13], describes the system in block diagram fashion. This scheme generates the sequence of inter-events intervals, according to the input-output relation of a system given by

$$\tau_n = g^{-1}(E_n; [\tau_1, \dots, \tau_{n-1}]), g(\tau_n; [\tau_1, \dots, \tau_{n-1}]) = \int_0^{\tau_n} \tilde{\mu}(\alpha; n; [\tau_1, \dots, \tau_{n-1}]) d\alpha \quad (1.1)$$

where $\tilde{\mu}$ denotes the intensity of the process re-expressed in terms of inter-event intervals. By supplying the signal $s(t)$ and specifying the functional dependence of the intensity on it, a non-stationary point process can be generated.

Several special point processes are important here.

- The *Poisson* process is the point process with the simplest statistical structure and the intensity function does not depend on previous events: $\mu(t; N_t; \mathbf{w}) = \lambda_t$. If λ_t is a constant, the process is termed *homogeneous* and is stationary. Stationary or not, the number of events occurring during an interval of time in a Poisson process is statistically independent of the number of events occurring in any other *disjoint* interval of time. The Poisson model was used by Siebert [31] as a workable approximation to auditory-nerve activity to derive how well an optimal processor could extract tonal frequency and amplitude information from populations of auditory-nerve fiber discharge patterns.
- In a *renewal* process, the inter-event intervals are statistically independent. *Non-stationary renewal* processes have been used to describe the responses of

Figure 1.1 The interval τ_n between events w_{n+1} and w_n result from the diagrammed system, where E_n , a sequence of statistically independent exponential random variables, and $s(t)$, representing the process's non-stationarities, serve as the inputs. The input-output relationship of the system is given by equation (1.1).

single auditory-nerve fibers [14]. Here, the intensity is given by

$$\mu(t; N_t; \mathbf{w}) = d(t)r(t - w_{N_t}).$$

The quantity $d(t)$ describes the temporal variation of the rate of discharge (*i.e.*, the rate at which action potentials are generated) and $r(\cdot)$ is the so-called *refractory* function. The interval $t - w_{N_t}$ denotes the time that has elapsed since the occurrence of the previous event. The probability of a neuron discharging is zero for a short period of time after it has discharged; this period of time is termed the *absolute deadtime* and is 0.5 ms to 1 ms for auditory-nerve fibers. Past the absolute deadtime, the probability of a second discharge gradually increases from zero to a sustained value; the duration of time for which the probability

is greater than zero but less than its maximum value is called the *relative dead-time*. For auditory-nerve fibers, the relative deadtime ranges between 25 ms to 40 ms. The refractory function describes both of these deadtimes - absolute and relative. The above model was used to study the effect of absolute and relative deadtimes on the representation of sinusoidal stimuli (tones) by single auditory-nerve fiber discharges [14].

- In a general point process, the probability of occurrence of events is influenced by events prior to the most recent; the discharge patterns of neurons located in the lateral superior olive (LSO) indicate such dependence. The intensity of the point process model that describes the activity of these neurons is influenced by the past two events [15]. The inter-event intervals formed a first-order Markovian time series. This serial dependence was modeled as a refractory effect [15].
- Of particular importance here is the *doubly stochastic* point process where the intensity function is itself a stochastic process. Here, $\{N_t; t \geq t_0\}$ is conditionally a point process given a sample function of the intensity process. The doubly stochastic Poisson process is the simplest of doubly stochastic point processes. Here, the intensity equals the sample function, which must be non-negative. The first two moments of the doubly stochastic Poisson counting process are

[32]

$$E(N_t) = E(\Lambda_t) \quad (1.2)$$

$$E(N_t^2) = E(\Lambda_t^2) + E(\Lambda_t) \quad (1.3)$$

where $\Lambda_t \triangleq \int_{t_0}^t \lambda_\sigma d\sigma$. Hence the variance of N_t is given by

$$\text{var}(N_t) = \text{var}(\Lambda_t) + E(\Lambda_t). \quad (1.4)$$

The *Fano factor* [7] of the point process (Snyder [32] calls it the index of dispersion) is given by

$$F(t) = \frac{\text{var}(N_t)}{E(N_t)} = 1 + \frac{\text{var}(\Lambda_t)}{E(\Lambda_t)} \quad (1.5)$$

For a stationary Poisson process, the second term in the above expression is zero and hence the Fano factor for a Poisson process is unity; for a doubly stochastic Poisson process the quantity is greater than unity. As will be seen, the Fano factor is a key quantity in identifying and modeling doubly stochastic Poisson processes.

1.2 Rate Variability in Auditory-Nerve Fiber Recordings

Five years ago, we studied the variation in the rate of discharge of single auditory-nerve fibers [16]. The rate of discharge of a response to a continuous tone is plotted against time in figure (1.2) for two different averaging intervals: $T = 5$ seconds and $T = 50$ seconds. The global mean was used in obtaining the average rate. Consider a

Figure 1.2 The rate of discharge of a continuous tone response from the unit 18-94 is plotted against averaging interval *index*. The estimates are obtained for two different averaging intervals: $T = 5$ seconds and $T = 50$ seconds. The average rates are normalized in both cases. The common average rate, $\lambda = 77.36$, is marked on the plots. The standard deviation of a stationary Poisson process with the same average rate, λ , as the data is given by $\sqrt{\lambda/T}$. The upper and lower bounds correspond to $\lambda \pm 2\sqrt{\lambda/T}$.

Table 1	
Unit	% of time the bounds were exceeded
Spontaneous activity:	
13-38	9.79
22-56	10.76
627-2	13.16
629-43	11.92
634-49	21.77
Continuous tone:	
18-94	27.79
22-56	15.79
24-122	13.50
628-45	67.33

Table 1.1 The fraction of time the rate of discharge, for different recordings from the auditory nerve, exceeded its bounds is tabulated here. An averaging interval of 1 second is used. The bounds used correspond to $\lambda \pm 2\sqrt{\lambda/T}$ where λ is the global mean and $T = 1$ second in this table. For a *stationary* Poisson process the estimates would exceed the bounds no more than 5 % of the time.

homogeneous Poisson process with parameter λ equaling the average discharge rate. The counting process (N_t) of a stationary Poisson process, centered and standardized (*i.e.*, zero mean and unit variance), converges asymptotically (*i.e.*, as $T \rightarrow \infty$) to a zero mean, unit variance, Gaussian random variable [32, Page 298]. Therefore we would expect Gaussian behavior for sufficiently large T . The standard deviation of a stationary Poisson process's intensity estimate equals $\sqrt{\lambda/T}$. The error bounds for the variation of the estimate about its mean, under asymptotic normality, are given by $\lambda \pm \sqrt{\lambda/T}$. For a stationary Poisson process, we would expect the bounds to be exceeded no more than 5% of the time. As shown in figure 1.2, the estimate exceeds the bounds far more frequently (27.79 % of the time). Furthermore, although the

second averaging interval is ten times the first, the variability of the estimate is not reduced significantly; we would expect this reduction for an asymptotically normal behavior. Table 1.1 tabulates the fraction of time the discharge rate exceeded these bounds for different recordings. It was evident that the rate estimates for both the spontaneous and driven discharges were too variable to be consistent with a stationary Poisson model.

In an initial attempt to analyze these discharge patterns, we employed a *doubly stochastic* Poisson process model. Using analysis procedures based on this model, we found that the excess variability could be consistently described by an intensity having a lowpass power spectrum (see figure 1.3). The increment process is computed by counting the number of spikes (*i.e.*, action potentials) that occur in adjacent, disjoint bins of duration one second. The spectrum of this increment process is plotted for driven discharges in figure 1.3 (Panel a). In panel b of the figure, the magnitude of the power spectrum is plotted for the increment process of a simulated stationary, renewal process having the same average rate (note the difference in the maximum values of the vertical axis). The spectra are very different in the two cases: the power is concentrated in frequencies less than 0.01 Hz for the data while the power is uniformly distributed over all frequencies for the simulation. These results indicate that the excess variability corresponded to a very low frequency variation far outside the acoustic range of the animal (cat). The correlation of the increment process is related to the correlation of the intensity process [16]. Hence, the correlation observed

in the spectrum of the increment process could be attributed to the correlation in the intensity process.

Teich [36] observed a similar variability in the discharges of auditory-nerve fibers using a different analytical tool: the pulse-number distribution (PND). The pulse-number distribution is an estimate of the counting probability $\Pr[N_T = n]$; the PND is obtained from the histogram of the number of events that occur in a fixed time interval, T . For a stationary Poisson process, the PND is given by

$$\Pr[N_T = n] = \frac{(\lambda T)^n}{n!} e^{-\lambda T}.$$

For increasing values of T , one would expect the PND to obey the law of large numbers and approach a normal distribution with the distribution mean equaling λT asymptotically. Teich found that the measured PND of the data did not conform to the theoretical predictions for a stationary renewal process.

These two independent observations support the notion that stationary point process models do not accurately describe the responses of the auditory-nerve fibers. All the responses studied at the two laboratories exhibited this unexplained variability. Preliminary investigation of responses of the LSO and the medial nucleus of the trapezoid body (MNTB) neurons recorded at a third laboratory also had this unexplained variability. Furthermore, Teich's observations, along with ours, covered a wide range of spontaneous discharge rates and characteristic frequencies (this frequency is a characteristic of the neuron and is measured by determining the stimulus frequency at which the level of the stimulus required to elicit a nominal response from that neuron

Figure 1.3 In Panel a, the estimate of the spectrum of the increment process of a continuous tone response of the unit 18-94 has been plotted against frequency (in Hz). The increment process is computed by counting the number of spikes that occurred in adjacent, non-overlapping bins of duration one second. The correlation in the intensity process is related to the correlation in the increment process (see [16]). Linear scales have been used on both axes. The spectral estimate in Panel b is that of the increment process of a simulated Poisson data having the same average rate as the continuous tone response. Note that the vertical scales are different in the two plots.

is a minimum). This variability is a phenomenon that appears to be a characteristic of the responses and not a consequence of experimental conditions. Thus, the point process models that describe the discharges must explain this observation.

1.3 The impact of the variability on signal processing by sensory systems

As noted before, preliminary analysis indicates that the variability is present in recordings from single neurons in the LSO as well as the MNTB. This indicates that the phenomenon is not an isolated occurrence in the auditory system but probably widely prevalent. In fact, the excess variability has been observed in other sensory systems, *e.g.*, recordings from the retina of goldfish [18]. Data from the optic nerve also indicates an excess variability similar to that found in auditory-nerve fiber data [2, 26]. Some preliminary results [23] indicate that this variability is not species-specific, *i.e.*, it is present in auditory-nerve recordings of Frogs and Gerbils as well.

The average discharge rate is often used in describing the representation of sensory stimuli in neural responses. The variability observed in the rate estimates implies that one can question the validity of any stimulus feature encoded in the average rate. As was observed earlier, the variability is related to a high concentration of power in very low frequencies. This indicates that the average rate variations span a very long time period and hence any estimate of the rate will converge very slowly. Thus a system that attempts to extract a stimulus feature from such highly variable data

must average the rate estimate over a number of neurons. But even this ensemble averaging will not eliminate the long-term variations. An important question in sensory processing systems is how are stimuli represented in discharge patterns that display such long term average-rate variability? Furthermore, can the traditional models be modified to take into account the variability or must we look for new models to describe neural discharges?

Two outstanding features of the auditory discharge patterns described earlier are their variability and the very low frequency components present in their spectra. From the expression for Fano factor of a doubly stochastic Poisson process (see equation (1.5)) we can see that the variability is well described by a doubly stochastic model. It will be seen that the low frequency spectrum is a characteristic of fractal noises; a doubly stochastic Poisson process with a “fractal” intensity process can characterize both the variability *and* low-frequency spectra of such data.

Chapter 2

Fractals

A time series can be classified as stochastic or deterministic. A deterministic time series could be chaotic or non-chaotic. Identical inputs to the external ear do not produce identical discharge patterns in auditory-nerve fibers; hence a simple, non-chaotic deterministic model is not appropriate. We will turn to fractals for a possible explanation of the variability observed in the recordings from auditory-nerve fibers. Mandelbrot [20] defines a *fractal* to be a set for which the Hausdorff-Besicovitch dimension - the fractal dimension - strictly exceeds the topological dimension. For a point, the topological dimension, $D_T = 0$; for a straight line or a circle or other standard curves, $D_T = 1$; and for a plane, $D_T = 2$. In all these cases, the fractal dimension, D_F , is an integer and equals D_T . The fractal dimension is bounded from above by the Euclidean dimension, D_E (i.e., $D_T \leq D_F \leq D_E$). A fractal could be stochastic or deterministic. The output of chaotic systems, though deterministic, is known to exhibit a stochastic-like behavior [9]. Furthermore, the correlation measure (defined in the next section) is known to display fractal characteristics in phase space. We will first investigate chaotic systems to test if a point process with a chaotic intensity function can describe the discharges of auditory-nerve fibers.

2.1 Chaos

Chaos results when a deterministic nonlinear system produces a seemingly random output that is exquisitely sensitive to its initial conditions. The theory of nonlinear dynamical systems is used in the analysis of chaotic activity. Such analysis is conducted in *phase space* [8]. If one assumes that the dynamics of the system is completely described by a set of n first-order differential equations, then the instantaneous state of the system at time t is given by the set of n variables $\{x_0(t), x_1(t), \dots, x_{n-1}(t)\}$ that satisfy these n equations. A single variable (say x_0), n^{th} -order differential equation is equivalent to this description. x_0 and its $n - 1$ derivatives can be obtained from a single time series. But a more convenient approach is to span the n -dimensional phase space by the set $\{x_0(t), x_0(t + \tau), \dots, x_0(t + (n - 1)\tau)\}$ (where τ is a fixed time increment) which is topologically equivalent to original set [34]. Following this latter approach, the set $(x_1, \dots, x_n)$ (here x_i denotes the i^{th} sample, $x_0(t + i\tau)$) represents the state of the system at any instant of time and defines a point in phase space. Subspaces of a chaotic system's phase space are self-similar, hence the link to fractals. The successive states of the system, as it evolves over time, constitute a curve in phase space called a "trajectory". With time, the trajectories either cover the entire phase space or they converge to a lower-dimensional subset called an "attractor". If the dimension of an attractor is a non-integer, the attractor is called a "strange attractor" and is associated with deterministic chaos. The strange attractor is a self-similar set.

Self-similar objects possess a fractal dimension [30]. According to Mandelbrot's definition of a fractal, every set with a non-integer D_F is a fractal; hence all strange attractors, by definition, are fractals. For a n -dimensional phase space, let $N(\epsilon)$ be the number of n -dimensional balls (or cubes) of radius ϵ required to cover an attractor. Then the fractal dimension D_F is defined as

$$D_F = \lim_{\epsilon \rightarrow 0} \frac{\log N(\epsilon)}{|\log \epsilon|}.$$

The concept of fractal dimensions can be generalized using the Renyi entropies of order q , defined in [12] as

$$I_q = - \lim_{\tau \rightarrow 0} \lim_{\epsilon \rightarrow 0} \lim_{d \rightarrow \infty} \frac{1}{d\tau} \frac{1}{q-1} \log \sum_{i_1, \dots, i_d} p^q(i_1, \dots, i_d), \quad (2.1)$$

where $p(i_1, \dots, i_d)$ is the joint probability that the instantaneous state of the system represented by $(x_1, \dots, x_d)$ has a configuration such that x_1 falls in cube i_1 , x_2 in cube i_2 , $\dots$, and x_d in cube i_d . ϵ is the size of the cube. Let us look at the limits in the above expression to get an intuitive feel what Renyi entropies measure in the limiting case. The logarithm of the summation of joint probabilities is a measure of information contained in the trajectory. The limit on τ implies that we are sampling the output of the dynamic system more frequently and by decreasing the value of τ we can track the trajectory more accurately. By letting ϵ go to zero, we can observe how this information varies with the box size used to cover the attractor and hence the dimension of the attractor. The limit on d enables us to see the behavior of the information for different embedding dimensions; from this we can determine if the

attractor is self-similar. The Renyi entropies satisfy the following inequality [12]

$$I_0 \geq I_1 \geq I_2 \geq \dots$$

Röschke and Başar [27] define the generalized dimensions, D_q , as

$$D_q = \lim_{\epsilon \rightarrow 0} \frac{I_q(\epsilon)}{|\log \epsilon|}.$$

The dimension, D_0 , identified with I_0 is the fractal dimension, D_F ; the dimension, D_1 , that corresponds to I_1 is called the “information dimension”; the dimension associated with I_2 , D_2 , is called the “correlation dimension”. These dimensions satisfy the relationship [11, 27] $D_0 (= D_F) \geq D_1 \geq D_2 \geq \dots$

2.1.1 Estimating Dimension of Attractors

Of the generalized dimensions, the correlation dimension D_2 is the easiest to estimate from attractors generated by experimental data. Furthermore, I_2 is infinite for random systems and $I_2 > 0$ is a sufficient condition for chaos; this provides us with a tool to determine if a given time series is random or chaotic. From equation (2.1), we see that to determine the correlation dimension (*i.e.*, $q = 2$) for $d = 1$ we only need to compute

$$\sum_i^{N(\epsilon)} p_i^2 \equiv C(\epsilon)$$

where p_i is the probability that an arbitrary point (of an attractor) falls into a cube of size ϵ and the summation is carried out over all non-empty cubes. $C(\epsilon)$ is also called the correlation measure. This quantity is easily computed for the time series

$(x_1, \dots, x_n)$.

$$C(\epsilon) = \lim_{N \rightarrow \infty} \frac{1}{N^2} [\text{number of pairs of points } (n, m) \text{ with } \|x_n - x_m\| < \epsilon].$$

When chaotic, it has been shown [11] that $C(\epsilon)$ behaves as a power of ϵ for small ϵ :

$$C(\epsilon) \sim \epsilon^{D_2}.$$

For any $d > 1$ consider

$$C_d(\epsilon) = \lim_{N \rightarrow \infty} \frac{1}{N_d^2} \{\text{number of pairs } (n, m) \text{ with } \|\mathbf{x}_n - \mathbf{x}_m\| < \epsilon\}$$

where $\mathbf{x}_n$ is a point in phase space and is constructed from the time series as $\mathbf{x}_n = \{x_n, x_{n+1}, \dots, x_{n+d-1}\}$. d is called the “embedding dimension”. $\|\cdot\|$ denotes the Euclidean distance. N_d is the number of points in the phase space of embedding dimension d . From [12] we have that

$$C_d(\epsilon) \simeq \sum_{i_1, \dots, i_d} p^2(i_1, \dots, i_d).$$

Combining the expression for I_2 (equation (2.1)) and the fact that $C(\epsilon)$ has a power law behavior for small ϵ , we obtain

$$C_d(\epsilon) \rightarrow \epsilon^{D_2} \exp(-d\tau I_2) \quad \text{as } d \rightarrow \infty, \text{ and } \epsilon \rightarrow 0$$

When the correlation measure, $C_d(\epsilon)$ is plotted against ϵ , $C_d(\epsilon)$ saturates at $1/N^2$ for very small values of ϵ and for very large values of ϵ it saturates at 1. For intermediate (yet small) values of ϵ (for a chaotic time series), $C_d(\epsilon)$ has a power-law behavior. If $C_d(\epsilon)$ is plotted against ϵ on a log-log scale for different values of d , we

would obtain curves that are parallel in this intermediate region and are displaced from one another by the quantity $\exp(-d\tau I_2)$; the correlation dimension is given by the slope in this intermediate region (for “reasonably” large values of d). For a chaotic time series, the slopes of the correlation measure, when plotted against the embedding dimension, should saturate at the value of the correlation dimension, D_2 . One note on the power-law behavior of the correlation dimension in phase space: this behavior holds only for small values of ϵ ; *i.e.*, it is a local phenomenon and does not encompass the whole phase space.

The correlation measure is computed for white noise and for the logistic system $x_{n+1} = 4x_n(1 - x_n)$ with a starting value of $x_1 = .2$; the quantity is plotted for each case in panel a and panel b of figure 2.1 respectively. The measure is computed for different embedding dimensions. The correlation dimension for each system is plotted in figure 2.2. White noise, clearly, has no attractor: by plotting the slope of the curves versus the embedding dimension, no saturation is observed. For the logistic map, $D_2 \approx 1$ (see figure 2.2).

2.1.2 Screening Auditory-Nerve Recordings for Chaotic Activity

In figure 2.3, we show a typical result of the correlation dimension analysis applied to auditory nerve data. We formed a time series by counting the number of discharges that occur in a sequence of relatively short bins [24]. This method will search the intensity for chaotic behavior. If the intensity of a point process is chaotic, the

Figure 2.1 The correlation measure, $C_d(\epsilon)$ is plotted against the cube size, ϵ , for increasing values of embedding dimension, d for a purely random time series (white noise) in panel a and for a purely chaotic system (the logistic map) in panel b. Logarithmic scales are used on both axes. The white noise is exponentially distributed with unit mean and variance. The logistic map is described by $x_{n+1} = 4x_n(1 - x_n)$; a starting value of $x_1 = .2$ is used.

Figure 2.2 The correlation dimension, D_2 is plotted against the embedding dimension, d , for white noise, the logistic map, and for the continuous tone response. This quantity is determined for each system by measuring the slopes of the correlation measure for different embedding dimensions. The correlation measure plots are found in figures 2.1 and 2.3.

Figure 2.3 The correlation measure $C_d(\epsilon)$ is measured from the point process by subdividing the time axis into equal duration bins (1 s) and produce a time series equal to the number of events occurring in each bin. This procedure is designed to study the chaotic nature of the intensity of the point process. The resulting variation of the correlation measure with the radius parameter ϵ and embedding dimension d is shown. The dataset is an auditory-nerve recording from unit 18-94.

measured correlation dimension (see figure 2.2)–the slope of the correlation measure on logarithmic scales–should not change as the imbedding dimension increases. As can be seen, the correlation measure obtained did not correspond to a chaotic phase space structure. Consequently, we interpret this result to indicate that the neural discharges of auditory nerve fibers are truly random, justifying the use of a stochastic fractal model for the point process. We will use yet another measure – the Fano factor (to be defined later) – to establish that a point process with a chaotic intensity function does not describe the data well.

We will now consider another possibility - the discharges of auditory nerve fibers are chaotic (modeled by a point process with a chaotic intensity function) but corrupted by noise while recording. Ben-Mizrachi and Procaccia [4] characterize the behavior of the correlation measure of a noisy trajectory which is obtained when the states of a chaotic system are measured in a noisy environment. The simplest model for such a system would be to add a noise component to the state vector. When the the random part dominates, the trajectory will fill space and hence the correlation measure, $C_d(\epsilon)$, will scale according to the embedding dimension d : $C_d(\epsilon) \sim \epsilon^d$. For scales above that characterizing the noise, the correlation measure should reflect chaotic behavior. The break point is determined by the noise strength: for higher noise strengths, the influence of noise is observed in larger scales. The break point is the same for all embedding dimensions, d . No such behavior is observed in the plots

of correlation measure for auditory-nerve data (see figure (2.3)). This result further confirms the use of a stochastic model for auditory-nerve recordings.

2.2 Stochastic Fractals

We will now investigate to see if a doubly stochastic Poisson process with a stochastic fractal intensity process can describe the variability and spectral characteristics of the the auditory nerve fiber discharges (see figures 1.2 and 1.3). For stochastic fractals, the concept of self-similarity is defined with respect to the sample function of the random process. The concept of dimension is associated with the graph of the sample function. We will define both these terms with specific reference to the *fractional Brownian motion* (fBm). Subsequently, we will derive the intensity process for a doubly stochastic point process from fractional Gaussian noise.

2.2.1 Fractional Brownian Motion and Fractional Gaussian Noise

The Brownian motion process, $B(t)$, has increments $B(t+s) - B(t)$ that are Gaussian with mean zero and variance s and that are independent for non-overlapping increments of time. A fractional Brownian motion can be defined as a moving average of the incremental process, $dB(t) = B(t+dt) - B(t)$, where the past increments of $B(t)$, $dB(s)$, are weighted by the fractional quantity $(t-s)^{H-1/2}$. Thus a fractional Brownian motion is defined by the integral

$$B_H(t) = \int_{-\infty}^t (t-s)^{H-1/2} dB(s), -\infty < t < \infty. \quad (2.2)$$

This integral is divergent and hence only increments of $B_H(t)$ are mathematically feasible. The increment process is given by

$$B_H(t) - B_H(0) = \int_{-\infty}^t (t-s)^{H-1/2} dB(s) - \int_{-\infty}^0 (-s)^{H-1/2} dB(s). \quad (2.3)$$

In this equation, the subtraction must take place before the lower limit is allowed to go to minus infinity. Incorporating the above constraints, we will use the following specialized definition of fractional Brownian motion, $\mathbf{B}_H \triangleq \{B_H(t); t \in \mathbf{R}^1\}$ [3]:

$$\begin{aligned} & B_H(t, \omega) - B_H(0, \omega) \\ &= \frac{1}{\Gamma(H + 1/2)} \left[\int_{-\infty}^0 (|t-s|^{H-1/2} - |s|^{H-1/2}) dB(s) + \int_0^t |t-s|^{H-1/2} dB(s) \right], t \in \mathbf{R}^1, \end{aligned} \quad (2.4)$$

where $\mathbf{B} \triangleq \{B(t); t \in \mathbf{R}^1\}$ is the standard Brownian motion and for $t < 0$ the integral $\int_0^t$ should be interpreted as $-\int_t^0$. $\Gamma(\cdot)$ is the gamma function. The above definition of fractional Brownian motion is valid only for H in the range $(0, 1)$ [21, 38]. Values of H in the range one-half to one correspond to, as will be seen, long range dependence in the increments of Brownian motion process, which is of interest to us in modeling recordings from auditory-nerve fibers. $H = 1/2$ corresponds to the standard Brownian motion. $0 < H < 1/2$ corresponds to short-term dependence in the increment process. Note that fBm is a non-stationary process. The covariance function of the fBm has a self-similar structure *i.e.*, $K_{B_H}(at, as) = a^{2H} K_{B_H}(t, s)$ (this definition of self-similarity is different from the one-variable case and requires identical scaling along both components).

The parameter of fBm, H , and the *fractal* dimension of the graph, $\{(t, B_H(t)); t \in (0, \infty)\}$, are related through the expression $D_F = 2 - H$ [35]. The topological dimension is 1 and the Euclidean dimension of $\mathbf{R}^2$ is 2. The fractal dimension of a normal, deterministic curve in $\mathbf{R}^2$ would be 2 and would equal the Euclidean dimension. The sample paths of fractional Brownian motion do not fill space completely; the dimension of the graph is, therefore, fractional, implying that fBm is a fractal process.

Starting with equation (2.4), it can be shown [21] that the increments of fBm are stationary (although fBm is non-stationary) *i.e.*,

$$B_H(t + \tau) - B_H(\tau) \stackrel{d}{=} B_H(t) - B_H(0). \quad (2.5)$$

and that they are self-similar [21] *i.e.*, for each $a > 0$ and $t_0 \in \mathbf{R}^1$,

$$B_H(t_0 + a\tau) - B_H(t_0) \stackrel{d}{=} a^H B_H(\tau). \quad (2.6)$$

Equality in the above expressions is in the distributional sense (denoted by $\stackrel{d}{=}$). The property of self-similarity implies that fBm is statistically the same on all time scales. Self-similarity also implies that the sample paths, as noted before, are fractals. The self-similarity parameter, H , is the same as the parameter used in defining fBm (see equation 2.4) and is related to the fractal dimension, $D_F = 2 - H$.

The covariance function of fBm is given by [3]

$$K_{\mathbf{B}_H}(s, t) = \frac{V_H}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H}), \quad s, t \in \mathbf{R}^1, \quad (2.7)$$

where

$$V_H \triangleq \text{var}[B_H(1)] = \frac{-\Gamma(2 - 2H) \cos(\pi H)}{\pi H(2H - 1)}.$$

Evaluating the covariance function at $s = t = T$, we obtain the variance of B_H to be $V_H T^{2H}$; the variance has a power-law behavior in that the variance increases as a power of T . The sequential range of the process is the peak to peak variation of a sample function of the process over a time period, T ; this quantity also obeys a similar power law (see [21]) and is of interest to hydrologists. Such power law behavior is a consequence of the fractal nature of the process. Since the variance goes to zero as T goes to zero, fBm is mean square continuous.

Mandelbrot [21] has shown that for *any* process with stationary and self-similar increments that is mean square continuous, H lies in the range $[0, 1)$. Hence, the restriction in the range of H in the definition of fBm in equation (2.4). Furthermore, the spectrum of the process is not well defined for $H \geq 1$. Since a Gaussian process is determined by its mean and covariance properties, it follows that fBm is the only non-constant Gaussian random function that has stationary and self-similar increments and is mean square continuous.

Mandelbrot [21] has shown that almost all sample paths of fBm for $0 < H < 1$ and t lying in a compact set are continuous. But the process is not differentiable with probability one. Therefore, we cannot obtain a derivative process. But the following process, termed the fractional Gaussian noise (fGn), is defined for any $\delta > 0$

$$\mathbf{B}_{H,\delta}(t) \triangleq \frac{B_H(t + \delta) - B_H(t)}{\delta}. \quad (2.8)$$

$\mathbf{B}_{H,\delta}(t)$ is a zero-mean stationary Gaussian process; using the above definition and the expression for the covariance function of fractional Brownian motion (see equation 2.7) the covariance function of the fractional Gaussian noise can be shown to be

$$K_{B_{H,\delta}}(\tau) = \frac{V_H \delta^{2H-2}}{2} \left[\left(\frac{|\tau|}{\delta} + 1 \right)^{2H} - 2 \left(\frac{|\tau|}{\delta} \right)^{2H} + \left| \frac{|\tau|}{\delta} - 1 \right|^{2H} \right]$$

where $\tau \in \mathbf{R}^1$ (2.9)

$$\cong V_H H(2H-1) |\tau|^{2H-2} \quad |\tau| \gg \delta. \quad (2.10)$$

Thus, the fractional Gaussian noise has a self-similar behavior for large lags. The power spectral density is given by

$$S_{B_{H,\delta}}(\omega) \cong |\omega|^{1-2H}, \quad \omega \in \mathbf{R}^1 \text{ and } 0 < \omega\delta \ll 1. \quad (2.11)$$

The correlation function of fractional Gaussian noise implies that the process is mixing and ergodic but not strongly mixing [28]. The Central Limit Theorem holds trivially because the increments of a Gaussian process are Gaussian. fGn is a good model for processes with strong long-term dependence and the expression for the spectral density reflects a “1/f” behavior (characterized by large amounts of low-frequency energy). The recordings from the auditory-nerve fibers, as noted before, are characterized by such long-term dependence and lowpass spectral behavior.

2.3 Generation of Fractional Gaussian Noise

We will now turn our attention to generating fractional Gaussian noise; the fractional Gaussian noise will later be used to obtain the intensity process for a doubly stochastic

Poisson process through appropriate nonlinear transformation. We will employ the *fast fractional Gaussian noise* (ffGn) generator proposed by Mandelbrot [22]. Chi *et al* [5] proposed an implementation of this generator with simpler parameter selection criteria. We shall follow and extend both of these references in subsequent discussion.

The theory of fractional Gaussian noise, discussed in the preceding section, was formulated in continuous time; this theory is realized in practice in discrete time. The discrete-time version of equation (2.9) is obtained by setting $\delta = 1$. Furthermore, if we standardize the variance of the fractional Gaussian noise *i.e.*, set $\text{var}\{\mathbf{B}_{H,\delta}(t)\} = V_H$, we obtain the following expression for the covariance for the standardized *discrete time fractional Gaussian noise* (dtfGn)

$$K(\tau, H) = \frac{1}{2}((\tau + 1)^{2H} - 2\tau^{2H} + |\tau - 1|^{2H}) \text{ for } \tau = 0, 1, \dots \quad (2.12)$$

The serial correlations are positive for all lags; they sum to infinity, indicating long-term dependence or “ $1/f$ ” spectral behavior. The ffGn, $X_f(t)$, we generate must have the above covariance structure. This ffGn is decomposed into two additive components:

$$X_f(t) = X_L(t) + X_H(t) \quad (2.13)$$

$X_H(t)$ will be a single non-standardized, independent, autoregressive Markov-Gauss process that approximates the small-lag (or high frequency behavior). $X_L(t)$ is the component that will approximate the large-lag behavior and will have covariance function

$$K'_L(\tau, H) = H(2H - 1)\tau^{2H-2} \quad (2.14)$$

which is the approximation of $K(\tau, H)$ for large values of τ . This approximation is expressed as a weighted sum of $N + 1$ standardized auto-regressive Markov-Gauss processes.

The approach is to obtain an accurate approximation of the large-lag behavior through a sum of independent Gauss-Markov processes:

$$X_L(t) = \sum_{n=0}^N W_n^{\frac{1}{2}} M_n(t), \quad (2.15)$$

where $M_n(t)$ denotes a Gauss-Markov process with correlation coefficient r_n and $W_n^{\frac{1}{2}}$ denotes the weighting factor. The covariance function of $X_L(t)$ is given by

$$K'_L(s, H) = \sum_{n=0}^N W_n r_n^s \text{ for } s = 0, 1, \dots$$

The approach is to determine the weighting factors after specifying a suitable sequence of correlation coefficients, r_n . By forcing the $\{r_n\}$ to approach one for increasing values of n , the low-frequency effects that we are looking for will result. Unequally-spaced sequences have been found to be most suitable for this purpose; Mandelbrot proposed

$$r_n = \exp(-B^{-n}) \quad (2.16)$$

with $B > 1$. The spacing in the values of correlation decreases in the range $(e^{-1}, 1)$ with increasing values of n (in the range $[0, \infty)$); this implies that e^{-1} is the point of separation between high and low frequencies. After some manipulation, Chi *et al* [5] determined the covariance of $X_L(t)$ to be

$$K'_L(s, H) = \sum_{n=0}^N W_n \exp(-sB^{-n}),$$

Figure 2.4 Fast fractional Gaussian noise is generated as a bank of $N + 2$ filters (see equation (2.20)). The low-frequency component, $X_L(t)$ is generated by the first $N + 1$ of these filters (see equation (2.15)) and the high-frequency component, $X_H(t)$ by the $(N + 2)^{th}$ filter. The impulse response of these filters is given by $M_n(t)$ which is the n^{th} Markov-Gauss process with a correlation coefficient, r_n . The input, $G_n(t)$, to each filter is comprised of independent and identically distributed Gaussian random variables that have zero mean and unit variance. $M_n(1) = G_n(1)$ is the initial condition for each filter. For the first $N + 1$ filters, $M_n(t) = r_n M_n(t - 1) + (1 - r_n^2)^{1/2} G_n(t)$ and $M_{N+1}(t) = G_{N+1}(t)$. The correlation values r_n are given in equations (2.16) and (2.19) and the weights are given in equations (2.17) and (2.18). The resulting fractional Gaussian noise has zero mean and unit variance.

where the weights are given by

$$W_n = \frac{H(2H-1)}{\Gamma(3-2H)} (B^{1-H} - B^{H-1}) B^{2(H-1)n} \quad (2.17)$$

We now obtain the variance (denoted by W_{N+1}) and serial correlation (denoted by r_{N+1}) of the high frequency component of ffGn, $X_H(t)$, by compensating for the corresponding deficiency in the covariance of the dtfGn *i.e.*,

$$W_{N+1} = K(0, H) - K'_L(0, H) = 1 - \sum_{n=0}^N W_n \quad (2.18)$$

$$\begin{aligned} r_{N+1} &= K(1, H) - K'_L(1, H) \\ &= \frac{1}{W_{N+1}} \left(2^{2H-1} - 1 - \sum_{n=0}^N W_n \exp(-B^{-n}) \right) \end{aligned} \quad (2.19)$$

One could add another high-frequency process to match the serial correlation of the data. We can rewrite the expression for fast fractional Gaussian noise in equation (2.13) as a sum of $N + 2$ filter-outputs

$$X_f(t) = \sum_{n=0}^{N+1} W_n^{\frac{1}{2}} M_n(t) \quad (2.20)$$

with the $N + 2$ weights given by equations (2.17) and (2.18). The impulse response of the first $N + 1$ filters is described by a Markov-Gauss process $M_n(t) = r_n M_n(t-1) + (1 - r_n^2)^{1/2} G_n(t)$ where $G_n(t)$ is white Gaussian noise; the correlation coefficients are given in equation (2.16). The impulse response of the $(N + 2)^{th}$ filter is given by $M_{N+1}(t) = G_{N+1}(t)$ and the correlation coefficient for this filter is given in equation (2.19). The input to these filters is a sequence of independent and identically distributed Gaussian random variables. The input and the resulting ffGn are zero

mean and unit variance. This scheme is described in figure 2.4. Mandelbrot [22] has found that values of 2,3, or 4 to be appropriate for B and Chi *et al* [5] have determined values in the range 15-20 to be suitable for N .

Before we present a simulation to test the above method, we define a measure to quantify the fractal behavior of waveform processes. We will use the *Allan variance* [1, 33] which is defined to be the mean squared value of the difference of the number of events in two djacent intervals of length T :

$$V(T) = \frac{1}{2T^2} E \left[((X_{t_0+T} - X_{t_0}) - (X_{t_0+2T} - X_{t_0+T}))^2 \right] \quad (2.21)$$

Here, $X_{t_0+T} - X_{t_0}$ denotes the number of events that occurred between time t_0 and the time $t_0 + T$. For large T , the Allan variance would be proportional to $1/T$ for a non-fractal process. Solo [33, equation (18a)] relates the Allan variance and the spectral density, $S(\cdot)$:

$$V(T) = \int_0^\infty \frac{(1 - \cos x)^2}{x^2} S\left(\frac{x}{T}\right) \frac{dx}{2\pi T} \quad (2.22)$$

The spectral density for the fractional Gaussian noise varies as ω^{1-2H} (see equation (2.11)). Thus the Allan variance would vary as T^{2H-2} for fGn. Figure 2.5 plots the Allan variance for simulated fractional Gaussian noise for two values of H : $H = .7$ and $H = .95$. The Allan variance plot indicates that the fractional Gaussian noise generated is not a *pure fractal* but has, as expected, power-law behavior for large values of T . This is not surprising because the fractional Gaussian noise, by definition

(see equation (2.10) or equation (2.14)), has a power-law behavior only for large lags.

Figure 2.5 The Allan variance for simulated fractional Gaussian noise is estimated by computing the mean squared value of the difference of the number of events in contiguous, non-overlapping blocks of duration T seconds. Logarithmic scales have been used along both axes. Fractional Gaussian noise has been simulated using the method of Chi *et al* (see equation (2.20) and figure 2.4). Independent, Gaussian random variables having zero mean and unit variance has been used in the simulation; the simulated fGn had zero mean and unit variance. The self-similarity parameter, H , was set equal to .7 and .95 respectively in each case. The parameter, B , was set equal to 4 and N was chosen to be 25 for both choices of H . One million points have been used in computing the Allan variance. The slopes of the straight lines equals $= 2H - 2$ (this is the expected slope - see main text) and is -0.6 for $H = .7$ and -0.1 for $H = .95$.

Chapter 3

Fractal Point Processes

As noted earlier, the auditory-nerve discharge patterns can be modeled by fractal point processes. A fractal point process is a doubly stochastic point process (defined in § 1.1) whose intensity process is a fractal waveform process. In this thesis, we are considering only point processes on the (non-negative) real line. This differs from the self-similar processes derived by O'Brien and Vervaat [10] from the counting process of point processes defined on the Poincaré half-plane.

From the definition of Fano factor (see equation (1.5)), it is clear that this quantity captures the variability of the counting process and the intensity process. We will now discuss as to how the Fano factor of point process data is measured; we will use this quantity extensively to test the different fractal point process models. We will also compare the Fano factor with the spectral density of the increment process and the Allan variance as measures of fractal activity in point process data.

3.1 Computation of the Fano Factor of data

The Fano factor is measured by forming the ratio of the estimated variance of the number of events in an interval of duration T seconds and the estimated mean of the

number. The variance and mean are estimated with simple averages: for a data of duration D seconds, the estimates are given by

$$\hat{m}(T) = \frac{1}{[D/T]} \sum_{n=1}^{[D/T]} (N_{nT} - N_{(n-1)T})$$

and

$$\hat{\sigma}^2(T) = \frac{1}{[D/T]} \sum_{n=1}^{[D/T]} (N_{nT} - N_{(n-1)T})^2 - \hat{m}^2(T).$$

The Fano factor for the data whose average rate is plotted in figure 1.2 is computed in the manner just described and is plotted in figure 3.1. The fractal characteristics of the response for large T is evident.

The covariance of the counting process, N_t , of a doubly stochastic Poisson process is related to the covariance of the stationary intensity process, λ_t [32, page 407]:

$$K_N(t, u) = \int_0^t \int_0^u K_\lambda(\alpha, \beta) d\alpha d\beta - \bar{\lambda}^2 tu + \bar{\lambda} \min(t, u).$$

where $\bar{\lambda}$ is the mean of the intensity process. The covariance of the increment process, $N_{s,t}$, denoted by $K_{N_{inc}}(s, t, u, v)$ is related to the covariance of the counting process:

$$K_{N_{inc}}(s, t, u, v) = K_N(t, v) + K_N(s, u) - K_N(t, u) - K_N(s, v) - \bar{\lambda}^2(t-s)(v-u)$$

where $\bar{\lambda}$ is the mean of the intensity process and $s \leq t \leq u \leq v$. From the above equations we can see that the spectral density of the increment process is related to the spectral density of the intensity process in a complicated fashion. Furthermore, it is not clear that this spectrum is even well defined. Thus, although the power spectrum of the increment process may capture fractal activity, it is not a good measure to quantify fractal effects of point process data.

Figure 3.1 The Fano factor of a continuous tone response (18-94) is plotted against T on logarithmic scales (panel a). The Fano factor is computed by forming the ratio of the estimated variance of the number of spikes falling in an interval of duration T seconds and the estimated mean of the number. The Fano factor of a point process with a chaotic intensity function is also plotted in panel a: the intensity function is described by the logistic map $x_{n+1} = 4x_n(1 - x_n)$ and a starting value of $x_1 = .2$ is used. In panel b, the Allan variance of the response 18-94 is plotted; this quantity is computed as described in § 2.3. Logarithmic scales are used on both axes.

The Allan variance (defined in equation 2.21) is another quantity that can be used to measure fractal activity in point process data. The Allan variance for the continuous tone response 18-94 is plotted in panel b figure 3.1. The Allan variance appears noisier than the Fano factor for large values of T : this is due to the fact that the Allan variance is decreasing while the Fano factor is growing. The Fano factor is a more robust measure of fractal activity than the Allan variance. The fact that the fractal point process is an impure fractal (this term will be defined later) accentuates this difference between the two measures.

Accurate measurement of Fano factor requires intervals at least ten times the fractal time (this term will be defined later in this section); the fractal time for auditory nerve discharges ranges between .1 and 5 seconds. To obtain statistically significant estimates for the largest desired value of T , the number of intervals must be at least ten. Thus, for a fractal time of 5 seconds we require 500 seconds (7.5 minutes) of data for accurate measurement of the Fano factor: this duration, in the context of auditory nerve discharges, is very long. Therefore, fractal analysis and modeling requires long-duration data records.

One other issue that must be addressed in context of using the Fano factor as an indicator of fractal activity: trends. Let us consider a Poisson process with a linearly decreasing intensity function: the rate decreases from an initial value of λ_i to a final value of λ_f . This kind of steady decrease is often seen in recordings and, as we shall show, adversely affects the measurement of the Fano factor. The measured

Fano factor is

$$F(T) = 1 + \frac{(\lambda_i - \lambda_f)^2}{6(\lambda_i + \lambda_f)} T \left(1 - \frac{T^2}{D^2}\right). \quad (3.1)$$

From our earlier discussion the ratio D/T must be at least 10 and therefore, the last term in the above equation is negligible. Therefore the Fano factor varies as T for large T . Consequently, the effect of trends cannot be distinguished from fractal effects having large fractal dimension; because of this artifact, data must be screened for trends before computing the Fano factor.

3.2 Fano Factors of Point Processes

To better understand how the Fano factor depends on fractal behavior as well refractory effects, Fano factors are computed for common point process models, some of which are not fractal (table 3.1). The baseline Fano factor value for all models is one, which occurs only for a stationary Poisson process (see figure 3.2). An absolute deadtime in a stationary renewal process forces the Fano factor to be less than one whenever the interval T is comparable to or greater than the deadtime. The asymp-

<i>Type of Point Process</i>	<i>Fano Factor $F(T)$</i>
Stationary Poisson	1
Renewal with Absolute Deadtime Δ	$\begin{cases} 1 & T \ll \Delta \\ (1 - \bar{\lambda}\Delta)^2 & T \gg \Delta \end{cases}$
Doubly Stochastic Poisson	$1 + \frac{2}{\bar{\lambda}T} \int_0^T (T - \tau) K_\lambda(\tau) d\tau$

Table 3.1 The Fano factors for several point processes are given. $K_\lambda(\cdot)$ is the covariance function of the stationary, intensity process λ_t .

otic value of the Fano factor for a renewal process equals the squared coefficient of variation of the intervals, defined as the ratio of the variance of an interval to the square of its mean [6, Page 72]; according to these authors, this result is a particular case of the following equation

$$\lim_{T \rightarrow \infty} F(T) = \lim_{n \rightarrow \infty} \frac{\text{var}(\tau_1 + \dots + \tau_n)}{nE^2(\tau)} \quad (3.2)$$

where $\text{var}(\cdot)$ denotes the variance of the partial sum of intervals and $E(\cdot)$ denotes the expected value of the intervals. The quantity on the right is an “index of dispersion” for intervals [6] or in our terminology it is a “Fano factor” for intervals. For the renewal case, Cox and Lewis [6] have shown that this quantity reduces to the square of the coefficient of variation for all values of n . Thus, asymptotically, one could conceivably use the two quantities interchangeably to study fractal behavior; but the Central Limit Theorem is used in deriving the above result; the significance of this comment will become apparent later. The Fano factor for a deadtime renewal process simulation is plotted in figure 3.2 and the reduction in Fano factor to below one for T in the range of the deadtime (that equaled 1 ms in the simulation) is seen. Furthermore, the asymptotic value of the Fano factor is found equal to the coefficient of variation of the intervals (0.84) - agreeing with the theoretical prediction.

We will now go back and address the possibility of a stationary point process with a chaotic intensity function describing the fractal behavior of data. The Fano factor of such a point process is plotted in figure 3.1. The intensity function is described by the logistic map: $x_{n+1} = 4x_n(1 - x_n)$ with a starting value of $x_1 = .2$. The Fano

factor of a continuous tone response is also plotted in the same figure. It is evident that the Fano factor of the chaotic point process does not describe the data well.

3.2.1 Fano Factors of Doubly Stochastic Point Processes

In the doubly stochastic Poisson case, the term corresponding to the variability of the *stationary* intensity process appears as an additive term to that of the stationary Poisson. Furthermore, the numerator of the second term in the expression is the integral of the windowed covariance function of the intensity process; this term relates the variability in the doubly stochastic point process to the spectrum of the intensity process. Thus, for an appropriate choice of intensity process, a doubly stochastic Poisson model could describe the variability and low-frequency characteristics of the data.

Let us consider an intensity process obtained by passing white Gaussian noise (zero-mean, unit variance) through a first order filter and then passing it through a square law transformation (such nonlinear transformations are discussed in greater detail later) to obtain a valid, non-negative intensity function. A doubly stochastic Poisson process with this intensity process is simulated by the method described in § 1.1. The filtered-Gaussian process has a covariance function equal to $e^{-|\tau|/a}$ where a is the first-order coefficient. When this process is square law transformed its covariance function is, as will be seen, $2e^{\frac{-2\tau}{a}}$ (see equation (3.20)). The Fano factor

Figure 3.2 Fano factors are plotted for a stationary Poisson process, a dead-time renewal process (with a deadtime of 1 ms), and a doubly stochastic Poisson process with the intensity process being the full-wave rectified, filtered Gaussian noise: zero-mean, unit variance independent Gaussian random variables (G_n) are passed through the first-order filter *i.e.*, $X_n = aX_{n-1} + G_n$, with $a = .8$. This filtered noise is then passed through a full wave rectifier to obtain an intensity process for the doubly stochastic Poisson process. Point processes are generated using the method of Johnson *et al* [13]. The intensity has a constant value of 100 spikes/s in the first two cases and an average rate of 100 spikes/s in third case. The simulated data is 1000 seconds long.

for this point process is obtained using the general expression in table 3.1

$$F(T) = 1 + \frac{2a}{\bar{\lambda}} + \frac{a^2}{\bar{\lambda}} \frac{e^{-2T/a} - 1}{T} \quad (3.3)$$

where $\bar{\lambda}$ is the average rate of the intensity. For large values of T , $F(T)$ is approximately equal to $1 + \frac{2a}{\bar{\lambda}}$. For $a = .8$ and an average rate of 100 spikes/s, the Fano factor is approximately 1.016 for all T . The Fano factor plot of the simulated process indicates that the variability of such a doubly stochastic Poisson process is, as expected, greater than for a stationary Poisson process (see figure 3.2). Ofcourse, it does not display long-term self-similar characteristics.

Now let us consider a stationary intensity process whose covariance function varies as τ^{-D} for large lags *i.e.*, it has a self-similar behavior, similar to that observed for fractional Gaussian noise (see equation (2.10)). Substituting this covariance function in the appropriate Fano factor expression in table 3.1 results in $F(T) = 1 + KT^{-D+1}$; for $0 < D < 1$, the Fano factor, plotted against T on logarithmic scales, will demonstrate the self-similar behavior of the point process. Figure 3.5 plots the Fano factor of a simulated doubly stochastic Poisson process (the point process is generated using the method of Johnson *et al* [13]) with the intensity derived by passing fractional Gaussian noise through a full-wave rectifier (this transformation serves to obtain a valid, non-negative intensity process): the self-similar behavior is clearly seen for values of T greater than a second.

The most pertinent case (in context of modeling auditory-nerve discharge patterns) is *not* contained in the table: a doubly stochastic renewal process having only

an absolute deadtime. By combining two of the table's entries, we can infer what the Fano factor might be for this case. The Fano factor for such a process can be written as

$$F(T) = C_{pp} + K_{pp}T^{D_{pp}} \quad (3.4)$$

where C_{pp} is the deadtime related term, K_{pp} is a constant and D_{pp} is the fractal dimension of the point process. C_{pp} would equal one for the doubly stochastic Poisson process and would be less than one for a doubly stochastic renewal process with deadtime. To study how the above expression for $F(T)$ varies with the different parameters, it is plotted on logarithmic scales varying one variable at a time. In panel a of figure 3.3 only the quantity C_{pp} is varied; as this quantity increases the *fractal time i.e.*, the time, T , before the onset of the self-similar behavior on the Fano factor plot. In panel b of the same figure only K_{pp} is varied and the fractal time decreases with increasing K_{pp} . In figure 3.4 D_{pp} is varied while C_{pp} and K_{pp} are held constant. Clearly, the slope of the Fano factor plot, for large T , increases with increasing D_{pp} . This slope which equals D_{pp} is termed the *fractal dimension* of the point process. Furthermore, the fractal time increases with increasing fractal dimension.

Thus, the two key quantities in describing the Fano factor of a fractal point process are fractal dimension and fractal time. It is, therefore, useful to define these quantities more precisely. The fractal dimension is the slope of the Fano factor plot (on logarithmic scales) once the power-law behavior dominates. The fractal time can

Figure 3.3 In panel a, the quantity $C_{pp} + 4T^{0.9}$ has been plotted against T on logarithmic scales for $C_{pp} = .2, .5, 1$. In panel b, the quantity $1 + K_{pp}T^{0.9}$ has been plotted for $K_{pp} = 1, 2, 4$

Figure 3.4 The quantity $1 + 4T^{D_{pp}}$ has been plotted for $D_{pp} = .5, .7, .9$

be defined using different criteria. We will define fractal time, T_F as the interval at which the slope of the tangent to the Fano factor is 90 percent of the value of the fractal dimension, D_{pp} . Mathematically, we obtain the fractal time by solving the nonlinear equation

$$\log(F(T_F) - C_{pp}) = .9D_{pp} \log T_F. \quad (3.5)$$

It will be useful to take a more careful look into the definition of fractal dimension of a point process and its relation to the fractal dimension of waveform processes. The term fractal point process, as defined in this document, is defined only for a doubly stochastic point process. The fractal nature of the point process is a consequence of the intensity process being a fractal waveform process with self-similar sample paths. For a given sample path of the intensity process, the counting process of a doubly stochastic Poisson process is a nonhomogeneous Poisson process. The counting probability for a nonhomogeneous Poisson process, N_t ; $0 \leq t_0 \leq t$, is [32]

$$\Pr[N_t = n] = \frac{1}{n!} \left(\int_{t_0}^t \lambda_\sigma d\sigma \right)^n \exp \left(- \int_{t_0}^t \lambda_\sigma d\sigma \right). \quad (3.6)$$

A little reflection will show that although the sample paths of the intensity process are self-similar, $a^H(N_{t_0,t} - N_{t_0})$ and $N_{t_0,at} - N_{t_0}$ are not equal in the distribution sense (this is the general definition of self-similarity in [21] modified for positive-valued processes) for arbitrary t_0 . But the variance of the counting process is related to the variance of the integrated intensity process through equation (1.5) and the variance of the integrated intensity process is related to the covariance of the intensity process (see table 3.1). Therefore, the self-similar property of the intensity process results

in a self-similar behavior in the second-order statistics of the counting process. The fractal dimension of the fractional Brownian motion is related to the self-similarity parameter (of sample paths), H , and is equal to $2 - H$ [20]. The variance of fBm varies as t^{2H} (see equation 2.7) and its exponent is related to the self-similarity parameter of the sample paths. Similarly the exponent of the covariance function of fractional Gaussian noise is related to the self-similarity parameter of its sample paths and is equal to $2H - 2$ (see equation (2.10)). But we have defined the exponent in the Fano factor to be fractal dimension, relating it to the covariance function and sample paths of the intensity process rather than the counting process. Furthermore, the fractal dimension is defined with respect to the second-order statistics of the intensity process (and equivalently, the counting process) rather than sample path characteristics (as is the case with fBm). Therefore, the Fano factor is an adequate measure of fractal characteristics.

3.3 Generating a Fractal Intensity Process

The intensity of a point process is, by definition, non-negative. Hence, the fractional Gaussian noise is not an appropriate intensity process. Mandelbrot [21] notes that if the process is not required to be mean-square continuous, we can use a non-Gaussian process with Levy-stable increments instead of Brownian motion in equation (2.4) to obtain self-similar processes. Lamperti [17] defines a Levy-stable distribution as follows: Given two independent random variables, X_1 and X_2 , from a Levy-stable

distribution, then for every pair of constants, $a_1 > 0$, $a_2 > 0$ we can find a constant $a_3 > 0$ such that $a_1 X_1 + a_2 X_2 = a_3 X_3$ where X_3 belongs to the same distribution. Levy-stable increments could, then, be used to define the intensity process. But such increment processes are known to have infinite variance. Lowen and Teich [19] propose an approximation to $1/f$ shot noise that has the desired spectral density, finite first-order amplitude statistics and a non-trivial amplitude distribution *i.e.*, the amplitude of the process is not infinite with probability one. In this thesis, we will attempt to derive the desired intensity process from fractional Gaussian noise using an appropriate transformation. It must be emphasized that the intensity process should be derived from the fractional Gaussian noise and not the fractional Brownian motion which is nonstationary. The integral of the intensity process, $\Lambda_t = \int_0^t \lambda_t dt$, occurs frequently in point process theory and therefore, it would be convenient to have it as a well-defined process - we could derive this process from fractional Gaussian process; but this is not an appropriate choice because the fractional Gaussian noise is not differentiable and the intensity of the point process would not exist.

The transformations of interest are those that result in a positive-valued, self-similar intensity process. We will consider only memoryless transformations (or time-invariant systems) *i.e.*, transformations for which the value of the output process at time t_1 , $Y(t_1)$, depends on the input process only through its value at time t_1 , $X(t_1)$. The output processes for such transformations are stationary in the strict sense if the input process is stationary in the strict sense [25, Page 306]; for fractional

Gaussian noise inputs, the output processes will be stationary in the strict sense for all such transformations. These transformations will result in processes that have a non-Gaussian probability density function. But because we are considering only memoryless transformations that have normally distributed inputs, the joint density of the output process is known (see [25, Page 306]).

The correlation structure completely describes a Gaussian process. Therefore, the self-similar property of the process is observed in the distribution of the sample paths (i.e., $X(at) \stackrel{d}{=} a^H X(t)$) and in the correlation function (for large lags). In this thesis, we will use two criteria to test if the resulting process, $Y(t)$, is self-similar: (i) Are the distributions of the time-scaled process $Y(at)$ and $a^{H_Y} Y(t)$ equal for every choice $a > 0$? (H_Y , the self-similarity parameter of the resulting process, could differ from H depending on the transformation) (ii) Does the covariance function of the resulting process reflect a self-similar behavior? In subsequent discussion, all results are derived assuming the parent process - fGn - has zero mean and unit variance.

In modeling data, we have access to the first and second order statistics of the counting process, N_t . The author is not aware of good estimates of the intensity process, λ_t , given a time series that is well described by a doubly stochastic Poisson process. But these statistics of N_t are directly related to the statistics of the *integrated* intensity process, Λ_t . Furthermore, even given the intensity process and given that the integrated intensity process exists, determining the distribution of Λ_t is difficult. The Fano factor relates the variability of the counting process to the covariance function

of the intensity process (see table 3.1). Therefore, it is of interest to see how the nonlinear transformation modifies the correlation structure of the input process - fractional Gaussian noise; this will provide a mathematical framework to select the intensity process of a fractal point process with the desired correlation structure and therefore the desired variability.

A theorem in [32] (this will be discussed in greater detail in a subsequent section) relates the asymptotic distributions of the integrated intensity process and the counting distributions (or PND's - defined in § 1.2). It is observed that the PND's of auditory-nerve recordings are only slightly asymmetric. It is also observed that an intensity process with a non-symmetric pdf results in a PND that is nonsymmetric. Hence the shape of the pdf provides a criterion in selecting an appropriate intensity process to model auditory discharges.

Analysis indicates that some of the transformations yield intensity processes that result in *mixed* fractal point processes: the Fano factor of the point process has components with different fractal exponents. This provides another criterion in selecting a transformation to derive the intensity process.

As we noted earlier (see § 2.2.1), the self-similarity parameter, H , of the fractional Gaussian noise is constrained to values in the range $(\frac{1}{2}, 1)$ for $1/f$ -like spectral behavior. Barton and Poor [3] note that $1/f$ -type spectral behavior corresponds to spectral densities varying as ω^{1-2H} for $1/2 < H < 3/2$. They further note that for $H \geq 1$ the spectrum is not clearly defined; no stationary L^2 process can possess such a spectrum.

In studying different nonlinear transformations, we will also constrain the resulting process to have a spectrum proportional to $\omega^{H'}$ with the exponent H' constrained to the range $(-1, 0)$; the exponent of the lag, τ , in the corresponding covariance function is constrained to values in the range $(-1, 0)$. This will put constraints on the allowable range of the input self-similarity parameter, H .

3.3.1 Memoryless Nonlinear Transformations

Full Wave Rectifier

The transformation $Y(t) = |X(t)|$ satisfies the self-similarity requirement and the resulting process, $Y(t)$, has the same self-similarity parameter, H , as $X(t)$. The covariance function of the output process is given by [25, Page 482]

$$K_Y(\tau) = \frac{2K_X(0)}{\pi}(\cos \alpha + \alpha \sin \alpha - 1) \quad (3.7)$$

where $\sin \alpha$ is the correlation coefficient and is given

$$\sin \alpha = \frac{K_X(\tau)}{K_X(0)} \quad -\frac{\pi}{2} < \alpha \leq \frac{\pi}{2}.$$

For small values of the correlation coefficient, $\sin \alpha$ is approximately equal to α and the covariance function of $Y(t)$ can be approximated by

$$K_Y(\tau) \approx \frac{1}{K_X(0)} K_X^2(\tau) \quad (3.8)$$

where we have adjusted the approximation to have the same variance as the true covariance of the full-wave rectified fGn. Thus, for small correlation in the input

process, the covariance function of the output process is proportional to the square of the covariance function of the input process. If the parent process is fractional Gaussian noise with a covariance function proportional to τ^{2H-2} , the covariance of the process $Y(t)$ would vary according to τ^{4H-4} although both processes have the same self-similarity parameter, H . To obtain a well-defined spectrum we require $-1 < 4H - 4 < 0$ which implies H should lie in the range $(\frac{3}{4}, 1)$. Thus the input process and output process can take on self-similarity parameter values in the range $(\frac{3}{4}, 1)$. Clearly, the covariance function of the input process (fGn) varies in this manner only for large lags; for large lags τ^{2H-2} is small and therefore, the above approximation of the covariance function is justified. For small lags, consider the expression for the covariance of dtfGn (equation (2.12)) which Chi *et al* [5] use to derive the fGn simulator: for $H=.95$, the first-lag covariance is 0.87 and the second-lag covariance is 0.8 and therefore the assumption of small correlation cannot be made.

For this intensity process, the Fano factor of the doubly stochastic Poisson process can be computed using the expression in Table 3.1: if the exact expression for $K_X^2(\tau)$, obtained by squaring the covariance function of fractional Gaussian noise in equation (2.9) is used, it is difficult to analytically evaluate the integral in the second term. Therefore, we expand the covariance function into a Taylor series about the origin and obtain the following approximation for large lags:

$$K_X^2(\tau) = \frac{B_H}{4} |\tau|^{4H-4}. \quad (3.9)$$

In the above approximation, we have set the quantity δ which appears in equation (2.9) equal to one because we simulate in discrete time (see discussion preceding equation (2.12)). The simulation technique of Chi *et al* [5] generates zero-mean, unit variance fGn and hence differs from the mathematical definition of fBm which is in continuous time; this difference affects only the variance of fGn which is seen to be V_H from equation (2.9). Since we are concerned with obtaining fractal processes, we can drop this quantity from detailed consideration. H is the self-similarity parameter of fGn, and B_H is a constant that depends only on H and is given by

$$B_H = \frac{2H(4H-1)(4H-2)(4H-3)}{6} + \frac{2H(2H-1)(2H-2)(2H-3)}{3} + 2H(2H-1). \quad (3.10)$$

Using this approximation in equation (3.8) we obtain the Fano factor of a doubly stochastic Poisson process with the intensity process $\lambda_t = \sqrt{\frac{\pi}{2}K_X(0)}\bar{\lambda}Y(t)$ where $\bar{\lambda}$ is average value of the intensity process. The covariance function of λ_t is $(\frac{\pi}{2K_X(0)})\bar{\lambda}^2$ times the covariance function of $Y(t)$. Using these results we obtain the Fano factor

$$F(T) = 1 + \frac{\pi\bar{\lambda}K_H}{4K_X^2(0)}T^{4H-3} \quad (3.11)$$

where K_H is a constant that depends only on H and is equal to

$$K_H = \frac{B_H}{(4H-3)(4H-2)} \quad (3.12)$$

and B_H is given in equation (3.10). Thus, the Fano factor is proportional to T^{4H-3} for large values of T . Note in the above expression that the factor scaling T^{4H-3} is determined by two factors - H and $\bar{\lambda}$.

In figure 3.5, the Fano factor is plotted for a simulated doubly stochastic Poisson process with a full-wave rectified fractional Gaussian noise as its intensity process. The point process is generated using the method described in § 1.1 and fGn is generated using the method of Chi *et al* [5]. The simulated fractional Gaussian noise has a self-similarity parameter $H = 0.95$. The Fano factor of the simulated doubly stochastic Poisson process is fitted with the generalized Fano factor form for such a process (see equation (3.4) - $C_{pp} = 1$ for the doubly stochastic Poisson case): $F(T) = 1 + 8.8T^{0.8}$ is found appropriate.

The probability density function (pdf) of $Y(t)$, $p_Y(y)$ is given by

$$p_Y(y) = 2p_X(y)u(y) \quad (3.13)$$

where $p_X(\cdot)$ is the pdf of $X(t)$ (normally distributed) and $u(\cdot)$ is the step function. This pdf is plotted in figure 3.6: zero mean, unit variance, white Gaussian noise is passed through a full-wave rectifier and the pdf of the resulting output is estimated.

One could also use a variation of this transformation *i.e.*, $Y(t) = |X(t) + C|$ where C is a constant; clearly, $Y(t)$ is not self-similar according to the first criterion. It is difficult to obtain an analytical expression for the correlation function of the transformed process. When the shift, C , is large compared to the mean and variance of the input process, the first and second order statistics of $|X(t) + C|$ can be approximated

Figure 3.5 Fano factors of doubly stochastic Poisson processes are plotted for two different intensity processes: (i) a full-wave rectified filtered-Gaussian process (denoted by the legend “Filtered”) (ii) a full-wave rectified fractional Gaussian noise (denoted by the legend “FWR”). Fractional Gaussian noise (zero mean, unit variance, $B = 4$, $N = 25$, $H = 0.95$) is simulated using the method of Chi *et al* [5]; the intensity process is then obtained by passing it through the appropriate transformation. The point process is generated using the method described in figure 1.1. An average rate of 100 spikes/s has been used in simulating the intensity processes. The simulation records are 1000 seconds long. The approximate theoretical Fano factor corresponding to the transformation, given by equation (3.11), is plotted (denoted by the legend “Th approx”). Curve fits to the simulated Fano factors are plotted: $F(T) = 1 + 8.8T^{0.8}$ is found appropriate for this transformation.

Figure 3.6 The marginal pdf of the output process, $Y(t)$, is plotted for the following cases: $Y(t) = |X(t)|$ (denoted by the legend “FWR”), $Y(t) = |X(t)+2|$ (denoted by the legend “Shifted”), and $Y(t) = \frac{1}{3} \sum_{n=1}^{n=3} |X_n(t)|$ (denoted by the legend “Sum”) The pdf is estimated by a histogram; a bin-width of 40 ms is used in each case. The expressions for the pdf for the single process transformations are given in equations (3.13) and (3.14). One million samples of $Y(t)$ is used in each estimate.

by those of $X(t) + C$ or $|X(t)| + C$ (note that these approximations are not self-similar according to the first criterion). The Fano factor, plotted in figure 3.7, indicates that $Y(t) = |X(t) + C|$ preserves the self-similarity property. The best fit for this Fano factor is $F(T) = 1 + 3.8T^{0.8} + 5T^{0.85}$, suggesting that the shifted transformations result in a fractal process comprised of two components - each having a different fractal dimension.

The marginal probability density function of the output process is given by

$$p_Y(y) = [p_X(y - C) - p_X(-y - C)]u(y) \quad (3.14)$$

where $p_X(\cdot)$ denotes the distribution of the input, normal process and $u(\cdot)$ denotes the step function. Thus, for larger C the density is more symmetric. The pdf estimate of zero mean, unit variance, white Gaussian noise passed through this transformation is plotted in figure 3.6. The shifted transformation, as expected, produces a more symmetric pdf. The variance has increased (as compared to the output of the unshifted full-wave rectifier). It is interesting to note that although the pdfs for the shifted and unshifted transformations are very different, the Fano factor plots are *not* very different (see figure 3.7).

Half Wave Rectifier

It can be verified that $Y(at) \stackrel{d}{=} a^H Y(t)$ for the transformation $Y(t) = \frac{X(t) + |X(t)|}{2}$, the self-similarity parameter for $Y(t)$ equals H . The covariance function, $K_Y(\tau)$, is

Figure 3.7 Fano factors of doubly stochastic Poisson processes are plotted for the following intensity process: a shifted ($C = 2$), full-wave rectified fractional Gaussian noise (denoted by the legend “Shifted”). Details of generating fractional Gaussian noise is given in the caption of figure 3.5. The intensity process is then obtained by passing it through the appropriate transformation. The point process is generated using the method of Johnson *et al* [13]. An average rate of 100 spikes/s has been used in simulating the intensity process. The simulation records are 1000 seconds long. Curve fits to the simulated Fano factor is plotted: $F(T) = 1 + 3.8T^{0.8} + 5T^{0.85}$ is found appropriate for this transformation. The Fano factor of a fractal point process with the intensity process obtained through full wave rectification of fGn, plotted in figure 3.5, is reproduced here for comparison.

determined to be [25, Page 483]

$$K_Y(\tau) = \frac{1}{4}(K_F(\tau) + K_X(\tau)) \quad (3.15)$$

where $K_F(\cdot)$ denotes the covariance function of the full-wave rectified process. With the assumption that the correlation of the input process is small (as discussed for the full wave rectifier the correlation of fGn is small for large lags), the approximation used in equation (3.8) holds and we obtain the following approximation for $K_Y(\tau)$

$$K_Y(\tau) \approx \frac{1}{4K_X(0)}K_X^2(\tau) + \frac{1}{4}K_X(\tau) \quad (3.16)$$

Using the large lag approximation for each term from equations (3.9) and (2.10) respectively, we obtain the following expression for the Fano factor of the doubly stochastic Poisson process with the intensity process equal to $\sqrt{\frac{2\pi}{K_X(0)}}\bar{\lambda}Y(t)$ where $Y(t)$ is a half-wave rectified fGn (zero mean) and $\bar{\lambda}$ is its average rate.

$$F(T) = 1 + \frac{\pi\bar{\lambda}K_H}{4K_X^2(0)}T^{4H-3} + \frac{\pi\bar{\lambda}}{2K_X(0)}T^{2H-1} \quad (3.17)$$

where K_H is given in equation (3.12). Note that the covariance of $Y(t)$ reflects the mixture of two fractal processes although the self-similarity parameters of the input and output sample paths are equal; the nonlinearity of the transformation has effected the correlation structure but not the sample paths of the process. If the input is fractional Gaussian noise whose covariance function is proportional to τ^{2H-2} , the output process would have a covariance that varies as $K_1\tau^{2H-2} + K_2\tau^{4H-4}$ where K_1 and K_2 are constants. If each component has a well-defined spectrum then the

mixture would have a well-defined spectrum; the component with a smaller range of allowable values for H would determine the allowable range for the mixture. Thus, H must lie in the range $(\frac{3}{4}, 1)$.

The Fano factor of a doubly stochastic Poisson process with the intensity process obtained through this transformation varies as a linear combination of T^{2H-1} and T^{4H-3} . The Fano factor for this simulated point process is plotted in figure 3.8. The simulation uses $H = .95$. The approximate theoretical expression for the Fano factor (given in equation 3.17) is also plotted in the figure. We fit the Fano factor for the simulation with an expression of the form in equation (3.4): the best fit is determined to be $F(T) = 1 + 80T^{0.8} + 10T^{0.85}$. This fit confirms the theoretical prediction of a mixture in the resulting fractal point process. But the fractal dimensions of the components in the simulation as indicated by the curve fit differ from the theoretical prediction. We would expect the component with the larger fractal dimension (0.85) to dominate for large T and the other component to dominate for small T (see figure 3.4). Theory predicts a higher “fractal strength” - this is seen in the fact that the Fano factor plot is shifted vertically with respect to the simulated plot.

The probability density function for the half-wave rectified process is given by [25, Page 481]

$$p_Y(y) = \frac{1}{2}\delta(y) + p_X(y)u(y) \quad (3.18)$$

where $\delta(\cdot)$ is the delta function. The pdf is plotted in figure 3.9.

Figure 3.8 Fano factors of doubly stochastic Poisson processes is plotted for intensity processes generated by passing fGn (simulation details are given in caption of figure 3.5) through a half wave rectifier. The average rate of the intensity process is 100 spikes/s. The simulation records are 1000 seconds long. The best fit of the simulated Fano Factor is determined to be $F(T) = 1 + 80T^{0.8} + 10T^{0.85}$ and is plotted. The approximate theoretical Fano factor, given in equation (3.17) is also plotted.

Figure 3.9 The marginal pdf is plotted for the following transformations: $Y(t) = \frac{X(t)+|X(t)|}{2}$ (denoted by the legend “HWR”), $Y(t) = \frac{(X(t)+2)+(|X(t)+2|)}{2}$ (denoted by the legend “Shifted”), and $Y(t) = \frac{1}{3} \sum_{n=1}^3 \left[\frac{X_n(t) + |X_n(t)|}{2} \right]$ (denoted by the legend “Sum”) where $X(t)$ and $X_n(t)$ are zero mean, unit variance white Gaussian processes and $X_n(t)$ are assumed to be independent. The histogram estimate of the rectified process is plotted; a binwidth of 40 ms is used in the estimate. The expressions for the pdf in the single process transformations are given in equations (3.18) and (3.19) (note that the theoretical expressions contain an impulse at the origin). The densities have an impulse at the origin. One million time samples of $Y(t)$ is used to estimate the pdf in each case.

One could, as in the case of the full-wave rectifier, consider a shifted half-wave rectifier as a possible transformation to obtain a positive-valued, self-similar intensity process. Analytical results are difficult to obtain for the covariance of the output process. A doubly stochastic Poisson process is simulated using this transformation to obtain the intensity process; the Fano factor for this point process is plotted in figure 3.10. The simulation indicates that the transformation results in a self-similar intensity process, although the sample paths of $Y(t)$ are not self-similar. We thus see that the resulting process has two components with different fractal dimensions. The weighting of the two components suggest that the component with fractal dimension 0.8 dominates for small and medium values of T while the other component will be the asymptotic fractal dimension of the mixture.

The pdf of the output process is given by

$$p_Y(y) = P_X(-C)\delta(y) + p_X(y - C)u(y) \quad (3.19)$$

where $P_X(\cdot)$ and $p_X(\cdot)$ denote the distribution function and pdf of the input process (fGn), $\delta(\cdot)$ is the delta function, and $u(\cdot)$ denotes the step function. In figure 3.9 the histogram estimate of the pdf of a half-wave rectified, zero-mean, unit variance white Gaussian noise is plotted. Again, the shifted transformation yields a density function that is more symmetric around the mode.

Figure 3.10 Fano factors is plotted for a doubly stochastic Poisson process with the intensity process generated by passing fGn (simulation details are given in caption of figure 3.5) through a shift operator ($C = 2$) and then a half wave rectifier. The average rate of the intensity process is 100 spikes/s. The simulation records are 1000 seconds long. The best fit of the simulated Fano Factor is determined to be $F(T) = 1 + 7T^{0.8} + 3T^{0.9}$ and is plotted. The Fano factor of the point process with the intensity process derived from the unshifted transformation, plotted in figure 3.8, is reproduced here for comparison.

Square Law

The sample paths of $Y(t) = X^2(t)$ are self-similar but the parameter is $2H$. The covariance function for $Y(t)$ is given as [25]:

$$K_Y(\tau) = 2K_X^2(\tau). \quad (3.20)$$

Thus, the covariance function of $Y(t)$ is proportional to τ^{4H-4} when the correlation of the input process (fractional Gaussian noise), $X(t)$, has a large-lag behavior of τ^{2H-2} . Here the nonlinearity of the transformation modifies the sample path behavior as well as the covariance function of the resulting process. The square law transformation produces a pure fractal process for large lags. For a fractional Gaussian noise input, we find that the output process has a well-defined spectrum when $3/4 < H < 1$. Using the approximation for $K_X^2(\tau)$ in equation (3.9), the Fano factor of the point process with the intensity process derived through a square law transformation of zero-mean, unit variance fGn: $\lambda_t = \frac{\bar{\lambda}}{K_X(0)}Y(t)$ where $\bar{\lambda}$ is the average rate of the intensity process is

$$F(T) = 1 + \frac{\bar{\lambda}K_H}{K_X^2(0)}T^{4H-3}. \quad (3.21)$$

The Fano factor thus varies as T^{4H-3} for large T . The Fano factor for a simulated doubly stochastic Poisson process with this intensity process is plotted in figure 3.11. The simulation uses $H = 0.95$, which implies that we would expect $F(T)$ to vary as $T^{0.8}$ for large time periods, T . The simulation confirms this large-lag behavior. The theoretical expression does not match the simulated Fano factor suggesting that the

large-lag approximation has limited validity. This suggests that the high-frequency components of the spectrum of the intensity process do make a significant contribution to the value of the Fano factor.

Papoulis [25, Page 481] gives us the pdf for the transformed process:

$$p_Y(y) = \frac{1}{\sqrt{2\pi K_X(0)y}} e^{-\frac{y}{2K_X(0)}} u(y) \quad (3.22)$$

This pdf is plotted in figure 3.12. Moments of all orders of a zero-mean Gaussian process is completely determined by the correlation function and easily computed [37, Page 229]. Hence moments of all orders of a square-law transformed Gaussian process is easily obtained.

As in the previous transformations, we could use a shifted square-law detector to obtain $Y(t)$ from $X(t)$. The sample paths of this process are not self-similar according to the first criterion. The covariance function for the process $Y(t) = (X(t) + C)^2$ is given by

$$K_Y(\tau) = 2K_X^2(\tau) + 4C^2 K_X(\tau). \quad (3.23)$$

The Fano factor when the intensity process is equal to $\frac{\bar{\lambda}}{(K_X(0)+C^2)}Y(t)$ is given by

$$F(T) = 1 + \frac{\bar{\lambda}K_H}{(K_X(0) + C^2)^2} T^{4H-3} + \frac{4C^2\bar{\lambda}}{(K_X(0) + C^2)^2} T^{2H-1} \quad (3.24)$$

where K_H is a constant that depends only the self-similarity parameter, H , $\bar{\lambda}$ is the average rate of the intensity process, and C is the shifting constant. The assumptions and approximations made here parallel the derivation for the full-wave rectifier and half-wave rectifier cases. Thus, using a shifted square-law detector results in a

Figure 3.11 Fano factor is plotted for a doubly stochastic Poisson process with the intensity process derived from fGn (simulation details are as in figure 3.5) with a square law transformation. The average rate is 100 spikes/s. The simulation records are 1000 seconds long. The theoretical expression for the Fano factor for this transformation (see equation (3.21)) is plotted for $\bar{\lambda} = 100$ and $H = .95$. The simulated Fano factor is best fit with the expression $F(T) = 1 + 33.5T^{0.8}$; this fit is also plotted.

Figure 3.12 The marginal pdf estimate of the output process, $Y(t)$, is plotted for the following cases: $Y(t) = X^2(t)$ (denoted by the legend “SL”), $Y(t) = (X(t) + 2)^2$ (denoted by the legend “Shifted”), and $Y(t) = \frac{1}{3} \sum_{n=1}^3 X_n^2(t)$ (denoted by the legend “Sum”) where $X(t)$ and $X_n(t)$ are zero-mean, unit variance white Gaussian noise and $X_n(t)$ are assumed independent. The expressions for the pdf for the single process transformations are given in equations (3.22) and (3.25). The expression for the density function for the sum of the squared processes is given in equation (3.37). The histogram is used to estimate the pdf for each case and a binwidth of 75 ms is used.

covariance function that is a mixture of two different decay rates for large lags. The asymptotic Fano factor is given by have two exponents - $4H - 3$ and $2H - 1$. The relative contribution of each component is determined by the value of the shift. The component with the fractal dimension $2H - 1$ will dominate asymptotically, but when it is the dominant component is determined by the value of C . This can be seen from our study of the variation of the Fano factor with K_{pp} (see § 3.2). Furthermore, their relative contributions to the Fano factor at small T changes with the value of C . A doubly stochastic Poisson process is simulated using this transformation (with $C = 2$) to derive the intensity process from fGn; the corresponding Fano factor is plotted in figure 3.13. Again we observe the discrepancy between the approximate theoretical expression of the Fano factor and the simulated Fano factor. The component with the higher fractal dimension would dominate for large T and the component with the smaller fractal dimension will dominate for small T ; the crossover value of T is determined by the relative strengths of the two components. Therefore, for larger C , the the second component in equation (3.24) will begin dominate at an earlier time, T . The marginal pdf for this transformation is easily obtained for a zero mean, unit variance Gaussian process, $X(t)$, using the Fundamental Theorem [25, Page 126]

$$p_Y(y) = \frac{1}{2\sqrt{y}} [p_X(\sqrt{y} - C) + p_X(-\sqrt{y} - C)] u(y) \quad (3.25)$$

Figure 3.13 Fano factor is plotted for a doubly stochastic Poisson process with the intensity process derived from fGn (simulation details are as in figure 3.5) by shifting it with a constant and then squaring it. The average rate is 100 spikes/s. The simulation records are 1000 seconds long. The theoretical expression for the Fano factor for this transformation (see equation 3.24) is plotted for $\bar{\lambda} = 100$ and $H = .95$. The simulated Fano factor is best fit with the expression $F(T) = 1 + 12.5T^{0.8} + 12.5T^{0.9}$; this fit is also plotted. The Fano factor for the unshifted transformation, plotted in figure 3.11, is reproduced here for comparison.

Exponentiation

This transformation described by $Y(t) = e^{X(t)}$ results in a positive-valued process.

From the Taylor series expansion for the output process

$$Y(t) = e^{X(t)} \approx 1 + X(t) + \frac{X^2(t)}{2!} + \frac{X^3(t)}{3!} + \dots \quad (3.26)$$

we see that we obtain the process whose sample paths are self-similar with a single parameter, H , only for $X(t)$ with small variance, for which the stochastic approximation $Y(t) = 1 + X(t)$ is valid. When the series can be approximated by finitely many terms, the output process is a linear combination of self-similar processes with parameters $H, 2H, 3H, \dots$. We can consider approximating the stochastic process with a different criterion *i.e.*, the magnitude of its correlation function. Using the definition of characteristic functions [37, Page 185], it can be shown that for zero-mean, unit variance fGn input the output process, has the covariance function

$$K_Y(\tau) = e \left[\exp \left(\frac{K_X(\tau)}{K_X(0)} \right) - 1 \right] \quad (3.27)$$

For small correlation in the input, the above expression can be approximated by $\frac{e}{K_X(0)} K_X(\tau)$. For fractional Gaussian noise input, this approximation is valid only for large lags.

The approximate Fano factor for the intensity process $\lambda_t = \bar{\lambda} e^{-1/2} Y(t)$ is given by

$$F(T) = 1 + \frac{\bar{\lambda} e^{-1}}{2} T^{2H-1}; \quad (3.28)$$

it varies as T^{2H-1} for large T . The Fano factor is computed for a simulated doubly stochastic Poisson process with the intensity process being the exponentiated frac-

tional Gaussian noise and is plotted in figure 3.14. The simulation uses $H = .95$. It is difficult to fit the entire curve with one component *i.e.*, with an expression of the form $1 + K_{pp}T^{D_{pp}}$; $1 + 15T^{.8} + 12.5T^{.85}$ describes the Fano factor plot well implying that the output process is a mixture of atleast two components with different fractal dimensions: the component with the smaller fractal dimension ($D_{pp} = 0.8$) is dominant for small T and the component with the larger fractal dimension ($D_{pp} = .85$) is dominant for large T . Furthermore, since the two components have similar strengths, the crossover between the two components takes place at about $T = 1$ second. This effect is to be expected from our analysis in § 3.2.1 and figure 3.4 which indicates that the crossover for constant C_{pp} and K_{pp} occurs at $T = 1$ second. The theoretical Fano factor (equation (3.28)) is also plotted in the same figure and the fact that it does not match the simulated Fano factor suggests that the large-lag approximation used in deriving the Fano factor is not accurate.

When $X(t)$ is normally distributed with zero mean and unit variance, $Y(t)$ has a log-normal distribution [25, page 99, second edition, 1984]:

$$p_Y(y) = \frac{1}{\sqrt{2\pi K_X(0)y}} \exp\left(-\frac{\ln^2 y}{2K_X(0)}\right). \quad (3.29)$$

The pdf is plotted in figure 3.15.

One could also consider similar transformations: $\exp(X^2(t))$, $\exp([X(t) + C]^2)$, $\exp(|X(t)|)$ and $\exp(|X(t) + C|)$. One obtains an approximate self-similar process for small values of input variance and C . The behavior of the output process of such transformations for large values of the input process or C will depend on how

Figure 3.14 The Fano factor of a doubly stochastic Poisson process with an intensity process generated using the Exponentiation transformation is plotted. The intensity process is generated by passing fractional Gaussian noise (see figure 3.5 for simulation parameters) through this transformation. The average rate of the intensity process is 100 spikes/s. The simulation record is 1000 seconds long. The function $1 + 15T^{.8} + 12.5T^{.85}$ is also plotted.

Figure 3.15 The marginal pdf estimate is plotted for the following transformations: $Y(t) = \exp(X(t))$ and $Y(t) = \frac{1}{3} \sum_{n=1}^3 \exp(X_n(t))$ where $X(t)$ and $X_n(t)$ are zero mean, unit variance Gaussian processes and $X_n(t)$ are assumed independent. The expressions for the pdf in the single process transformation given in equation (3.29). The histogram, with a binwidth of 150 ms, is used to estimate the pdf from one million time samples of $Y(t)$.

the variance of $X(t)$ compares to the value of C (assuming zero-mean for the input process); we can also obtain stochastic approximations with small input correlation as criterion which is a very appropriate in context of “1/f” spectral behavior of fractals whose large-lag covariance is well approximated by τ^{2H-2} (for fGn) and is small.

3.3.2 Combinations of Nonlinear Transformations

We will now consider a different class of transformations - one that involves taking product of independent, self-similar processes: $Y(t) = |X_1^2(t)X_2^2(t)|$. The resulting process is positive-valued and has self-similar sample paths with parameter $2(H_1 + H_2)$ where H_1 and H_2 are the self-similarity parameters of the input processes $X_1(t)$ and $X_2(t)$ respectively. This transformation multiplies two square law transformed processes. With the assumption that the two input processes are not correlated, the covariance of the output process is equal to the product $K_{SL_1}(\tau)K_{SL_2}(\tau)$ where $K_{SL_i}(\tau)$, $i = 1, 2$ are the covariance functions of the processes obtained by square law transforming $X_1(t)$ and $X_2(t)$ respectively. The covariance function of the resulting process is obtained using the result for a single square law transformed process (see equation (3.20)).

$$K_Y(\tau) = 2K_{X_1}^2(\tau) + 2K_{X_2}^2(\tau) + 4K_{X_1}^2(\tau)K_{X_2}^2(\tau) \quad (3.30)$$

where we have, as before, used the fact that the input - fGn - is zero mean. Using the large-lag approximation of equation (3.9), we obtain the following expression for

the covariance of the output process

$$K_Y(\tau) = \frac{B_{H_1}}{2}|\tau|^{4H_1-4} + \frac{B_{H_2}}{2}|\tau|^{4H_2-4} + \frac{B_{H_1}B_{H_2}}{4}|\tau|^{4(H_1+H_2)-8} \quad (3.31)$$

If the output process is to be a stationary L^2 process, the sum $H_1 + H_2$ is constrained to lie in the range $(\frac{7}{4}, 2)$ and the individual parameters are constrained to $\frac{3}{4} < H_1, H_2 < 1$.

The Fano factor for $\lambda_t = \bar{\lambda}Y(t)$ (as before $\bar{\lambda}$ denotes the average rate of the intensity process) is computed to be

$$F(T) = 1 + \frac{\bar{\lambda}K_{H_1}}{K_{X_1}^2(0)}T^{4H_1-3} + \frac{\bar{\lambda}K_{H_2}}{K_{X_2}^2(0)}T^{4H_2-3} + \frac{\bar{\lambda}K_H^{prod}}{2K_{X_1}^2(0)K_{X_2}^2(0)}T^{4(H_1+H_2)-7} \quad (3.32)$$

where K_H^{prod} is equal to

$$\frac{B_{H_1}B_{H_2}}{(4(H_1 + H_2) - 7)(4(H_1 + H_2) - 6)}, \quad (3.33)$$

K_{H_i} is defined in equation (3.12), and B_{H_i} are defined according to equation (3.10).

This transformation allows us to control the fractal dimension of the three components in the output process and obtain the desired impure fractal process. The Fano factor is plotted for a simulated doubly stochastic Poisson process with the intensity process derived according to this product transformation in figure 3.16. The simulation results are presented for two cases: $H_1 = H_2 = .95$ and $H_1 = .85, H_2 = .95$. The best fit for the Fano factor with equal self-similarity parameters is found to be $F(T) = 1 + 10T^{.68}$. The best fit for the second case is determined to be $F(T) = 1 + 20T^{0.6} + 5T^{0.95}$.

The probability density function for the intensity process derived in this manner is given by [25, Page 205]

$$p_Y(y) = \int_{-\infty}^{\infty} \frac{1}{|w|} p_{Y_1}(w) p_{Y_2}(z/w) dw; \quad (3.34)$$

Figure 3.16 Fano factor of a doubly stochastic Poisson process with the intensity process being the product of two square law transformed, independent fractional Gaussian noise processes (zero mean, unit variance, $B=4$, $N=25$). The Fano factors are computed for two cases: $H_1 = H_2 = .95$ and $H_1 = .85, H_2 = .95$. The average rate of the intensity process is 100 spikes/s. The simulation records are 1000 seconds long. The best fits for each case are plotted: for $H_1 = H_2$, it is given by $1 + 10T^{0.68}$ and for $H_1 \neq H_2$ it is given by $1 + 20T^{0.6} + 5T^{0.95}$.

where $p_{Y_i}(\cdot)$ denote the pdf of independent, square law transformed zero mean, unit variance Gaussian processes. This expression is difficult to evaluate analytically; it is plotted in figure 3.17.

Similar transformations such as $Y(t) = |X_1(t)X_2(t)|$, $Y(t) = |(X_1(t) + C)(X_2(t) + C)|$, and $Y(t) = (X_1(t) + C)^2(X_2(t) + C)^2$ can be used. The transformation with shifts would not have self-similar sample paths for arbitrary values of C but the correlation structure of the resulting process may yet be approximately self-similar.

Now we will consider a transformation of the form

$$Z_N(t) = \frac{1}{N} \sum_{n=1}^{n=N} Y_n(t) \quad (3.35)$$

where N is finite. $Y_n(t)$ are independent processes and identically distributed with identical self-similarity parameters. These processes are obtained by passing fractional Gaussian noise through one of the nonlinear transformations described earlier in this section. Clearly, the process $Z_N(t)$ has sample paths that are self-similar with the same parameter as the sample paths of the individual processes. Furthermore, since the processes $Y_n(t)$ are independent, the covariance function of $Z_N(t)$ is

$$K_{Z_N}(\tau) = \frac{1}{N} K_Y(\tau), \quad (3.36)$$

where $K_Y(\tau)$ is the common covariance function of the input processes. Therefore, this transformation causes a downward scaling of the covariance function. The effect of this scaling on the Fano factor is maximal for small T and delays the onset of self-similar behavior: it increases the fractal time.

Figure 3.17 The pdf estimate is plotted for the product of two independent Gaussian processes (zero mean, unit variance), and for the transformation $Y(t) = \frac{1}{3} \sum_{n=1}^{n=3} X_{1,n}^2(t) X_{2,n}^2(t)$. $X_{1,n}(t)$ and $X_{2,n}(t)$ are pairwise independent for all n and are zero mean, unit variance. The histogram estimator is used with a binwidth of 40 ms. One million samples of $Y(t)$ are used in the estimate.

The summed square law transformation is used to generate an intensity process with 15 processes in the summation. A doubly stochastic Poisson process with this intensity process is generated and its Fano factor computed. $F(T)$ is plotted in figure 3.18. Comparing with the Fano factor plot for $\lambda_t = X^2(t)$, we see that the summation increases the fractal time or delays the onset of power-law behavior.

It is also observed that the shifted transformations, for the shift magnitudes considered here, produce a mixture of fractal components with different dimensions. It does not affect the fractal time.

Note that the distribution of the random variable $Z_N(t_1) = \frac{1}{N} \sum_{n=1}^N Y_n(t_1)$ is simply the convolution of N independent random variables, meaning that the pdf of $Z_N(t_1)$ varies with the number N of terms in the sum. Let us consider the case of $Z_N(t) = \frac{1}{N} \sum_{n=1}^N X_n^2(t)$. Assuming that each $X_n(t)$ is a fractional Gaussian noise process with identical parameter, H , the random variable $Z_N(t_1)$ (for every time t_1) is a sum of squared Gaussian random variables. The pdf for the resulting random variable, $Z_N(t_1)$ is the well-known χ^2 -distribution which, for zero-mean, unit variance fGn input is [25, Page 250]:

$$p_{Z_N}(z) = \frac{1}{2^{\frac{N-1}{2}} (K_X(0)/N)^{(N-1)/2} \Gamma(\frac{N-1}{2})} z^{(N-3)/2} e^{-Nz/2K_X(0)} u(z) \quad (3.37)$$

where $\Gamma(\cdot)$ is the gamma function and $u(\cdot)$ denotes the step function. The pdf estimate for a sum of $N = 3$ square law transformed processes is plotted in figure 3.12; this estimate is also plotted for the single square law transformed fGn process. The estimate for the sum has been normalized to have the same mean as the single com-

ponent case. The estimates confirm the theoretical prediction that summing identical processes results in a more symmetric density for the intensity process. The pdf estimate of the process obtained by summing $N = 3$ outputs of each of the following nonlinear transformations: the Full-wave rectifier, the Half-wave rectifier, the exponentiation, and the product transform is computed and plotted in figures 3.6, 3.9, 3.15, and 3.17 respectively. In each case, the summing results in a pdf that is more symmetric than the individual density functions.

In obtaining $Z_N(t)$, *all* the components must be self-similar. For example, let us consider the process derived by summing a fGn and four zero-mean, unit variance white Gaussian processes. The covariance of this sum has the same structure as for the sum of 5 fGn processes with the scaling factor being different - in the former case it is $1/N^2$ while in the latter it is $1/N$. The Fano factor plot for this point process (see figure 3.18) indicates that the self-similar behavior may set in after an increased fractal time. However, a result presented later suggests that the mixture process may not be self-similar asymptotically, suggesting caution in adding processes that do not have identical self-similar parameters; it might be better to use the product transformation.

The half-wave rectifier and the product transformation yield mixed fractals: the Fano factor of the point process generated using the intensity process derived through these transformations is described by multiple fractal exponents. The shifted versions of the full-wave rectifier, the half-wave rectifier, and the square law transformation

Figure 3.18 The Fano factor is plotted for a simulated doubly stochastic Poisson process with the intensity obtained through three types of transformations: the intensity process is obtained by passing fractional Gaussian noise through a Square Law transformation, *i.e.*, $Y(t) = X_n^2(t)$, indicated by the legend “SL” and the second intensity process is the sum of 15 such independent processes, *i.e.*, $Y(t) = \frac{1}{15} \sum_{n=1}^{15} X_n^2(t)$ where $X_n(t)$ is a fractional Gaussian noise process generated using the method of Chi *et al* [5] with parameters $B = 4$, $N = 25$, and $H = .95$; the fGn process has zero mean and unit variance. The plot is the Fano factor of the point process whose intensity is the sum of a squared fGn (with the above characteristics) and four zero-mean, unit variance *white* Gaussian noise processes.

all yield mixed fractals. But the Fano factors for the shifted and unshifted versions are significantly different only for the half wave rectifier. Using the large-lag approximation of the covariance function of the intensity process results in an erroneous expression for the Fano factor. The half-wave rectifier, the full-wave rectifier, the square law transformation, and the product transformation yield highly non-symmetric probability density functions while Exponentiation yields a more symmetric pdf for the intensity process. The summation law results in a symmetric intensity pdf; this transformation increases the fractal time. Simulations indicate that using the summed square law transformation on one fGn and four non-fractal (Gaussian) processes does not yield a fractal intensity process.

3.4 Existence of a Well-Defined Point Process

One important mathematical question to be answered is whether the intensity processes generated through the different transformations give rise to a well defined doubly stochastic Poisson process. One of the sufficient conditions for a counting process to be a Poisson counting process [32, page 51] pertains to the intensity function: it requires that $\int_a^b \lambda_\sigma d\sigma$ exist and be finite for all finite intervals $[a,b)$, $t_0 \leq a \leq b$. By definition of a doubly stochastic Poisson process, almost every sample path of the intensity process should give rise to a valid Poisson process. Therefore, almost every sample path must satisfy the above condition. In other words, we require that the intensity process be integrable. We will assume that mean-square (MS) integrabil-

ity suffices. The intensity process is mean square integrable if [25, page 256, second edition, 1984]

$$\int_a^b \int_a^b |K_\lambda(t_1, t_2)| dt_1 dt_2 < \infty. \quad (3.38)$$

For a positive-valued, stationary intensity process, which is what we are interested in, the above condition is equivalent to having [25]

$$\int_0^T (T - \tau) K_\lambda(\tau) d\tau < \infty \quad (3.39)$$

where $T = b - a$. We will establish this condition by obtaining bounds on the covariance function of the intensity process obtained from fractional Gaussian noise using different transformations.

The covariance function of the intensity process that is obtained by passing fGn through a full-wave rectifier, given in equation (3.7), is bounded by $K_\lambda(\tau) \leq K_X(0)$; this bound follows from observing that $\sin \alpha$ and $\cos \alpha$ are bounded in magnitude by one and α is bounded, in magnitude, by $\frac{\pi}{2}$. Therefore, this intensity process satisfies the mean square integrability condition trivially.

Consider the covariance of the intensity process obtained using a half-wave rectifier (see equation (3.15)): from the MS integrability of the full wave-rectified process, it follows that this intensity process will be MS integrable if fractional Gaussian noise is MS integrable. Consider the covariance of fGn in equation (2.9). It can be bounded as follows:

$$K_\lambda(\tau) \leq 2 \left(1 + \frac{|\tau|}{\delta} \right)^{2H} \text{ for every } \tau \in \mathbf{R}^1. \quad (3.40)$$

For $1/2 < H < 1$ - our range of interest - it follows that

$$\int_0^T (T - \tau) K_\lambda(\tau) \leq 2 \int_0^T (T - \tau) \left(1 + \frac{|\tau|}{\delta}\right)^{2H} < \infty. \quad (3.41)$$

We have the covariance function for the square law transformation from equation (3.20): $K_\lambda(\tau) = 2K_X^2(\tau)$ where $K_X(\cdot)$ denotes the covariance of fGn. $K_X(\tau)$ has been demonstrated to be bounded for every τ in $\mathbf{R}^1$; it follows that this squared covariance is bounded by the square of the quantity on the right hand side of equation (3.40). From this, the MS integrability of the intensity process follows. The covariance of the intensity process derived by the shifted square law transformation is a linear combination of $K_X(\tau)$ and $K_X^2(\tau)$ and the combination is bounded because each component is bounded from which the MS integrability follows.

The covariance of the intensity process resulting from exponentiating fGn (see equation 3.27) is bounded by $e(e - 1)$ for all τ and hence the MS integrability follows. The MS integrability for the intensity process derived through finite products and sums of the above transformations follows easily.

The other condition to be satisfied is that the integral be finite for finite intervals. For a doubly stochastic Poisson process this is trivially satisfied. When the intensity used in the model is of the form $\lambda_F s(t)$ where λ_F is a stationary, fractal intensity process the finiteness criterion is satisfied when $s(t)$ integrates to a finite value over every finite interval $[a, b)$.

3.5 Convergence of Integrated Intensity Process

In doubly stochastic Poisson process theory, the integrated intensity process, $\Lambda_t = \int_0^t \lambda_t dt$, occurs frequently. For example, the moments of the counting process are defined in terms of the moments of Λ_t (see equations (1.2) and (1.3)). Although we have demonstrated the existence of the integral of the intensity process (obtained through different transformations of fGn), it is difficult to obtain the distribution of this integrated process. We will now investigate the asymptotic behavior of this integrated process, Λ_t for the fractal processes generated through the different transformations. For the square law transformation, Rosenblatt [28, 29] has shown that the discrete version of

$$\tilde{\Lambda}_T = \frac{\Lambda_T - E(\Lambda_T)}{\sigma(\Lambda_T)}$$

where $\sigma(\cdot)$ denotes standard deviation, converges asymptotically to a non-normal distribution. For the other transformations it is difficult to directly determine this convergence and hence we will take an indirect approach. With $\tilde{\Lambda}_T$ as just defined and the centered and normalized version of N_T given

$$\tilde{N}_T = \frac{N_T - E(N_T)}{\sigma(N_T)},$$

we have the following theorem due to Snyder [32, Page 298].

Theorem: *Suppose $E(\Lambda_T) \rightarrow \infty$ and $F(T) \rightarrow \infty$ as $T \rightarrow \infty$. If $\tilde{\Lambda}_T$ converges in distribution to some random variable ξ as $T \rightarrow \infty$, then $\tilde{N}_T$ also converges in distribution to ξ as $T \rightarrow \infty$.*

This theorem states that for the given conditions, if the normalized integrated intensity process, $\tilde{\Lambda}_T$, converges in distribution to some random variable, then the PND, which is the estimate of the counting probability $\Pr[N_T = n]$, converges in distribution to the *same* random variable. The doubly stochastic Poisson processes generated using the fractal intensity processes derived from fractional Gaussian noise through the different transformations satisfy the requirements on $E(\Lambda_T)$ and the Fano factor, $F(T)$. Thus, if the PND estimate for a simulated process does not equal the density estimate of Λ_T for sufficiently large T , it implies that $\tilde{\Lambda}_T$ does not converge in distribution as $T \rightarrow \infty$. In figures 3.19 through 3.23 the density estimate of Λ_T is plotted with the pulse number distribution for the corresponding doubly stochastic Poisson process (note that the estimates plotted have not been centralized *i.e.*, made zero mean but the estimates of the PND and Λ_t have the same mean value): the two plots match for all the transformations considered in this thesis.

In figure 3.24 the estimates of the PND and integrated intensity process is plotted for two cases where the estimates do not match: in panel a the estimates are plotted for a full-wave rectified, filtered white Gaussian noise and in panel b the estimates are plotted for the sum of 5 processes one of which is a fGn and the other four are squared white Gaussian noise processes. This disparity does not imply that the integrated intensity process does not converge asymptotically in these two cases but rather the condition $F(T) \rightarrow \infty$ as $T \rightarrow \infty$ is not satisfied in these cases. In fact, from equation (3.3) it is seen that $F(T) \rightarrow c$ as $T \rightarrow \infty$ where c is a constant. Snyder

[32] has proven a result similar to the one stated here for the case when the Fano factor approaches a constant as $T \rightarrow \infty$: the result predicts that asymptotically the two quantities - $\tilde{N}_t$ and $\tilde{\Lambda}_t$ - converge to different distributions. The simulation result for the summation of a fGn and four non-fractal processes suggests that the Fano factor of the summed process may not be fractal as $T \rightarrow \infty$ and hence care must be taken when choosing the components when we desire the resulting process to be self-similar (see also discussion in § 3.3.2).

In modeling data, the quantity that can be measured is the PND. The theorem just stated relates the estimated PND and the pdf of the integrated intensity process; empirically, we can determine the relationship between the pdfs of the intensity process and the integrated intensity process and thereby relating the estimated PND and the the pdf of the intensity process. The shifted transformations yield intensity processes that result in fractal point processes with symmetric PND's. Note that the shift in the transformation causes an increase in the variance of the PND's. The summation transformation also results in a fractal point process with a symmetric PND; the variance is less than for a shifted transformation (compare the PND's for the summed square law and shifted square law transformations). Comparing the characteristics of the PND's with the probability density functions of the corresponding intensity process, we note that symmetric (nonsymmetric) intensity pdf's yield point processes with symmetric (nonsymmetric) PND's. Therefore, the pdf of the intensity process of a doubly stochastic Poisson process provides us with a criterion to select a

particular transformation to derive the intensity process from fGn. The PND of the continuous tone response 18-94, plotted in figure 3.25, is symmetric suggesting that the intensity process used in the fractal point process model of this response must have a symmetric pdf.

3.6 Ergodicity of Fractal Intensity Process

From a modeling viewpoint, we must determine if time averages of the intensity converge to ensemble averages for the different transformations studied. As noted before, the transformations considered in this section all yield a stationary intensity process. The stationary intensity process, λ_t , is mean-ergodic if and only if the variance of

$$\frac{1}{T} \int_0^T \lambda_t dt$$

goes to zero as $T \rightarrow \infty$ [25, page 326]. The variance of the above quantity is given by

$$\frac{2}{T^2} \int_0^T (T - \tau) K_\lambda(\tau) d\tau.$$

The intensity processes from all the transformation are shown to be mean-ergodic by establishing that

$$\lim_{T \rightarrow \infty} \frac{1}{T^2} \int_0^T (T - \tau) K_X(\tau) d\tau = 0$$

and

$$\lim_{T \rightarrow \infty} \frac{1}{T^2} \int_0^T (T - \tau) K_X^2(\tau) d\tau = 0.$$

Figure 3.19 The densities of the *uncentered* and normalized versions of Λ_T (denoted by the legend “INT”) and N_T (denoted by the legend “PND”) for a simulated doubly stochastic fractal Poisson process are plotted together. In the top panel the intensity process is generated using the full-wave rectifier and in the lower panel the intensity process is derived using the shifted ($C = 2$) full-wave rectifier. The fractional Gaussian noise (zero mean, unit variance) is generated using the method of Chi *et al* [5] with parameters $B = 4$, $N = 25$, $H = .95$. The PND is estimated with $T = 1$ second. The normalized density of Λ_T is estimated over blocks of 5000 spikes.

Figure 3.20 The densities of the *uncentered* and normalized versions of Λ_T (denoted by the legend “INT”) and N_T (denoted by the legend “PND”) for a simulated doubly stochastic fractal Poisson process are plotted together. In the top panel the intensity process is generated using the half-wave rectifier and in the lower panel the intensity process is derived using the shifted ($C = 2$) half-wave rectifier. The fractional Gaussian noise (zero mean, unit variance) is generated using the method of Chi *et al* [5] with parameters $B = 4$, $N = 25$, $H = .95$. The PND is estimated with $T = 1$ second. The normalized density function of Λ_T is estimated over blocks of 5000 spikes.

Figure 3.21 The densities of the *uncentered* and normalized versions of Λ_T (denoted by the legend “INT”) and N_T (denoted by the legend “PND”) for a simulated doubly stochastic fractal Poisson process are plotted together. In the top panel the intensity process is generated using the square law transformation and in the lower panel the intensity process is derived using the shifted ($C = 2$) square law transformation. The fractional Gaussian noise (zero mean, unit variance) is generated using the method of Chi *et al* [5] with parameters $B = 4$, $N = 25$, $H = .95$. The PND is estimated with $T = 1$ second. The normalized probability density of Λ_T is estimated over blocks of 5000 spikes.

Figure 3.22 The densities of the *uncentered* and normalized versions of Λ_T (denoted by the legend “Int” in panel a and “INT” in panel b) and N_T (denoted by the legend ‘PND’) for a simulated doubly stochastic fractal Poisson process are plotted together. In the top panel the intensity process is generated using exponentiation and in the lower panel the intensity process is obtained through the transformation $Z_N(t) = \frac{1}{15} \sum_{n=0}^{n=15} X_n^2(t)$. The fractional Gaussian noise (zero mean, unit variance) is generated using the method of Chi *et al* [5] with parameters $B = 4$, $N = 25$, $H = .95$. The PND is estimated with $T = 1$ second. The normalized density function of Λ_T is estimated over blocks of 5000 spikes.

Figure 3.23 The densities of the *uncentered* and normalized versions of Λ_T (denoted by the legend “INT”) and N_T (denoted by the legend “PND”) for a simulated doubly stochastic fractal Poisson process are plotted together. In the top panel the intensity process is generated using the product transformation with $H_1 = H_2 = 0.95$ and in the lower panel the intensity process is derived using the same transformation but with $H_1 = 0.85$ and $H_2 = 0.95$. The fractional Gaussian noise (zero mean, unit variance) is generated using the method of Chi *et al* [5] with parameters $B = 4$, $N = 25$, $H = .95$. The PND is estimated with $T = 1$ second. The normalized density of Λ_T is estimated over blocks of 5000 spikes.

Figure 3.24 The densities of the *uncentered* and normalized versions of Λ_T (denoted by the legend “INT”) and N_T (denoted by the legend “PND”) for a simulated doubly stochastic fractal Poisson process are plotted together. In the top panel the intensity process is obtained by first passing fGn through a first-order filter with a coefficient of 0.8, the output of which is then passed through a square law transformation. In the lower panel the intensity process is derived through the transformation $Y(t) = \frac{1}{5} \sum_{n=0}^{n=5} X_n^2(t)$ where $X_1(t)$ is fractional Gaussian noise (zero mean, unit variance) that is generated using the method of Chi *et al* [5] with parameters $B = 4$, $N = 25$, $H = .95$. $X_i(t)$ for $i = 2, 3, 4, 5$ are zero-mean, unit variance, white Gaussian processes. The PND is estimated with $T = 1$ second. The normalized density of Λ_T is estimated over blocks of 5000 spikes.

Figure 3.25 The PND estimate of the continuous tone response 18-94 is plotted against the number of events falling in each bin of duration $T = 1$ second.

This easily follows by noting that $K_X(\tau) < |\tau|^{2H-2}$ ($1/2 < H < 1$) and $K_X^2(\tau) < |\tau|^{4H-4}$ ($3/4 < H < 1$) for all $\tau \in \mathbf{R}^1$.

The stationary intensity process is autocorrelation-ergodic [25, Page 329] if the variance of the quantity

$$R_T(\tau) = \frac{1}{T} \int_0^T \lambda(t+\tau)\lambda(t)dt$$

goes to zero as $T \rightarrow \infty$. Establishing this ergodicity requires knowledge of fourth order moments; this ergodicity was established only for the square law transformed fGn process, by using the result in [37, page 229] that factors all higher order moments of Gaussian process in terms of its second order moments. Rosenblatt [28] has shown, for the discrete case, the process resulting from the square law transformation of fractional Gaussian noise does not obey the Central Limit Theorem; when the partial sum is normalized appropriately it converges to a non-normal distribution.

3.7 Convergence of Intensity Estimates

The time-averaged intensity, λ_t , is estimated by $\hat{\Lambda}_t = \frac{N_{t,t+T}}{T}$ where $N_{t,t+T}$ is the increment process (see § 1.2 for definition). By definition, the doubly stochastic Poisson process is conditionally a Poisson process given a sample function λ_t . The increments of a Poisson process are independent and hence for a given sample function, λ_t , the estimate of this nonstationary intensity at time t is

$$\hat{\Lambda}_T = \frac{N_T}{T}. \quad (3.42)$$

The variance of this estimate is given by

$$\text{var}(\hat{\lambda}) = \frac{1}{T^2} \text{var}(\hat{N}_T) \quad (3.43)$$

The variance of the increment process is obtained from the expression for the Fano factor (which is nothing but the ratio of variance to mean of the counting process) in table 3.1 which results in the following expression for the variance of the intensity estimator:

$$\text{var}(\hat{\lambda}) = \frac{1}{T^2} \left[\bar{\lambda}T + 2 \int_0^T (T - \tau) K_\lambda(\tau) d\tau \right] \quad (3.44)$$

where $K_\lambda(\cdot)$ is the covariance function of the intensity process. The rate of convergence of this estimator is easily computed for a doubly stochastic Poisson process. When the intensity process is comprised of uncorrelated time samples the second term in the above expression is zero and therefore the convergence rate is $O(T^{-1})$. Let us consider the intensity process obtained by passing white Gaussian noise through a first-order filter and then through a square law transformation. From the discussion preceding equation (3.3), it follows that the variance of the intensity estimator being discussed here is

$$\text{Var}(\hat{\lambda}_t) = \frac{1}{T^2} \left[\bar{\lambda}T + 2aT + a^2(e^{-2T/a} - 1) \right]$$

Obtaining a Taylor's series expansion of $e^{-\frac{2T}{a}}$ about the origin and letting T go to ∞ , we can see that the second and third terms to yield a $O(1)$ convergence rate; therefore, the first term determines the rate of convergence of the estimator and is $O(T^{-1})$.

When the intensity process is the full-wave rectified, fractional Gaussian noise process its covariance is (see equation (3.8)) proportional to $K_X^2(\tau)$; for large lags this varies as τ^{4H-4} . Thus the variance of the estimator is proportional to $\frac{1}{T^2}T^{4H-2} = T^{4H-4}$. As we noted earlier the range of H is restricted to $\frac{3}{4} < H < 1$. Thus for $H = .95$ the estimate's convergence rate is $O(T^{-0.2})$; for a non-fractal intensity process the convergence rate is, as we just saw, T^{-1} . The dominant convergent rates of this estimator for intensity processes derived using other nonlinear transformations is tabulated in table 3.2.

The intensity estimate is plotted for a simulated doubly stochastic Poisson process with the intensity process obtained by square law transformation. The fractional Gaussian noise used to simulate the intensity process has a self-similarity parameter, H , equal to .95. The intensity process is estimated with two different averaging intervals: $T = 1$ seconds and $T = 25$ seconds. These estimates are plotted in figure 3.26. Note the variability of the estimates. Following the discussion in § 1.2 the standard deviation of a stationary Poisson process's intensity estimate equals $\sqrt{\lambda/T}$ which for $\lambda = 80$ is 4 spikes/second for $T = 5$ seconds. The intensity estimates for the doubly stochastic Poisson process far exceed these bounds. Furthermore, although the second averaging interval is ten times the first, the variability of the estimate is not reduced significantly. This variability is observed in the auditory-nerve fiber recordings (see figure 1.2) and can be described by fractal processes.

Figure 3.26 The intensity estimate is plotted for a doubly stochastic Poisson process; the intensity process is obtained by passing fractional Gaussian noise (zero mean, unit variance, $B = 4$, $N = 25$, and $H = .95$) through a Square Law transformation. The estimate is the average number of events occurring in nonoverlapping intervals of T seconds. The estimates are obtained for two different averaging intervals: $T = 1$ second and $T = 25$ seconds. The average rate for the 25-second averaging interval is normalized to be equal to average rate for the 1-second case. The average rate, $\lambda = 100$, is marked on the plots. The standard deviation of a stationary Poisson process with the same average rate, λ , as the data is given by $\sqrt{\lambda/T}$. The upper and lower bounds correspond to $\lambda \pm 2\sqrt{\lambda/T}$.

<i>Transformation</i>	<i>Dominant Convergence Rate</i>
Full-Wave Rectifier	$O(T^{4H-4})$
Half-Wave Rectifier	$O(T^{2H-2})$
Square Law	$O(T^{4H-4})$
Exponentiation	$O(T^{2H-2})$
Product Square Law	$\min(O(T^{4H_1-4}), O(T^{4H_2-4}))$
Summed Square Law	$O(T^{2H-2})$

Table 3.2 The dominant convergence rate is tabulated for the intensity estimator given in equation (3.42): the rates are computed for intensity processes obtained through different nonlinear transformations. In deriving the convergence rate, the input is taken to be fractional Gaussian noise. H , H_1 , H_2 denote the self-similarity parameters of fGn. Large-lag approximation of the covariance function of the intensity process is used to derive the rates. The dominant rate is the slower of the rates of the two components.

3.8 Analysis of Fano Factors of Fractal Point Processes

In modeling data, the factors that modify the fractal dimension and the fractal time must be categorized. These factors can be classified into two broad categories: factors consequent of the intensity process and factors inherent to the point process itself. Under the first category, we will study the effect of the choice of self-similarity parameter of fractional Gaussian noise, the choice of nonlinearity (in generating the intensity process), and the average rate of the intensity process. It is well known that refractory effects are present in neural firing. Thus, absolute and relative deadtimes must be incorporated into the model resulting in a doubly stochastic renewal process model. It has been observed that auditory-nerve discharges with a high average rate display short-term Markovian dependence. Such dependence has been observed in LSO discharges as well. A general doubly stochastic point process model can model

such dependencies. Under the second general category, we will study the effects of absolute deadtime, relative deadtime and Markovian dependence in interspike intervals.

From our analysis of the different transformations (see § 3.3.1), we note that the Full-Wave rectifier yields a single component process for low correlation in the input process and the shifting always results in processes with mixed exponents while the square law transformation yields a single exponent in the output process. The summation law does not modify the self-similarity of the input process. As we noted earlier, the product transformation results in a process whose sample paths have a self-similarity parameter that is a sum of the input parameters. The exponents of the output process are tabulated in table 3.3; it is evident that the fractal dimension in the case of a single component process and the exponents in the case of a mixed process increase with the self-similarity parameter of the input process. The variation of the Fano factor with H was predicted by our typical model in equation (3.4) and figure 3.4. This is tested through simulations and the Fano factor is plotted for doubly stochastic Poisson processes with intensity process obtained by square law transforming fGn: the simulations were performed for $H = .9$ and $H = .95$. If the intensity process is well described by a doubly stochastic (deadtime) renewal process of the form in equation 3.4, the Fano factor would vary with H as indicated in figure 3.4: the plots for different exponents will crossover with the larger exponent dominating for larger time-scales and the smaller exponent dominating for smaller time-scales. Figure 3.27

<i>Type of Transformation</i>	<i>Fractal Exponents</i>
Full Wave Rectifier	$4H-3$
Half Wave Rectifier	$4H-3, 2H-1$
Square Law	$4H-3$
Shifted Square Law	$4H-3, 2H-1$
Exponentiation	$2H-1$
Product Law	$4H_1 - 3, 4H_2 - 3, 4(H_1 + H_2) - 7$
Summation Law	H_{input}

Table 3.3 The Fractal exponents for different point processes are given; the intensity processes are derived by the corresponding transformation above. H_{input} is the common self-similarity parameter of the input processes.

indicates that the larger exponent dominates for all time-scales. This a could be a consequence of the fact that fGn is not a pure fractal or could be a consequence of the nonlinearity of the transformations. The other possibility is that the Fano factor of a doubly stochastic renewal process is not well described by a linear combination of a deadtime term and the Fano factor expression for a doubly stochastic Poisson process.

We note from table 3.3 that the different transformations result in Fano factors that comprise of one or both of the exponents: $4H - 3$ and $2H - 1$. From the expressions for the covariance of the output processes we note that the coefficient of the component $2H - 1$ is never influenced by H but the quantity K_H appears in the coefficient of the exponent term. Therefore, the self-similarity may affect the fractal time; to study the variation of K_H with H we plot these quantities in figure 3.28. Clearly for the region of interest $-3/4 < H < 1$, K_H is almost constant. Thus the parameter H does not significantly affect the fractal time.

Figure 3.27 The Fano factor of doubly stochastic Poisson processes is plotted for two values of the self-similarity parameter of the input (fGn process): $H = .9, .95$. Square law is the transformation used to generate the intensity process. The fractal dimensions of the resulting point process are 0.6 and 0.8 respectively. The parameters of fGn are $B = 4, N = 25$ and is zero mean, unit variance. The average rate of the intensity process in each case is 100 spikes/s. 200 seconds of simulated data is used.

Figure 3.28 The quantity K_H is plotted against H for the relevant range $-3/4 < H < 1$. The expression for K_H is from equation (3.12).

The effect of the variation of the average rate can be seen by noting that the average rate (denoted by $\bar{\lambda}$) occurs in the numerator of the coefficient of the exponent in the Fano factor expressions (see, for example, equation 3.11 which is the Fano factor for the full-wave rectified case). Our analysis of a typical Fano factor of a doubly stochastic Poisson process in § 3.2 predicts the variation of the Fano factor plot with the average rate. This theoretical variation was plotted in figure 3.3. We can thus conclude that increasing the average rate increases the fractal time but has no effect on the fractal dimension. Simulations confirm this prediction: we use the summed square law (with $N = 6$) transformation on fractional Gaussian noise to obtain the intensity process; the self-similarity parameter of fGn is held constant at $H = .95$ as the average rate of the intensity process is varied. The doubly stochastic point process is simulated and the Fano factor computed for three different average rates: 50 spikes/s, 100 spikes/s, and 150 spikes/s. The results are plotted in figure 3.29.

A summed square law transformed fractional Gaussian noise process with an average rate of 100 spikes/s serves as the input to the point process generator described in figure 1.1. A doubly stochastic renewal process is obtained as the absolute deadtime is varied. The Fano factors corresponding to three different values for this deadtime are plotted in figure 3.30. The deadtime causes the Fano factor to decrease below one (the constant Fano factor of a stationary Poisson process) for values of T comparable to the deadtime. For the transformation used here the Fano factor varies as

Figure 3.29 Fano factor is plotted for doubly stochastic Poisson processes for three different average rates. Summed ($N = 6$) square Law transformation is used to derive the intensity process from fractional Gaussian noise (zero mean and unit variance); the parameters of fGn are $B = 4$, $N = 25$, $H = .95$.

T^{4H-3} and for $H = .95$, $F(T) \propto T^{0.8}$. The simulations indicate that the asymptotic slope of the Fano factor *i.e.*, the fractal dimension of the point process, is the same for all three choices of deadtime. Increasing the deadtime increases the fractal time. Furthermore, increasing the deadtime increases the minimum value of $F(T)$. The $F(T)$ for a stationary renewal process with only an absolute deadtime is given by (see table 3.1) $(1 - \bar{\lambda}\Delta)^2$ and hence a larger deadtime implies a smaller minimum $F(T)$. Similarly, having a fixed nonzero absolute deadtime, and increasing the average rate should decrease the minimum value of $F(T)$; Panel b of 3.30 demonstrates this phenomenon. Thus this phenomenon predicted for a stationary renewal process also holds for a doubly stochastic renewal process. But note that as the rate is varied for a constant, nonzero absolute deadtime, the Fano factor curves crossover at about $T \approx 80ms$; this behavior parallels the effect of varying K_{pp} in the typical Fano factor expression for a doubly stochastic Poisson process (see equation 3.4 and figure 3.3). This suggests that the Fano factor of a doubly stochastic renewal process cannot be described by a simple linear combination of a deadtime term and the Fano factor expression for a doubly stochastic Poisson process; the combined effect of the different factors is, perhaps, more complex.

Next, we investigate the effect of introducing a relative deadtime into the point process through simulations. The doubly stochastic renewal process is generated using the method proposed by Johnson *et al* [13]; the relative deadtime is varied and the Fano factor computed. The relative deadtime is varied by shifting the time

Figure 3.30 In Panel a the Fano factors of doubly stochastic renewal processes corresponding to absolute deadtimes of .5 ms, 1 ms, and 1.5 ms are plotted here. The intensity process is derived from a zero mean, unit variance fGn process with parameters $B = 4$, $N = 25$ and a self-similarity parameter, $H = .95$; the summed ($N = 6$) square law transformation is used. The average rate of the intensity process is 100 spikes/s. In Panel b the Fano factor is plotted for a doubly stochastic renewal process with the absolute deadtime kept constant at 1 ms and the average rate of the intensity process varied: rates of 50 spikes/s, 100 spikes/s, and 150 spikes/s are used.

past the occurrence of the previous event (assuming zero absolute deadtime) at which maximum probability of a new event occurring is attained. Figure 3.31 indicates that the relative deadtime does not affect the fractal dimension but increases the fractal time of the process. Increasing the relative deadtime increases minimum value of $F(T)$.

We now consider the effect of Markovian dependence on the self-similar behavior of the point process. We introduce a first order Markovian dependence of the type discussed in [15] into the point process; the resulting inter-spike intervals have a dependence structure described by $E[\tau_{n+1}|\tau_n] = (a - b\tau_n)u(a - b\tau_n) + K$ where $u(\cdot)$ is the step function and K is a constant. We choose the following values of a and b that were used by Johnson *et al* [15] to model responses from the LSO: $a = .0025$, $b = .25$; $a = .00176$, $b = .25$; $a = .0025$, $b = .42$. The resulting Fano factor is plotted in figure 3.32. This dependence does effect the Fano factor plot and causes its value to dip below one for large dependence but the effect is small. There is a slight increase in the overall fractal time; this is a characteristic of the dependence structure and is therefore expected.

One final note: some of the factors may not appear to have significant effect on the Fano factor plot individually but their effect is magnified when combined with one or more factors. For example, the deadtime effects are enhanced by higher average rates.

Figure 3.31 The Fano factors of doubly stochastic renewal processes is plotted for different relative deadtimes. There is no absolute deadtime. The probability of an event occurring increases linearly since the occurrence of the previous event, attaining to a maximum in one case 25 ms later; in the second case this maximum is achieved 35 ms later and in the third case the maximum probability is achieved 45 ms after the occurrence of the previous event. The characteristics of intensity process is as described in figure 3.30 (Panel a). The average rate of intensity process is 100 spikes/s.

Figure 3.32 The Fano factor of a doubly stochastic renewal process is plotted. The fractal intensity process is derived from fGn using the summed ($N = 6$) square law transformation. The point process is generated using the method of Johnson *et al* [13]. The Markovian dependence between inter-spike intervals is given by $E[\tau_{n+1}|\tau] = (a - b\tau_n)u(a - b\tau_n)$ where $u(\cdot)$ is the step function. a and b take on the values $a = .0025$, $b = .25$; $a = .00176$, $b = .25$; $a = .0025$, $b = .42$. fGn is generated using the method of Chi *et al* [5]. The characteristics of the fGn used is described in figure 3.30. An average rate of 100 spikes/s is used.

Chapter 4

Conclusions and Discussion

In this thesis we have defined a fractal point process to be a doubly stochastic point process whose intensity process is a fractal waveform process. We showed that the choice of intensity process is crucial to explain the observed *very* low frequency spectral density and the variability. A point process with a chaotic intensity function does not describe these characteristics of the data. When a first-order noise is appropriately transformed to obtain a non-negative intensity function, it results in a doubly stochastic Poisson process that is characterized by a variability greater than that of a stationary Poisson process and by a low-frequency spectrum. However, the neural data to be modeled had much higher variability and far lower frequency spectral components than could be described by a doubly stochastic Poisson process with a intensity process having a simple Markovian dependence. In fact, no strongly mixing process is an appropriate intensity process: the Fano factor of point processes with strongly mixing intensity process will approach a constant value for large values of T . Fractal processes differ from such processes in that this correlation function decays more slowly at infinity

A fractal point process is a fractal process only by its association with a fractal waveform intensity process. The counting process itself, unlike fractional Brownian motion (fBm), does not have self-similar sample paths. Thus, a fractal point process is not a *pure* fractal as it displays self-similar characteristics only over large time scales. It displays self-similar characteristics for time-periods larger than the *fractal time*. We determined through analysis and simulation that the fractal time is a function of the average rate of the intensity process, the fractal dimension, the absolute deadtime, the relative deadtime, and short-term serial dependence. Of these, the average rate and the deadtimes - especially the relative deadtime - are observed to have the greatest impact. We predict that, all other things being equal, the spontaneous activity recorded from a neuron will have a larger fractal time than a stimulated response because stimulation increases the average rate.

Rather than the power spectrum of the intensities, the Fano factor is the quantity that captures this “impure” self-similar characteristic. The exponent of the Fano factor’s power-law behavior for intervals larger than the fractal time measures the *fractal dimension* of the point process. It should be noted that this definition of fractal dimension differs from that for waveform processes, *e.g.*, fBm where the fractal dimension is directly related to the self-similarity parameter of the sample paths of the process. The fractal dimension is independent of the average rate, value of the deadtime - absolute or relative, and serial dependence and is determined *only* by the fractal nature of the intensity process. Thus, the Fano factor serves to discriminate

between fractal and non-fractal processes, such as a doubly stochastic process with non-fractal intensities and singly stochastic point processes.

This work explores different alternatives to generating non-negative, fractal intensity processes. We have taken the route of using memoryless, nonlinear transformations to derive the intensity process from fractional Gaussian noise [5, 21]. Many of these transformations preserve the self-similarity property and thus cannot be distinguished solely on measurements of fractal dimension. Our simulations indicate that only the square law transformation produces a pure fractal while the other transformations result in processes that are mixtures of two or more exponents. When the shifted version of the transformations is used to derive the intensity process, the resulting point process contains a mixture of components with different exponents. Thus although the self-similarity property is preserved, the resulting process cannot be described by a single exponent. The characteristics of the data determine which transformation will describe them: if the data is best described by a single exponent, then it would not be appropriate to use a transformation that results in a mixed process. It is observed that the shifted and unshifted versions of the transformations result in Fano factors that are significantly different only in the case of the half-wave rectifier.

A theorem due to Snyder [32] proves that, for a doubly stochastic Poisson process with fractal intensity process, the centered and normalized counting process, N_t , and integrated intensity process, Λ_t , converge to the same distribution asymp-

totically under certain conditions. All the transformations considered in this work demonstrated this theorem. Therefore, in modeling fractal point process data, where we have access only to the PND estimate of N_t , this asymptotic equivalence provides us with information on the shape of the integrated intensity process by estimating the PND over large bins (to obtain an asymptotic estimate). We observed that the PND estimates of recorded auditory-nerve data were slightly asymmetric. Under a doubly stochastic Poisson assumption this result implies that the integrated intensity process has a similar asymptotic distribution. However, the transformations yielded intensity processes that were highly skewed; the integrated intensity process for our data was more symmetric. When the summation transformation was used with the square law transformation the symmetry of the PND estimate was similar to that of the data. The shifting transformation also yields nearly symmetrical PND estimates. Consequently, symmetry of the intensity process is another criterion that can be used to select an appropriate transformation to derive the intensity process. This criterion suggests that the shifted transformations and the summation transformation are suitable for modeling auditory nerve discharges. Thus, the PND can be used to select models for fractal intensity generation that yield similar fractal dimension.

There is another selection criterion which we found to be useful: the fractal time. Most of the auditory-nerve data we analyzed had fractal times in the range .5 to 2 seconds. The absolute deadtime for these neurons was in the range of 0.5 ms to 1 ms and the relative deadtime varied in the range 25 ms to 45 ms. We tried to model the

auditory data with a doubly stochastic renewal model with the estimated refractory times and average rate of the data with a square law transformed fractional Gaussian noise serving as the fractal intensity process. The fractal time of the simulation was much less than that of the data. Also, the sum of a finite number of such processes resulted in a fractal point process with the appropriate fractal time. Using shifted transformations to derive the intensity process affects the fractal time of the resulting point process to a measure but it does not provide us with the desired process. To examine this issue more carefully, let us consider the expression for the covariance of a shifted, square law transformed fGn in equation (3.23). Taking the cue from the averaged summation transformation *i.e.*, (defined in equation (3.35)) we could scale the component with the exponent $2H - 2$ by choosing small values of C . But the first component (with exponent $4H - 4$) cannot be adjusted without tampering with the average rate in which case the modeling becomes difficult if not impossible. The interesting feature of the averaged summation transformation is that it provides an additional degree of freedom and unlike the shifted transformation, it results in a process that has a single component. This criterion indicates that the summation transformation is the appropriate transformation to obtain the intensity process of the fractal point process model for auditory nerve discharges.

Long-term trends, related to adaptation of the discharges, are sometimes present in auditory nerve recordings. The Fano factor of data with linearly varying trends varies as T . Such data are not easily distinguished from fractal activity corresponding

to large fractal dimensions and therefore the data must be tested for trends before starting the fractal analysis. The Fano factor is not a reliable indicator of fractal activity of such data. We have discarded such data from analysis until a different measure of fractal activity that is robust to trends is developed.

We noted in the introductory part of this thesis that the variability observed was not unique to auditory nerve fiber discharges: single unit recordings from the lateral superior olive (LSO) and the medial nucleus of the trapezoid body (MNTB) contained similar variability. Similar observations have been made in the visual system. If indeed this variability can be described by fractal behavior, it would suggest that the origin of fractals is perhaps at the local level: it originates at the spike generator of individual neurons. If this is the case, the simultaneous recordings from parallel fibers should be statistically independent and therefore this hypothesis can be experimentally tested but not completely validated as other hypotheses may yield similar results. The fractal characteristics would provide us a tool with which to analyze these characteristics. It has been observed that the LSO neuron receives one ipsilateral input via numerous terminal boutons that are distributed over its dendrites. One neural model proposes that the small excitatory postsynaptic potentials (EPSP) summate to produce an analog signal. This signal arrives at the spike generator, low pass filtered, and diminished in magnitude. When a sufficient number of such EPSP waveforms are added together and the spike generator threshold is exceeded, a spike is generated. This model is consistent with some of the results of this thesis: the

refractory effects of the recorded data do not account for the fractal time observed in discharges recorded from the LSO. Using a sum of a finite number of square law transformed fGn described the data, lending credence to the spike generator model just discussed.

Preliminary investigation indicates that the responses of neurons in the LSO and MNTB display fractal characteristics. We will now consider the constraints imposed on the inputs to these neurons under the condition that the LSO merely averages the inputs – each of which is a point process. When analyzing the summation transformation, we noted that if four white Gaussian noise processes were added to a fractional Gaussian noise process, the resulting process is not likely to be self-similar at large time scales. Thus, if we view the spike generator as merely adding point processes, we will constrain the component point processes. We know that adding N stationary Poisson processes results in a Poisson process that has an intensity function equal to the sum of the component intensity functions. If a sufficient number of the input point processes are not fractal the resulting point process will be, under the Poisson assumption, a Poisson process with an intensity function that is a sum of the sample paths of the component intensity processes. Thus summing together fractal and non-fractal waveform processes would soon destroy the self-similar property of the discharges (unless of course, the fractal activity is generated locally at the spike generator). Consequently, all inputs to a neuron must necessarily be fractal processes to yield a fractal process. We have seen that adding together identical fractal waveform

processes results in a process that has self-similar sample paths and the self-similar correlation structure. But the case when the inputs have dissimilar self-similarity parameters must be analyzed to determine if a neuron, in a fractal environment, can combine dissimilar inputs to yield a self-similar output. The preceding discussion suggests that under the condition that fractal activity originates at the cochlea or is of some other global origin, we require *all* the inputs to the LSO to be fractal point processes. This requirement implies that the cochlear nucleus neurons providing this input must all display fractal characteristics. This prediction should be verified experimentally. It then follows that all the auditory nerve discharges must be fractal; *all* the responses analyzed at three different laboratories seem to confirm this hypothesis. Arguing in the reverse direction, we can conclude that if all the auditory nerve discharges display fractal behavior, the discharges of all the higher order neurons will exhibit fractal activity (under the assumption of a global fractal origin and the neuron performing simple nonlinear transformations discussed in this thesis).

It is commonly assumed in sensory processing theory that the stimulus magnitude is encoded in the average rate of single neuron discharges. As the rate estimates of a fractal point process converge very slowly, average rate sampled over short periods of time would provide highly varying values. To overcome this problem the rate estimates may be obtained through ensemble averaging rather than time averaging. If the averaging is over an ensemble of fractal discharges, the rate estimate converges faster but convergence will still be slow because ensemble averaging is done only over

a few neural outputs. If the ensemble is a mixture of fractal and nonfractal processes, the average rate will converge faster if the contribution of the nonfractal components is greater.

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