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POST BUCKLING BEHAVIOR OF A CIRCULAR ROD CONSTRAINED WITHIN
A CIRCULAR CYLINDER

Rice University

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RICE UNIVERSITY

POST BUCKLING BEHAVIOR OF A CIRCULAR ROD
CONSTRAINED WITHIN A CIRCULAR CYLINDER

by

KENNETH G. SORENSON

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

DOCTOR OF PHILOSOPHY

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April, 1984

**POST BUCKLING BEHAVIOR OF A CIRCULAR ROD
CONSTRAINED WITHIN A CIRCULAR CYLINDER**

KENNETH G. SORENSON

ABSTRACT

It has been previously demonstrated that an axially loaded weightless circular rod buckles helically when constrained within a circular cylinder. The effects of pinned and fixed end conditions on the behavior of such a helically buckled rod are investigated. Both end conditions locate the rod end on the cylinder axis. The pinned end condition considers the rod end free to rotate about all three axes, allows axial translation, and prevents radial movement. The fixed end condition allows axial rotation and translation, requires the rod end to be tangent to the cylinder axis, and prevents radial movement. Both end conditions are found to perturb the helix in an exponentially decaying manner for a distance of less than one pitch length. Far from the end, the rod behaves as an undisturbed constant-pitch helix.

The distance from the rod end to the point of initial contact with the cylinder wall is calculated. Closed form analytical solutions are obtained for the deflected shapes and internal reactions of the end sections. The maximum internal bending moment for the pinned-end case is 19.5% higher in the end section than in the helical section. The maximum bending moment for the fixed-end case also occurs in

the end section and is 1% higher than in the helical section. The solution procedure applies to rods of either finite or infinite length.

Experimental results confirm the analytical findings.

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TABLE OF CONTENTS

ABSTRACT	ii
ACKNOWLEDGEMENTS	iv
LIST OF TABLES	vii
LIST OF FIGURES	viii
NOMENCLATURE	x
1.0. INTRODUCTION	1
2.0. ANALYSIS OF A PINNED-END HELICALLY BUCKLED ROD	5
2.1 Analysis of the Contact Section	6
2.2 Analysis of the End and Transition Sections	10
2.3 Maximum Bending Moment in the End Section	11
2.4 Boundary Conditions	12
2.5 Solution Method	15
3.0 ANALYSIS OF A FIXED-END HELICALLY BUCKLED ROD	18
3.1 Boundary Conditions	18
3.2 Solution Method	19
4.0 RESULTS	22
5.0 EXPERIMENTAL INVESTIGATION	31
6.0 DISCUSSION	33
7.0 SUMMARY AND CONCLUSIONS	43
8.0 REFERENCES	52

TABLES	54
FIGURES	57
APPENDIX I Derivation of Governing Beam-Column Equation	70
APPENDIX II Derivation of Boundary Condition at Rod Half-Length Point	78
APPENDIX III Rod Deformed Into Constant Pitch Helix	80
APPENDIX IV Pinned-End Analysis with End Section Connected Directly to Helical Section	86
APPENDIX V Pinned-End Analysis with End Section Connected to Perturbed Helical Section	93
APPENDIX VI Fixed-End Analysis with End Section Connected Directly to Helical Section	99
APPENDIX VII Fixed-End Analysis with End Section Connected to Perturbed Helical Section	105
APPENDIX VIII Pinned-End Analysis Computer Listing . . .	110
APPENDIX IX Fixed-End Analysis Computer Listing . . .	124
APPENDIX X Pinned-End Analysis Results	131
APPENDIX XI Fixed-End Analysis Results	142
APPENDIX XII Two-Dimensional Analogy	152

LIST OF TABLES

TABLE 1.	Selected Pinned-End Results	54
TABLE 2.	Selected Fixed-End Results	55
TABLE 3.	Comparison Between Current Results and Those of Reference [10]: Fixed End Condition	56
TABLE I-1	Local-Global Coordinate System Transformation	76

LIST OF FIGURES

FIGURE 1.	Pinned-End Rod Geometry and Nomenclature	57
FIGURE 2.	Rod Displacement Results In R - θ Coordinates	58
FIGURE 3.	Rod Displacement Results In x - y Coordinates	59
FIGURE 4.	Local Helix Angle Results	60
FIGURE 5.	Bending Moment Results	61
FIGURE 6.	Radial and Tangential Bending Moment Components	62
FIGURE 7.	Shear Force Results	63
FIGURE 8.	Radial and Tangential Shear Force Components	64
FIGURE 9.	Wall Contact Force Results	65
FIGURE 10.	External Reactions for Pinned-End Case	66
FIGURE 11.	External Reactions for Fixed-End Case	67
FIGURE 12.	Schematic Results for Numerical Example #1 - Drill String at the Bit . . .	68
FIGURE 13.	Schematic Results for Numerical Example #2 - Tubing at a Packer	69
FIGURE I-1.	Differential 3-Dimensional Rod Element	77

FIGURE I-2.	Sign Convention For Bending	
	Moment And Shear Force	77
FIGURE II-1.	Finite-Length Rod Geometry	79
FIGURE II-2.	Finite-Length Rod Mid-Point	
	Boundary Condition	79
FIGURE III-1.	Helically Buckled Rod	85
FIGURE III-2.	Helix Angle Definition	85
FIGURE IV-1.	Pinned End Section Connected	
	Directly to Helical Section	92
FIGURE IV-2.	End Section Reactions.	92
FIGURE V-1.	Rod Displacement Results	97
FIGURE V-2.	Local Helix Angle Results	98
FIGURE V-3.	Wall Contact Force Results	98
FIGURE VI-1.	Fixed End Section Connected	
	Directly To Helical Section	104
FIGURE VII-1.	Rod Displacement Results	108
FIGURE VII-2.	Local Helix Angle Results	109
FIGURE VII-3.	Wall Contact Force Results	109
FIGURE XII-1.	Two-Dimensional Analogy	155
FIGURE XII-2.	Two-Dimensional Rod	
	Displacement Results	156
FIGURE XII-3.	Two-Dimensional Internal	
	Bending Moment Results	156
FIGURE XII-4.	Maximum Bending Moment For	
	Various End Fixity Conditions	157

NOMENCLATURE

α	$[W/EI]^{1/2}, (1/\text{length})$
$a_{ij}, b_{ij}, c_{ij}, d_{ij}, \left. \begin{matrix} \\ \\ \end{matrix} \right\}$	Constant parameters in the rod deflection equations, (non-dimensional)
EI	Rod bending stiffness, (force-length ²)
$\hat{F}$	External rod reactions, (force)
F	$\hat{F}/\alpha W r$, (non-dimensional)
γ	Local helix angle (see Figure III-2)
L	Rod length from rod end to initial wall contact in Appendices IV, V, VI, and VII, (length)
L_1, L_2	Rod length from rod end to the first and second wall contact points, (length)
L_C	Rod contact section half-length for finite length rod, (length)
L_S	Rod half-length for finite-length rod, (length)
λ	$2\pi/\alpha p$, (non-dimensional)
$\hat{M}_j$	Rod internal bending moment; $j = 1$ for $x - z$ plane, $j = 2$ for $y - z$ plane (force-length)
M_j	$\hat{M}_j/Wr$, (non-dimensional)
$\hat{M}_R, \hat{M}_\Theta$	Rod internal bending moments in the $R-z$ and $\Theta-z$ planes, (force-length)
M_R, M_Θ	$M_R = \hat{M}_R/Wr, M_\Theta = \hat{M}_\Theta/Wr$, (non-dimensional)
$\hat{M}$	Rod total internal bending moment, (force-length)
M	$\hat{M}/Wr$, (non-dimensional)

p	Helix pitch, (length)
ϕ	αL , Rod length to initial wall contact point in Appendices IV, V, VI, and VII, (non-dimensional)
ϕ_i	αL_i , Rod length to first ($i = 1$) and second ($i = 2$) contact points, (non-dimensional)
ϕ_s	Rod half-length for finite-length rod case, (non-dimensional)
ϕ_c	Rod contact section half-length for finite-length rod case, (non-dimensional)
$\hat{q}_j$	Rod distributed force acting in positive x-direction for $j = 1$ and y-direction for $j = 2$, (force/length)
r	Radial clearance between rod and cylinder, (length)
$\hat{R}$	Radial position of rod centerline, (length)
R	$\hat{R}/r$, (non-dimensional)
θ	Circumferential position of rod centerline measured clockwise from the x-axis, (radians)
θ_1, θ_2	Circumferential position of the first and second contact points
$\hat{u}_j$	rod centerline position; $j = 1$ for the x-direction, $j = 2$ for the y-direction used in Appendices IV and VI, (length)

$\hat{u}_{ij}$	Rod centerline position; $i = 1$ for the end section, $i = 2$ for the transition section; $j = 1$ for the x-direction, $j = 2$ for the y-direction; (length)
u_j	$\hat{u}_j/r$, (non-dimensional)
u_{ij}	$\hat{u}_{ij}/r$, (non-dimensional)
$\hat{v}_j$	Rod centerline position in the contact section; $j = 1$ for the x-direction, $j = 2$ for the y-direction; (length)
v_j	$\hat{v}_j/r$, (non-dimensional)
$\hat{V}_j$	Rod internal shear force; $j = 1$ for the x-direction, $j = 2$ for the y-direction, (force)
V_j	$\hat{V}_j/\alpha W r$, (non-dimensional)
V_{ij}	Same as V_j where $i = 1$ for the end section, and $i = 2$ for the transition section, (length)
$\hat{V}_R, \hat{V}_\theta$	Rod internal shear force in the radial and tangential directions, (non-dimensional)
V_R, V_θ	$V_R = \hat{V}_R/\alpha W r$, $V_\theta = \hat{V}_\theta/\alpha W r$, (non-dimensional)
$\hat{w}$	Wall contact force per unit length acting radially inward on rod, (force/length)
w	$\hat{w}/\alpha^2 W r$, (non-dimensional)
W	Axial compressive force applied to rod end, (force)

x, y, z	Cartesian coordinate system positioned at rod-end, z aligned with the cylinder axis (see Figure III-2)
ζ	αz , axial distance from rod end, (non-dimensional)
prime	$d/d\zeta$, derivative with respect to ζ

1.0 INTRODUCTION

The buckling behavior of an axially loaded circular rod constrained within a circular cylinder is a problem which has been of interest to the oil industry for many years. This problem relates to the buckling of drill pipe near the bit and to the problem of tubing sealed by a packer inside casing. Knowledge of buckling behavior is important for structural design of the drill string elements, tubing elements, and packer assemblies. In addition, drill string buckling results in side forces on the drill bit which can affect hole deviation behavior.

Early investigations treated the problem as 2-dimensional and considered vertical or near-vertical holes. Two analyses by Lubinski [1], and Lubinski and Woods [2] established the configuration of the drill string near the bit for the case of a drill string buckled in a plane. Further work by Woods and Lubinski [3] extended these results to include the effects of a stabilizer near the bit. Lubinski and Blenkarn [4] considered the related situation of tubing buckling.

A classic analysis by Lubinski, Althouse, and Logan [5] in 1962 considered the 3-dimensional buckling behavior of tubing sealed in packers and established the helical post-buckled configuration of an axially loaded weightless circular rod constrained in a circular cylinder. This analysis

addressed the post-buckled , torque-free, static rod behavior away from the rod end and assumed frictionless constraint in a straight hole. Fluid pressure effects were considered and were found equivalent to fictitious externally applied axial forces. Fluid flow effects were considered and were found to have negligible effect on rod behavior. The relationship between the pitch of the helix and the axial force was found to be

$$p^2 = \frac{8\pi^2 EI}{W} \quad (1.1)$$

where W = axial compressive force on the rod

p = helix pitch

EI = rod bending stiffness.

Subsequent analysis by Cheatham and Pattillo [6] showed that the pitch-force relationship for the helically buckled rod is more completely described by

$$\frac{4\pi^2 EI}{W} \leq p^2 \leq \frac{8\pi^2 EI}{W}, \quad (1.2)$$

where the upper limit applies during loading and the lower limit during unloading.

Analyses by Paslay and Boggy [7], Dawson and Paslay [8], and Paslay and Dawson [9] consider the buckling problem using elastic stability theory, and investigate the effect

of hole inclination on buckling initiation and behavior. It is found that for practical drilling situations, very high axial loads become necessary for helical buckling at hole inclinations greater than 10 degrees.

With the exception of references [1-3], which consider planar buckling only, and an investigation by Mitchell [10], previous buckling analyses have concentrated on rod behavior remote from end effects such as will occur at the bit or stabilizer in a drill string, or at a packer in a tubing assembly. A detailed review of Mitchell's results shows that, while the analysis assumed frictionless contact between the rod and cylinder wall, the results included a large tangential concentrated contact force where the tubing first contacts the cylinder wall (see Appendix VI).

A summary review of the technical literature on drill string buckling is presented by Paslay [9]. Extensive bibliographies on drill string and tubing analysis in general are presented in references [14] and [15].

The purpose of this investigation is to determine the effect of end conditions on the helical post-buckled configuration of an axially loaded circular rod constrained in a circular cylinder. The post-buckling equilibrium analyses which follow assume that the constant diameter circular rod is weightless and static, and that displacements are small. Contact between the rod and the smooth cylinder wall is

frictionless. The effects of torque and axial variation in the axial force are neglected. Fluid effects are neglected except that static pressures can be treated as externally applied forces as in reference [5]. Results are presented for both a pinned-end case, such as a drill string at the bit, and fixed-end case, such as tubing at a packer, or a drill string at closely spaced stabilizers.

2.0 ANALYSIS OF A PINNED-END HELICALLY BUCKLED ROD

The deflected centerline of a helically buckled rod with a pinned-end condition is shown schematically in Figure 1. The pinned-end condition positions the rod end on the cylinder axis and allows axial translation and rotation about all three axes. The effective radius, r , of the circular cylinder on which the rod centerline lies is equal to the difference between the radii of the constraining cylinder and the rod. The centerline of the continuous circular rod is described by three sections: an end section, a transition section, and a contact section. The end section is that portion of the rod from the rod end to the point of initial wall contact at A. The transition section is an intermediate portion of the rod which is not on the cylinder wall and connects the end section at A to the contact section at B. The contact section is on the straight circular cylinder wall everywhere along its length.

Initial analysis showed that the force system which acts on the end section is incompatible with the internal reactions of a rod deformed into a simple helix (see Appendix IV). This finding suggested that the end section connects to a perturbed helix which has a variable helix angle. Analysis of a helically buckled rod with a variable helix angle, however, resulted in wall contact forces which acted to pull, rather than push, on the rod near the initial

contact point at A (see Appendix V). This behavior further suggested that an intermediate transition section which does not contact the wall, is present as shown in Figure 1.

2.1 Analysis of the Contact Section

The governing differential equations for the 3-dimensionally buckled beam-column for the assumptions specified above, reduce to a set of standard beam-column equations applied in the x - z and y - z planes (see Appendix I);

$$EI \frac{d^4 \hat{v}_j}{dz^4} + W \frac{d^2 \hat{v}_j}{dz^2} - \hat{q}_j = 0 \quad (2.1)$$

where $\hat{v}_j$ describe the position of the rod centerline with $j = 1$ for the x direction and $j = 2$ for the y direction.

The distributed lateral forces $\hat{q}_1$ and $\hat{q}_2$ which act on the rod are the x and y components of the wall contact force and are

$$\hat{q}_1 = -\hat{w} \cos \theta \quad (2.2a)$$

and

$$\hat{q}_2 = -\hat{w} \sin \theta \quad (2.2b)$$

where $\hat{w}$ is the wall contact force per unit length acting radially inward on the rod, and θ is the circumferential

position of the rod centerline measured from the x axis as shown in Figure 1.

Incorporating equation (2.2), equation (2.1) can be rewritten in a non-dimensional format as

$$v_1'''' + v_1'' + w \cos \theta = 0 \quad (2.3a)$$

$$v_2'''' + v_2'' + w \sin \theta = 0, \quad (2.3b)$$

where $v_j = \hat{v}_j/r$ is the non-dimensional position of the rod centerline in the x ($j = 1$) and y ($j = 2$) coordinate directions;

$w = \hat{w}/\alpha^2 W r$ is the non-dimensional distributed wall contact force;

$\zeta = \alpha z$ is the non-dimensional position along the z axis;

$$\alpha = [W/EI]^{1/2};$$

and where a prime designates $d/d\zeta$, differentiation with respect to the new independent variable ζ .

The equation for a curve lying on the surface of a circular cylinder is written parametrically as

$$\hat{v}_1 = r \cos \theta \quad (2.4a)$$

$$\hat{v}_2 = r \sin \theta \quad (2.4b)$$

$$z = g(\theta). \quad (2.4c)$$

Introducing the non-dimensional notation, differentiating, and treating θ as $\theta(\zeta)$ results in

$$v_1 = \cos \theta \quad (2.5a)$$

$$v_1' = -\theta' \sin \theta \quad (2.5b)$$

$$v_1'' = -\theta'' \sin \theta - (\theta')^2 \cos \theta \quad (2.5c)$$

$$v_1''' = [(\theta')^3 - \theta'''] \sin \theta - 3\theta' \theta'' \cos \theta \quad (2.5d)$$

$$v_1'''' = [6(\theta')^2 \theta'' - \theta'''''] \sin \theta + [(\theta')^4 - 4\theta' \theta''' - 3(\theta'')^2] \cos \theta \quad (2.5e)$$

$$v_2 = \sin \theta \quad (2.6a)$$

$$v_2' = \theta' \cos \theta \quad (2.6b)$$

$$v_2'' = \theta'' \cos \theta - (\theta')^2 \sin \theta \quad (2.6c)$$

$$v_2''' = -[(\theta')^3 - \theta'''] \cos \theta - 3\theta' \theta'' \sin \theta \quad (2.6d)$$

$$v_2'''' = -[6(\theta')^2 \theta'' - \theta'''''] \cos \theta + [(\theta')^4 - 4\theta' \theta''' - 3(\theta'')^2] \sin \theta \quad (2.6e)$$

Substitution of these relations into either of equations (2.3) results in two equations for $\theta(\zeta)$;

$$\theta'''' + [1 - 6(\theta')^2] \theta'' = 0 \quad (2.7)$$

and

$$w + (\theta')^4 - (\theta')^2 - 4\theta' \theta''' - 3(\theta'')^2 = 0 \quad (2.8)$$

Equation (2.7) is the governing differential equation for $\theta(\zeta)$ in the contact section. Once $\theta(\zeta)$ is determined by solution of equation (2.7), equation (2.8) can be solved separately for the wall contact force $w(\zeta)$.

Note that it is not possible to linearize equation (2.7) by neglecting the $6(\theta')^2$ term relative to unity. As shown in Appendix III, the solution for the helically buckled rod away from the end gives values of $0.707 \leq \theta' \leq 1.0$ [5,6], such that $3 \leq 6(\theta')^2 \leq 6$ which is clearly not small relative to unity. It is necessary, therefore, to solve the non-linear fourth order differential equation (2.7) in order to obtain $\theta(\zeta)$ for the contact section.

The internal reactions of the rod in the contact section are expressed in terms of the rod deflections in Appendix I and are

$$V_j = -(v_j'''' + v_j') \quad (2.9)$$

$$M_j = v_j''' \quad (2.10)$$

where $V_j = \hat{V}_j / \alpha W r$ is the non-dimensional internal shear force corresponding to the dimensional shear force $\hat{V}_j$;

$M_j = \hat{M}_j / W r$ is the non-dimensional internal bending moment corresponding to the dimensional bending moment $\hat{M}_j$;

and where $j = 1$ for the x-direction component, and $j = 2$ for the y-direction component.

These can be transformed to R, θ coordinates to obtain

$$V_R = 3\theta'\theta'' \quad (2.11a)$$

$$V_\theta = (\theta')^3 - \theta' - \theta''' \quad (2.11b)$$

$$M_R = -(\theta')^2 \quad (2.12a)$$

$$M_\theta = \theta'' \quad (2.12b)$$

where $V_R = \hat{V}_R/\alpha W r$

and

$$V_\theta = \hat{V}_\theta/\alpha W r$$

are the non-dimensional radial and tangential shear force components corresponding to the dimensional forms $\hat{V}_R$ and $\hat{V}_\theta$,

and where $M_R = \hat{M}_R/W r$

and

$$M_\theta = \hat{M}_\theta/W r$$

are the non-dimensional radial and tangential bending moment components corresponding to $\hat{M}_R$ and $\hat{M}_\theta$.

2.2 Analysis of the End and Transition Sections

Since there is no distributed force acting on the end and transition sections, $w = 0$ and the governing differential equations become

$$u_{ij}'''' + u_{ij}'' = 0 \quad (2.13)$$

where $u_{ij} = \hat{u}_{ij}/r$ is the non-dimensional position of the rod centerline in the end ($i = 1$) and transition ($i = 2$) sections,

and where $j = 1$ and 2 for the x and y directions as before.

The general solution of equation (2.13) is

$$u_{ij} = a_{ij} + b_{ij}\zeta + c_{ij} \cos \zeta + d_{ij} \sin \zeta \quad (2.14)$$

where a_{ij} , b_{ij} , c_{ij} , and d_{ij} are unknown constants.

2.3 Maximum Bending Moment in the End Section

The magnitude of the internal bending moment in the end section is

$$M = [(u'_{11})^2 + (u'_{12})^2]^{1/2} \quad (2.15)$$

where $M = \hat{M}/Wr$ is the non-dimensional bending moment corresponding to $\hat{M}$.

Extremum values of M occur where $dM/d\zeta = 0$, which requires

$$u'_{11} u'''_{11} + u'_{12} u'''_{12} = 0 \quad (2.16)$$

Substitution of equation (2.14) into equation (2.16) shows that the moment is an extremum where

$$f(\zeta) = (d_{11}^2 + d_{12}^2 - c_{11}^2 - c_{12}^2) \sin \zeta \cos \zeta + (c_{11} d_{11} + c_{12} d_{12}) (\cos^2 \zeta - \sin^2 \zeta) = 0. \quad (2.17)$$

For the pinned-end problem, $c_{11} = c_{12} = 0$, resulting in the condition

$$\sin \zeta \cos \zeta = 0 \quad (2.18)$$

which gives an extremum moment location of $\zeta = \pi/2$. As will be seen in Section 4, this is the location of maximum moment, and the maximum moment for the pinned-end case is

$$M = [d_{11}^2 + d_{12}^2]^{1/2} \quad (2.19)$$

2.4 Boundary Conditions

Boundary conditions at the rod end are those for a pinned connection in the x - z and y - z planes located on the cylinder axis:

$$u_{1j}(0) = 0 \quad (2.20)$$

$$u_{1j}''(0) = 0 \quad (2.21)$$

The boundary conditions at the first contact point, $\zeta = \phi_1$, require the rod to contact the cylinder wall tangent to the wall surface. Continuity of position, slope, curvature, and tangential shear between the end and transition sections is also required:

$$u_{11}(\phi_1) = \cos \theta_1 \quad (2.22a)$$

$$u_{12}(\phi_1) = \sin \theta_1 \quad (2.22b)$$

$$u_{11}'(\phi_1) = -\theta_1' \sin \theta_1 \quad (2.23a)$$

$$u_{12}'(\phi_1) = \theta_1' \cos \theta_1 \quad (2.23b)$$

$$u_{21}(\phi_1) = \cos \theta_1 \quad (2.24a)$$

$$u_{22}(\phi_1) = \sin \theta_1 \quad (2.24b)$$

$$u'_{21}(\phi_1) = -\theta'_1 \sin \theta_1 \quad (2.25a)$$

$$u'_{22}(\phi_1) = \theta'_1 \cos \theta_1 \quad (2.25b)$$

$$u''_{11}(\phi_1) = u''_{21}(\phi_1) \quad (2.26a)$$

$$u''_{12}(\phi_1) = u''_{22}(\phi_1) \quad (2.26b)$$

$$\begin{aligned} & u'''_{11}(\phi_1) \sin \theta_1 - u'''_{12}(\phi_1) \cos \theta_1 \\ & = u'''_{21}(\phi_1) \sin \theta_1 - u'''_{22}(\phi_1) \cos \theta_1 \end{aligned} \quad (2.27)$$

where θ_1 and θ'_1 are constants; $\theta_1 = \theta(\phi_1)$, $\theta'_1 = \theta'(\phi_1)$. A discontinuous radial shear force is allowed since a concentrated radial contact force can occur as demonstrated with the 2-dimensional analogy of Appendix XII.

The boundary conditions at the second contact point, $\zeta = \phi_2$, require that the rod contact the wall at a position of $\theta(\phi_2) = 0$ and that there be continuity of position, slope, curvature, and tangential shear between the transition and contact sections:

$$u_{21}(\phi_2) = 1 \quad (2.28a)$$

$$u_{22}(\phi_2) = 0 \quad (2.28b)$$

$$u'_{21}(\phi_2) = 0 \quad (2.29a)$$

$$u'_{22}(\phi_2) = \theta'_2 \quad (2.29b)$$

$$u''_{21}(\phi_2) = -(\theta'_2)^2 \quad (2.30a)$$

$$u''_{22}(\phi_2) = \theta''_2 \quad (2.30b)$$

$$-(u_{22}'' + u_{22}') = (\theta_2')^3 - \theta_2' - \theta_2''' \quad (2.31)$$

where θ_2' , θ_2'' , θ_2''' are unknown constants; $\theta_2' = \theta'(\phi_2)$, $\theta_2'' = \theta''(\phi_2)$, $\theta_2''' = \theta'''(\phi_2)$. The specification of $\theta(\phi_2) = 0$ is arbitrary since the rod is assumed weightless.

An additional boundary condition is necessary in the contact section. The problem being solved is either that of a finite length rod loaded axially at both ends, or that of a semi-infinite rod. For the semi-infinite case it is required that the pitch of the contact section be some constant value far from the rod end; i.e., $\theta''(\infty) = 0$. It is shown in Appendix II that this requirement is also the mid-length condition for the finite length rod. The final boundary condition is, therefore,

$$\theta''(\phi_s) = 0 \quad (2.32)$$

where $\zeta = \phi_s$ is the half length of the finite length rod, or is infinity for the semi-infinite rod case.

A count of the boundary conditions gives a total of 23 equations in the 23 unknowns;

$$\begin{aligned} & a_{ij}, b_{ij}, c_{ij}, d_{ij} \quad i = 1, 2; j = 1, 2 \\ & \phi_1, \phi_2 \\ & \theta_1, \theta_1' \end{aligned} \quad (2.33)$$

and

$$\theta_2', \theta_2'', \theta_2'''.$$

2.5 Solution Method

The solution procedure is comprised of two major parts; 1) solution of the set of simultaneous non-linear algebraic equations resulting from application of the boundary conditions for the end and transition sections at $\zeta = 0$, $\zeta = \phi_1$, and $\zeta = \phi_2$, and 2) solution of the non-linear fourth order differential equation of the contact section such that $\theta''(\phi_s) = 0$. The first part is accomplished by assuming a value for ϕ_1 and using the first 22 boundary conditions, equations (2.20-2.31) to solve for the remaining 22 unknowns. This calculation generates values for θ_2' , θ_2'' , and θ_2''' for a given value of ϕ_1 . These are then used in the second part of the solution as initial values, along with $\theta(\phi_2) = 0$, for the numerical integration of the non-linear equation (2.7). Integration is continued along ζ to $\zeta = \phi_s$, and the results are checked to determine if $\theta''(\phi_s) = 0$. ϕ_1 is adjusted until $\theta''(\phi_s) = 0$ is satisfied. The overall approach is, therefore, a shooting method for the solution of the non-linear boundary value problem on θ in the contact section. The adjustment of ϕ_1 is performed using an interval halving procedure.

Treating ϕ_1 as known, application of equations (2.20) and (2.21) shows that

$$a_{1j} = c_{1j} = 0 \quad (2.34)$$

for $j = 1, 2$.

Equations (2.22) and (2.23) give

$$b_{11} = \frac{-\cos \theta_1 \cos \phi_1 - \theta_1' \sin \theta_1 \sin \phi_1}{\sin \phi_1 - \phi_1 \cos \phi_1} \quad (2.35a)$$

$$b_{12} = \frac{-\sin \theta_1 \cos \phi_1 + \theta_1' \cos \theta_1 \sin \phi_1}{\sin \phi_1 - \phi_1 \cos \phi_1} \quad (2.35b)$$

$$d_{11} = \frac{\cos \theta_1 + \phi_1 \theta_1' \sin \theta_1}{\sin \phi_1 - \phi_1 \cos \phi_1} \quad (2.36a)$$

$$d_{12} = \frac{\sin \theta_1 - \phi_1 \theta_1' \cos \theta_1}{\sin \phi_1 - \phi_1 \cos \phi_1} \quad (2.36b)$$

Equations (2.25) through (2.32) require that

$$a_{21} + b_{21} \phi_1 + c_{21} \cos \phi_1 + d_{21} \sin \phi_1 - \cos \theta_1 = 0 \quad (2.37a)$$

$$a_{22} + b_{22} \phi_1 + c_{22} \cos \phi_1 + d_{22} \sin \phi_1 - \sin \theta_1 = 0 \quad (2.37b)$$

$$b_{21} - c_{21} \sin \phi_1 + d_{21} \cos \phi_1 + \theta_1' \sin \theta_1 = 0 \quad (2.37c)$$

$$b_{22} - c_{22} \sin \phi_1 + d_{22} \cos \phi_1 - \theta_1' \cos \theta_1 = 0 \quad (2.37d)$$

$$(\cos \theta_1 - \phi_1 \theta_1' \sin \theta_1) \frac{\sin \phi_1}{\sin \phi_1 - \phi_1 \cos \phi_1} - c_{21} \cos \phi_1 - d_{21} \sin \phi_1 = 0 \quad (2.37e)$$

$$(\sin \theta_1 - \phi_1 \theta_1' \cos \theta_1) \frac{\sin \phi_1}{\sin \phi_1 - \phi_1 \cos \phi_1} - c_{22} \cos \phi_1 - d_{22} \sin \phi_1 = 0 \quad (2.37f)$$

$$\theta_1' \frac{\sin \phi_1}{\sin \phi_1 - \phi_1 \cos \phi_1} + b_{21} \sin \theta_1 - b_{22} \cos \theta_1 = 0 \quad (2.37g)$$

$$a_{21} + b_{21} \phi_2 + c_{21} \cos \phi_2 + d_{21} \sin \phi_2 - 1 = 0 \quad (2.37h)$$

$$a_{22} + b_{22} \phi_2 + c_{22} \cos \phi_2 + d_{22} \sin \phi_2 = 0 \quad (2.37i)$$

$$b_{21} - c_{21} \sin \phi_2 + d_{21} \cos \phi_2 = 0 \quad (2.37j)$$

$$b_{22} - c_{22} \sin \phi_2 + d_{22} \cos \phi_2 - \theta_2' = 0 \quad (2.37k)$$

$$c_{21} \cos \phi_2 + d_{21} \sin \phi_2 - (\theta_2')^2 = 0 \quad (2.37l)$$

$$\theta_2'' + c_{22} \cos \phi_2 + d_{22} \sin \phi_2 = 0 \quad (2.37m)$$

$$b_{22} + (\theta_2')^3 - \theta_2' - \theta_2''' = 0 \quad (2.37n)$$

Solution for the unknowns of equation (2.37) requires simultaneous solution of the set of 14 non-linear algebraic equations. This is accomplished numerically using the Modified Quasilinearization Algorithm (MQA) [16].

Integration of equation (2.7), once the values of θ_2' , θ_2'' , and θ_2''' are determined from solution of equations (2.37), is performed numerically using a fixed-step fourth order Runge-Kutta integration procedure [17].

The computer program used to perform the solution of the MQA procedure, the Runge-Kutta procedure, and shooting on ϕ_1 , is included as Appendix VIII.

3.0 ANALYSIS OF A FIXED-END HELICALLY BUCKLED ROD

The fixed-end analysis is identical to that for the pinned-end case except that the boundary conditions at the rod end are modified to require the rod end to be tangent to the cylinder axis.

As with the pinned-end case, initial analysis indicated that a configuration similar to that of Figure 1 results. An analysis following that by Mitchell [10] shows that the end section cannot connect to a constant pitch helix at the point of wall contact unless an unrealistic concentrated tangential contact force is introduced (see Appendix VI). Further analysis, assuming that the end section connects to a perturbed helix with no intermediate transition section, results in an unrealistic distributed wall contact force which pulls the rod against the wall near the contact point (see Appendix VII). As with the pinned-end analysis, this result suggested that an intermediate transition section is present as shown in Figure 1.

3.1 Boundary Conditions

Boundary conditions at the rod end are those for a built-in end in the x - z and y - z planes located on the cylinder axis:

$$u_{1j}(0) = 0 \quad (3.1)$$

$$u'_{1j}(0) = 0 \quad (3.2)$$

All other boundary conditions, equations (2.22-2.32), remain the same as for the pinned-end case.

3.2 Solution Method

The problem solution procedure of Section 2.5 is also used for the fixed-end case.

Equations (3.1), (3.2), (2.22), and (2.23) require that

$$a_{11} = \frac{\cos\theta_1(1-\cos\phi_1) + \theta'_1 \sin\theta_1(\phi_1 - \sin\phi_1)}{2 - 2\cos\phi_1 - \phi_1 \sin\phi_1} \quad (3.3a)$$

$$b_{11} = \frac{-\theta'_1 \sin\theta_1(1-\cos\phi_1) - \sin\phi_1 \cos\theta_1}{2 - 2\cos\phi_1 - \phi_1 \sin\phi_1} \quad (3.3b)$$

$$c_{11} = -a_{11} \quad (3.3c)$$

$$d_{11} = -b_{11} \quad (3.3d)$$

$$a_{12} = \frac{\sin\theta_1(1-\cos\phi_1) - \theta'_1 \cos\theta_1(\phi_1 - \sin\phi_1)}{2 - 2\cos\phi_1 - \phi_1 \sin\phi_1} \quad (3.4a)$$

$$b_{12} = \frac{\theta'_1 \cos\theta_1(1-\cos\phi_1) - \sin\theta_1 \sin\phi_1}{2 - 2\cos\phi_1 - \phi_1 \sin\phi_1} \quad (3.4b)$$

$$c_{12} = -a_{12} \quad (3.4c)$$

$$d_{12} = -b_{12} \quad (3.4d)$$

Equations (2.24-2.30a) require

$$a_{21} + b_{21} \phi_1 + c_{21} \cos \phi_1 + d_{21} \sin \phi_1 - \cos \theta_1 = 0 \quad (3.5a)$$

$$a_{22} + b_{22} \phi_1 + c_{22} \cos \phi_1 + d_{22} \sin \phi_1 - \sin \theta_1 = 0 \quad (3.5b)$$

$$b_{21} - c_{21} \sin \phi_1 + d_{21} \cos \phi_1 + \theta_1' \sin \theta_1 = 0 \quad (3.5c)$$

$$b_{22} - c_{22} \sin \phi_1 + d_{22} \cos \phi_1 - \theta_1' \cos \theta_1 = 0 \quad (3.5d)$$

$$c_{21} \cos \phi_1 + d_{21} \sin \phi_1 - \left[\frac{\cos \theta_1 (1 - \cos \phi_1) + \theta_1' \sin \theta_1 (\sin \phi_1 - \phi_1 \cos \phi_1)}{2 - 2 \cos \phi_1 - \phi_1 \sin \phi_1} \right] = 0 \quad (3.5e)$$

$$c_{22} \cos \phi_1 + d_{22} \sin \phi_1 - \left[\frac{\sin \theta_1 (1 - \cos \phi_1) - \theta_1' \cos \theta_1 (\sin \phi_1 - \phi_1 \cos \phi_1)}{2 - 2 \cos \phi_1 - \phi_1 \sin \phi_1} \right] = 0 \quad (3.5f)$$

$$\begin{aligned} & -(c_{21} \sin \theta_1 - c_{22} \cos \theta_1) \sin \phi_1 + (d_{21} \sin \theta_1 - d_{22} \cos \theta_1) \cos \phi_1 \\ & + \theta_1' \frac{(1 - \cos \phi_1 - \phi_1 \sin \phi_1)}{(2 - 2 \cos \phi_1 - \phi_1 \sin \phi_1)} = 0 \end{aligned} \quad (3.5g)$$

$$a_{21} + b_{21} \phi_2 + c_{21} \cos \phi_2 + d_{21} \sin \phi_2 - 1 = 0 \quad (3.5h)$$

$$a_{22} + b_{22} \phi_2 + c_{22} \cos \phi_2 + d_{22} \sin \phi_2 = 0 \quad (3.5i)$$

$$b_{21} - c_{21} \sin \phi_2 + d_{21} \cos \phi_2 = 0 \quad (3.5j)$$

$$b_{22} - c_{22} \sin \phi_2 + d_{22} \cos \phi_2 - \theta_2' = 0 \quad (3.5k)$$

$$c_{21} \cos \phi_2 + d_{21} \sin \phi_2 - \theta_2'^2 = 0. \quad (3.5l)$$

Equations (2.30b) and 2.31) are used to solve directly for θ_2'' and θ_2''' :

$$\theta_2'' = -c_{22} \cos \phi_2 - d_{22} \sin \phi_2 \quad (3.6a)$$

$$\theta_2''' = b_{22} + (\theta_2')^3 - \theta_2' \quad (3.6b)$$

Solution for the 12 unknowns of equation (3.5) and integration of equation (2.7) were accomplished with the same techniques as described in Section 2.5. The computer program used to perform the calculations is included in Appendix IX.

4.0 RESULTS

The analyses described in Sections 2 and 3 are formulated such that a value for ϕ_c , the contact section half-length for a finite-length rod loaded axially at both ends, must be specified. Review of Tables 1 and 2, which contain summaries of selected results for $\phi_c = 2, 4$, and 6 , shows that the results are relatively insensitive to ϕ_c and indicates that $\phi_c = 6.0$ can be used to represent the semi-infinite rod situation. While the analyses of Sections 2 and 3 are applicable to the finite-length rod problem, the behavior of a semi-infinite rod is of primary interest here and the results which are presented in the remainder of this section and in Section 6.0 are for $\phi_c = 6.0$. Computer printouts of the results for $\phi_c = 6.0$ are included as Appendix X for the pinned-end problem and as Appendix XI for the fixed-end problem.

Figures 2 through 11 present the key results of both the pinned-end and fixed-end analyses. All results are presented in the non-dimensional format used throughout this analysis and are completely general with regard to rod size, rod stiffness, cylinder size, and compressive axial load magnitude.

The geometry of the buckled rod is shown in Figures 2 and 3. For the pinned-end case, initial wall contact occurs at

$$\zeta = \phi_1 = 2.46867, \quad (4.1a)$$

and a second contact point occurs at

$$\zeta = \phi_2 = 4.38105. \quad (4.1b)$$

The pinned-end rod is in contact with the cylinder wall everywhere along its length for $\zeta \geq 4.38105$. For the fixed-end case, the initial wall contact occurs at

$$\zeta = \phi_1 = 3.81729, \quad (4.2a)$$

and a second contact point occurs at

$$\zeta = \phi_2 = 5.24273. \quad (4.2b)$$

The fixed-end rod is in contact with the cylinder wall everywhere for $\zeta \geq 5.24273$. The deflected shape of the end section for the pinned-end rod is planar.

In Figure 2a, the radial position of the rod in the transition section for both the pinned and fixed cases appears to be on the wall. The numerical results of Appendix X and XI show that the transition sections are actually off the wall by a very small amount. The maximum clearance for the pinned-end case is approximately 0.002 r and for the fixed-end case is 0.0005 r. While the transition section separation from the cylinder wall is small, allowing the rod to be off the wall for some length inboard of the first contact point eliminates the negative wall contact force

present in the results of the continuous contact analyses (Appendices V and VII). Comparison of the results from the continuous contact analyses with those of this section shows that, with the exception of the wall contact force which becomes zero, the rod behavior in the transition section is nearly identical to the behavior of the contact sections of the continuous contact analyses over the same rod segment. This is consistent with the very small change in radial position which results from relaxation of the wall contact requirement.

Figures 2 and 4 show that away from the rod end, the rod is buckled into a constant pitch helical configuration with $\theta' = 1/\sqrt{2} = .70711$. The contact section of the rod can therefore be described by two subsections; a perturbed helix section and a helical section. The perturbed helix section is that portion of the rod from the second contact point at $\zeta = \phi_2$ to the point $\zeta = \phi_h$, at which

$$\theta'(\phi_h) = 0.999 \times \frac{1}{\sqrt{2}}, \quad (4.4)$$

where the choice of the factor 0.999 is arbitrary. The helical section is defined as the portion of the rod at $\zeta \geq \phi_h$. Using this definition to determine the length of the perturbed section, the pinned-end results show that the helical section begins at

$$\zeta = \phi_{h, \text{ pinned}} = 6.75105, \quad (4.5a)$$

and the fixed-end results show the helical section beginning at

$$\zeta = \phi_{h, \text{ fixed}} = 7.79273. \quad (4.5b)$$

The total non-helical length of rod from the rod end to the beginning of the helical section can be expressed as a fraction of the helix pitch by

$$\phi_h / \alpha p_{\text{pinned}} = 0.760 \quad (4.6a)$$

$$\phi_h / \alpha p_{\text{fixed}} = 0.877. \quad (4.6b)$$

As shown in Appendix III, a constant value of $\theta' = .70711$ is the result obtained in [5] for a helically buckled rod away from any end effects. It is interesting to note that while the present analysis does force the contact section solution to a constant value of θ' by requiring that $\theta''(\phi_s) = 0$, it does not force the solution to any particular value of θ' . The fact that the solution converges to the value predicted in [5] provides corroboration of these results.

It is also shown in Appendix III that the parameter θ' is related to the helix pitch, p , and the helix angle, γ (see Figure III-2), by

$$p = 2\pi / \alpha \theta'$$

and

$$\gamma = \alpha r \theta'.$$

Figures 5 and 6 show the internal bending moments, M , M_R , and M_θ , along the rod. For the pinned-end case, the maximum bending moment occurs in the end section at

$$\zeta(M_{\max, \text{pinned}}) = \pi/2 = 1.57080, \quad (4.7)$$

as demonstrated in Section 2.3, and is

$$M_{\max, \text{pinned}} = 0.59727 \quad (4.8)$$

which is 19.5% greater than the bending moment in the helical portion of the rod. The maximum bending moment in the fixed-end rod also occurs in the end section, at

$$\zeta(M_{\max, \text{fixed}}) = 2.99229, \quad (4.9)$$

and is

$$M_{\max, \text{fixed}} = 0.50491 \quad (4.10)$$

which is only 1% greater than in the helical portion. Note that the bending moment at the fixed rod end equals that of the helical portion.

The internal shear force along the rod is shown in Figures 7 and 8. The maximum shear force occurs in the helical portion of the rod and is

$$V_{\max} = 0.35355 \quad (4.11)$$

for both the pinned-end and fixed-end cases.

The wall contact force is shown in Figure 9. The pinned-end contact force is

$$w(\phi_2)_{\text{pinned}} = .14672 \quad (4.12)$$

at the beginning of the contact section. The fixed-end force starts at

$$w(\phi_2)_{\text{fixed}} = .12050 \quad (4.13)$$

For both cases, the contact force increases along the length until it reaches a maximum value of

$$w_{\text{max}} = 0.25 \quad (4.14)$$

in the helical portion.

Figures 10 and 11 show an end view of the concentrated external reactions acting on the rod. For the pinned-end case, there is a radial force acting at the rod end of

$$F(0)_{\text{pinned}} = .32670 \quad (4.15)$$

Concentrated radial contact forces associated with the radial shear discontinuities at the first and second contact points are

$$F(\phi_1)_{\text{pinned}} = 0.61028 \quad (4.16a)$$

$$F(\phi_2)_{\text{pinned}} = 0.04726. \quad (4.16b)$$

The fixed-end rod has an end force and moment of

$$F(0)_{\text{fixed}} = 0.19187 \quad (4.17a)$$

$$M(0)_{\text{fixed}} = 0.50000. \quad (4.17b)$$

Concentrated radial contact forces of

$$F(\phi_1)_{\text{fixed}} = 0.38202 \quad (4.18a)$$

and

$$F(\phi_2)_{\text{fixed}} = 0.02557 \quad (4.18b)$$

result as shown.

The analysis of Sections 2 and 3 was formulated to produce closed-form solutions for the deflected shapes of the end and transition sections for both the pinned-end and fixed-end cases. The rod geometry is described by

$$u_{ij} = a_{ij} + b_{ij}\zeta + c_{ij} \cos\zeta + d_{ij} \sin\zeta \quad (2.14)$$

where $i = 1$ for the end section, $i = 2$ for the transition section, and $j = 1, 2$ for the x, y directions. The internal rod reactions are

$$V_{ij} = -(u_{ij}'' + u_{ij}') = -b_{ij} \quad (4.19)$$

and

$$M_{ij} = u_{ij}'' = -c_{ij} \cos\zeta - d_{ij} \sin\zeta. \quad (4.20)$$

Parameter values for the pinned-end problem are:

$$\begin{aligned}
a_{11} &= 0 & a_{12} &= 0 \\
b_{11} &= 0.22147 & b_{12} &= -0.24017 \\
c_{11} &= 0 & c_{12} &= 0 \\
d_{11} &= -0.26884 & d_{12} &= -0.53335 \\
a_{21} &= 0.57127 & a_{22} &= -1.39407 \\
b_{21} &= -0.00994 & b_{22} &= 0.32454 \\
c_{21} &= -0.14423 & c_{22} &= 0.35197 \\
d_{21} &= -0.44980 & d_{22} &= -0.09174
\end{aligned} \tag{4.21}$$

Parameter values for the fixed-end problem are:

$$\begin{aligned}
a_{11} &= -0.12662 & a_{12} &= -0.48370 \\
b_{11} &= 0.19104 & b_{12} &= 0.01782 \\
c_{11} &= 0.12662 & c_{12} &= 0.48370 \\
d_{11} &= -0.19104 & d_{12} &= -0.01782 \\
a_{21} &= 0.78238 & a_{22} &= -1.62402 \\
b_{21} &= -0.04708 & b_{22} &= 0.31654 \\
c_{21} &= 0.27556 & c_{22} &= 0.29687 \\
d_{21} &= -0.37685 & d_{22} &= 0.21527
\end{aligned} \tag{4.22}$$

The results of the fixed-end analysis can be compared directly with the results of Mitchell [10]. Table 3 presents a comparison of selected results and shows that the value for ϕ_1 presented in reference [10] is significantly greater than the result obtained here. It is also observed that the results of reference [10] overpredict $V(0)$ and

underpredict $M(0)$, $\theta'(\infty)$, and $w(\infty)$. The value for the helix pitch away from the rod end is significantly overpredicted by reference [10], which will lead to a calculation of lower than actual bending stresses in the helical portion of the rod.

5.0 EXPERIMENTAL INVESTIGATION

An experimental investigation was conducted to evaluate the rod behavior predicted analytically in the previous sections. The test apparatus consisted of a 3/16" diameter, 6 foot long, aluminum rod constrained in a 1" diameter straight tube. The tube was constructed of five sections coupled together; four precisely machined 6" long plexiglass tube segments and a three foot long section of same diameter clear PVC pipe. Three of the 6" tube segments were fabricated with a series of closely spaced aluminum rings on the inside diameter. The rod was loaded axially with a hydraulic jack. Axial load was measured with a calibrated strain gage load transducer.

The rod behavior characteristics which can be most easily compared with the analytical results are the contact point locations and the pitch of the helical section. The contact locations were identified by checking electrical continuity between the rod and the tube rings. Measurements at several axial loads gave results of $2.44 < \phi_1 < 2.85$ and $4.1 < \phi_2 < 4.7$ for a pinned-end configuration. These values compare well with analytical results (4.1); $\phi_1 = 2.46867$, $\phi_2 = 4.38105$. Fixed-end rod measurements showed that $3.65 < \phi_1 < 3.89$ and $5.2 < \phi_2 < 5.6$. These results can be compared with analytical results (4.2) for the fixed-end rod; $\phi_1 = 3.81729$ and $\phi_2 = 5.24273$. The effects of

friction were minimized by rotating the rod slowly during and after loading. Pitch length measurements in the helical portion of the rod gave $0.63 < \theta' < 0.71$, which compares well with the analytical result $\theta' = .70711$. Visual confirmation of the presence of a transition section was obtained by moving the rod tangentially outboard of the first contact point and observing that the rod tended to pivot about the contact point, demonstrating no wall contact for some distance inboard of the contact point.

Comparison of the analytical and experimental results shows good agreement and provides verification of key aspects of the analytical results. The location of the contact points for both the pinned-end and fixed-end cases was confirmed. The presence of a transition section and the helical behavior away from the end effects was also verified.

6.0 DISCUSSION

Two numerical examples are presented; one for a pinned-end rod, and one for a fixed-end rod. The pinned-end example is representative of a drill string near the bit. The fixed-end example is representative of tubing sealed in casing by a packer.

Example #1: Pinned End Condition - Drill String Near the Bit

6-3/4" O.D., 2-3/4" I.D., steel drill collar

8-3/4" diameter hole

50,000 pounds weight-on-bit

This rod cross-section has a calculated weight of 101.4 lb/ft, and a bending stiffness of

$$EI = 2.973 \times 10^9 \text{ lb-in},$$

which along with an axial force of

$$W = 50,000 \text{ lb}$$

gives

$$\alpha = [W/EI]^{1/2} = 4.101 \times 10^{-3} \text{ in}^{-1}.$$

The effective radius, r , is the radial clearance between the drill collar and the hole and is

$$r = 1.0 \text{ in.}$$

Rod geometry results are shown schematically in Figure 12. The distance from the bit (i.e., the rod end) to the initial wall contact point is, from equation (4.1a),

$$L_1 = 2.46867/\alpha = 50.2 \text{ ft.}$$

The distance from the bit to the second wall contact point is, from equation (4.1b),

$$L_2 = 4.38105/\alpha = 89.0 \text{ ft.}$$

The distance from the bit to the constant-pitch helical section is, from equation (4.5a),

$$L_h = 6.75105/\alpha = 137.2 \text{ ft.}$$

The maximum separation between the drill collar and the wall in the transition section (i.e., between the first and second wall contact points) is $0.002r = 0.002$ inches.

The helical section has a pitch, from equation (III.10), of

$$p = 2\pi/0.70711\alpha = 180.6 \text{ ft,}$$

a helix angle, from equation (III.9), of

$$\begin{aligned} \gamma &= 0.70711\alpha r = 0.00290 \text{ radians} \\ &= 0.166 \text{ degrees,} \end{aligned}$$

and a distributed wall contact force, from equation (4.14), of

$$\hat{w} = 0.25000 \alpha^2 W_r = 2.5 \text{ lb/ft.}$$

Concentrated radial contact forces occur at the bit, and at the first and second wall contact points. The side force at the bit is, from equation (4.15),

$$\hat{F}(0) = 0.32670 \alpha W_r = 67.0 \text{ lb.}$$

The radial contact forces at the first and second contact points are found from equations (4.16) to be

$$\hat{F}(L_1) = 0.61028 \alpha W_r = 125.1 \text{ lb.}$$

and

$$\hat{F}(L_2) = 0.04726 \alpha W_r = 9.7 \text{ lb.}$$

The maximum bending moment occurs in the end section at a distance of

$$L(M_{\max}) = \pi/2\alpha = 31.9 \text{ ft.}$$

from the bit, and is

$$\hat{M}_{\max} = 0.59727 W_r = 2490 \text{ ft-lb,}$$

based upon equations (4.7) and (4.8).

The maximum shear force occurs in the helical section and is given by equation (4.11) as

$$\hat{V}_{\max} = 0.35355 \alpha W r = 72.5 \text{ lb.}$$

Example #2: Fixed End Condition - Tubing Sealed In a Packer

2-7/8", 6.5 lb/ft steel tubing

7" casing

20,000 pound axial force

This size tubing is 2-7/8" I.D. with 0.217" wall thickness, resulting in

$$EI = 7.594 \times 10^7 \text{ lb-in,}$$

which along with

$$W = 20,000 \text{ lb,}$$

gives

$$\alpha = [W/EI]^{1/2} = 0.01623$$

The effective hole radius is $r = 1.846$ inches.

The rod geometry is shown schematically in Figure 13. The first contact point occurs at

$$L_1 = 3.81729/\alpha = 19.6 \text{ ft,}$$

based upon result (4.2a). The second contact point occurs at

$$L_2 = 5.24273/\alpha = 26.9 \text{ ft,}$$

based upon result (4.2b). The distance from the bit to the beginning of the constant-pitch helical section is, from equation (4.5b),

$$L_h = 7.79273/\alpha = 40.0 \text{ ft.}$$

The maximum separation between the tubing and the wall in the transition section is $0.00046r = 0.00085$ inches.

The helical section has a pitch, from equation (III.10), of

$$p = 2\pi/0.70711\alpha = 45.6 \text{ ft,}$$

a helix angle, from equation (III.9), of

$$\begin{aligned}\gamma &= 0.70711\alpha r = 0.02119 \text{ radians} \\ &= 0.728 \text{ degrees}\end{aligned}$$

and a distributed wall contact force, from equation (4.14), of

$$\hat{w} = 0.25000 \alpha^2 W r = 29.2 \text{ lb/ft.}$$

A concentrated radial moment and force occur at the packer. Their magnitudes are given by equation (4.17) as

$$\begin{aligned}\hat{M}(0) &= 0.50000 \alpha W r = 1538 \text{ ft-lb,} \\ \hat{F}(0) &= 0.19187 \alpha W r = 115.0 \text{ lb.}\end{aligned}$$

The concentrated radial contact forces at the first and second contact points are obtained using equations (4.18);

$$\hat{F}(L_1) = 0.38202 \alpha W_r = 228.9 \text{ lb}$$

$$\hat{F}(L_2) = 0.02557 \alpha W_r = 15.3 \text{ lb.}$$

The maximum bending moment occurs in the end section at a distance of

$$L(M_{\max}) = 2.99229/\alpha = 15.4 \text{ ft.}$$

from the bit, and is

$$\hat{M}_{\max} = 0.50491 W_r = 1553 \text{ ft-lb.,}$$

based upon results (4.9) and (4.10).

The maximum shear force occurs in the helical section and is given by equation (4.11) as

$$\hat{V}_{\max} = 0.35355 \alpha W_r = 211.8 \text{ lb.}$$

Several assumptions are made in the analyses of Sections 2 and 3 which might affect application of these results to drill string or tubing analyses:

- 1) Slender beam theory applies.
- 2) The circular rod has constant outside diameter and uniform stiffness.
- 3) Axial force variation is neglected.
- 4) Torque effects are neglected.
- 5) Fluid flow effects are neglected.
- 6) Dynamic effects are neglected.

- 7) Rod contact with cylinder wall is frictionless.
- 8) The rod is weightless.
- 9) The cylinder wall is circular, smooth, and straight.

The first and second assumptions will hold for practical situations given the small deflections encountered and drill collar or tubing geometries typically used.

The effects of axial variation in the axial force may be important under certain circumstances. If the axial force is large and the rod has a low weight per unit length, the variation of the axial force within the non-helical portion of the rod will be small. An example of this situation is provided by Example #2 above. The total length of the rod from the rod end to the beginning of the helical section is $L_h = 40.0$ ft. The weight of the rod is 7.2 lb/ft, so that the total variation in the axial load over the non-helical portion of the rod is only 290 pounds. For the assumed axial force of 20,000 pounds, this small variation in axial force is not likely to affect results significantly. Alternatively, if the axial force is small relative to the weight of the non-helical sections, the axial force variation will be relatively large and may significantly affect the results. An example of this situation is provided in Example #1 above. The length of the non-helical portion of the rod is 137.2 ft. The weight of the rod is

101.4 lb/ft, so that the total weight of the non-helical portion is 13,910 pounds. This weight is not negligible compared to the axial force of 50,000 pounds, and the axial force variation may affect the results.

For practical situations, it has been shown that the effects of torque are small according to Dawson and Paslay [8], and Walker and Friedman [12]. Fluid flow effects are shown negligible in reference [5].

The effects of bottom hole assembly dynamics have been analyzed in reference [13]. It is possible that the combination of rod rotation and rod-cylinder friction will cause the bottom hole assembly of a drill string to walk around the inside of the hole. This effect was observed in the laboratory experimental investigation and will likely lead to a quasi-static situation. Given the relatively slow rotational speeds of less than several hundred rpm for drill strings and the absence of rotation for tubing, neglecting dynamic effects appears reasonable.

It was observed in the experimental investigation that friction can affect, very significantly, the behavior of the buckled rod. In the experimental model, friction effects tended, however, to dissipate as the rod was rotated. Frictional effects in a static situation such as with tubing may be significant and require further investigation.

As shown in references [7-9], gravity effects can significantly affect the initiation and characteristics of rod buckling. The effects of gravity on the results obtained here will be significant in situations where the axial force is small relative to the weight of the non-helical portion of the rod, and require further study. Results presented here are, therefore, valid only for vertical or slightly inclined holes.

In practical drilling situations, a drill string will frequently incorporate stabilizers near the bit. The condition of fixity at a stabilizer is greater than that at a pinned-end, but less than that of a fixed-end. Based upon the results of the 2-dimensional analogy in Appendix XII, it is anticipated that the condition at a stabilizer is bounded by the pinned-end and fixed-end problems analyzed here. If multiple stabilizers are used, closer spacing results in a situation which will more closely match the fixed-end problem. While the results obtained here should bound the stabilizer situation, the presence of stabilizers, hole non-uniformities, and hole curvature all require further investigation.

The discussion above has identified several areas for further study:

1. Gravity effects; both for axial force variation and hole inclination considerations.

2. Rod-cylinder frictional effects.
3. Hole curvature effects.
4. The effects of stabilizers or hole surface non-uniformities.

7.0 SUMMARY AND CONCLUSIONS

The buckling behavior of an axially loaded circular rod constrained within a circular cylinder is investigated. This problem relates to the buckling of drill pipe near the bit and to the problem of tubing sealed by a packer inside casing. Knowledge of buckling behavior is important for structural design of drill string elements, tubing elements, and packer assemblies. In addition, drill string buckling results in side forces on the drill bit which can affect hole deviation behavior.

With the exception of early investigations by Lubinski [1], Lubinski and Woods [2], and Woods and Lubinski [3], which consider planar buckling only, and an investigation by Mitchell [10], previous buckling analyses have concentrated on rod behavior remote from end effects such as will occur at the bit or stabilizer in a drill string, or at a packer in a tubing assembly.

The purpose of this investigation is to determine the effect of end conditions on the helical post-buckled configuration of an axially loaded circular rod constrained in a circular cylinder. The post-buckling equilibrium analysis assumes that the circular rod is weightless and static, and that displacements are small. Contact between the rod and the smooth cylinder wall is frictionless. The effects of torque and axial variation in the axial force are neglected.

Fluid effects are neglected, except that static pressures can be treated as externally applied forces as in reference [5]. Results are obtained for both a pinned-end case, such as a drill string at the bit, and a fixed-end case, such as tubing at a packer, or a drill string at closely spaced stabilizers.

The deflected centerline of a helically buckled rod with a pinned-end condition is shown schematically in Figure 1. Both the pinned- and fixed-end conditions position the rod end on the cylinder axis and allow axial translation and rotation. The pinned-end condition allows rotation about all three axis. The fixed-end condition requires the rod end to be tangent to the cylinder axis. The effective radius, r , of the circular cylinder on which the rod centerline lies is equal to the difference between the radii of the constraining cylinder and rod.

The centerline of the continuous circular rod is described by three sections; an end section, a transition section, and a contact section. The end section is that portion of the rod from the rod end to the point of initial wall contact. The transition section is an intermediate portion of the rod which is not on the cylinder wall and connects the end section to the contact section. The contact section is on the straight circular cylinder wall everywhere along its length.

Both the pinned- and fixed-end analyses incorporate the same analytical approach. The governing differential equations for the 3-dimensionally buckled beam-column are obtained from equilibrium considerations and reduce to a set of standard beam-column equations based upon slender beam theory. These equations are applied to each of the three rod sections individually. Application of the beam-column equations to the contact section results in two equations for $\theta(\zeta)$. Equation (2.7) is the governing differential equation for $\theta(\zeta)$ in the contact section and is solved numerically. Equation (2.8) is solved separately for the wall contact force $w(\zeta)$. Application of the beam-column equations to the end and transition sections results in the set of equations (2.13).

Boundary conditions are defined which represent the pinned- and fixed-end conditions at the rod end, and which require continuity of position, slope, curvature, and tangential shear force at the end-transition section connection and at the transition-contact section connection. In addition, the boundary conditions require that the rod deflection conform to the cylinder surface at the first and second contact points. A final boundary condition is specified to require a constant helix angle ($\theta'' = 0$) far from the rod end (at $\zeta = \phi_s$).

The solution procedure is comprised of two major parts:

- 1) solution of the set of simultaneous non-linear algebraic equations resulting from application of the boundary conditions for the end and transition sections at the rod end and, at the first and second contact points; and
- 2) solution of the non-linear fourth order differential equation of the contact section such that the helix angle is constant at a specified distance from the rod end, $\Theta''(\phi_s) = 0$. The first part is accomplished by assuming a value for the distance to the first contact point, ϕ_1 , which is not known, and using the end and transition section boundary conditions to solve for the remaining end and transition section unknowns. This requires simultaneous solution of a set of non-linear algebraic equations, which is accomplished numerically using the Modified Quasilinearization Algorithm (MQA) [16]. This calculation generates values for the end conditions of the contact section for a given value of ϕ_1 . These values are then used in the second part of the solution as initial values for the numerical integration of the non-linear equations (2.7) using a fixed-step fourth-order Runge-Kutta algorithm. Integration is continued along ζ to $\zeta = \phi_s$, and the results are checked to determine if $\Theta''(\phi_s) = 0$. ϕ_1 is adjusted until $\Theta''(\phi_s) = 0$ is satisfied. The adjustment of ϕ_1 is performed using an interval-halving procedure.

Figures 2 through 11 present the key results of both the pinned-end and fixed-end analyses. All results are presented in a non-dimensional format and are completely general with regard to rod size, rod stiffness, cylinder size, and compressive axial load magnitude.

The contact section is found to consist of two sub-sections. A perturbed helix sub-section begins at the second contact point and is defined to end where the helix angle becomes 0.999 times the helix angle far from the bit. A helical sub-section is defined as that portion of the rod far from the rod end, where the helix pitch is constant.

For the pinned-end case the end section is planar. Initial wall contact occurs at

$$\alpha z = 2.46867.$$

The second wall contact point occurs at

$$\alpha z = 4.38105,$$

and the rod centerline is a constant pitch helix for

$$\alpha z \geq 6.75105.$$

The non-dimensional pitch of the helical section for both the pinned- and fixed-end conditions is

$$\alpha p = \sqrt{2} \, 2\pi = 8.88577,$$

which is the value obtained by Lubinski, Althouse and Logan [5].

The non-dimensional concentrated radial contact forces present at the rod end, and at the first and second contact points are, for the pinned-end case,

$$\begin{aligned}\hat{F}(0)/\alpha W r &= 0.32670 \\ \hat{F}(\phi_1)/\alpha W r &= 0.61028 \\ \hat{F}(\phi_2)/\alpha W r &= 0.04726.\end{aligned}$$

The maximum pinned-end bending moment occurs in the end section at $\alpha z = \pi/2 = 1.57080$, and is

$$\hat{M}_{\max}/W r = 0.59727,$$

which is 19.5% higher than the value of

$$\hat{M}_{\text{helix}}/W r = 0.50000$$

in the helical section of the rod for $\alpha z \geq 6.75105$.

For the fixed-end case, initial wall contact occurs at

$$\alpha z = 5.24273,$$

and the rod centerline is a constant pitch helix for

$$\alpha z \geq 7.79273.$$

For the fixed-end case, the non-dimensional concentrated radial contact forces present at rod end, and at the first

and second contact points are

$$\hat{F}(0)/\alpha W r = 0.19187$$

$$\hat{F}(\phi_1)/\alpha W r = 0.38202$$

$$\hat{F}(\phi_2)/\alpha W r = 0.02557.$$

A concentrated radial moment is present at the fixed-end of the rod and is

$$\hat{M}(0)/W r = 0.50000,$$

which is the same as the bending moment in the helical section.

The maximum fixed-end bending moment occurs in the end section at $\alpha z = 2.99229$, and is

$$\hat{M}_{\max}/W r = 0.50491,$$

which is only 1% higher than in the helical section of the rod.

The maximum shear force and the maximum wall contact force occur in the helical section for both the pinned- and fixed-end conditions and are

$$\hat{V}_{\max}/\alpha W r = 0.35355$$

and

$$\hat{w}_{\max}/\alpha^2 W r = 0.25000.$$

Comparison of the pinned- and fixed-end results shows that the general behavior is very similar for the two cases. Both can be described by an end section, a transition section, and a contact section. Both have the same helix pitch far from the rod end. The rod first contacts the wall farther from the rod end for the fixed-end case, and the transition section is shorter. The distance from the rod end to the beginning of the helical section is longer for the fixed-end case than for the pinned-end case. The magnitudes of the concentrated contact forces for the fixed-end case are approximately 60% of those for the pinned-end case. The maximum bending moment occurs in the end section for both cases, and the magnitude is approximately 18% higher for the pinned-end case. The maximum shear force and maximum wall contact force occur in the helical section and are the same for both cases.

In conclusion, the effect of pinned- and fixed-end conditions on the post-buckling behavior of a circular rod constrained within a circular cylinder have been investigated subject to certain idealizations. Closed-form analytical solutions are obtained for the deflected shape of the end portions of the rod. Results obtained have practical applications to the structural design of drill string and tubing assemblies. The rod is found to buckle into a helical configuration far from the end, with the helix pitch

given by Lubinski, Althouse and Logan [5]. The pinned-end condition is more severe than the fixed-end condition. The maximum bending moment increases by 20% due to end effects for the pinned-end condition, and increases by less than 1% for the fixed-end condition.

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TABLE 1 SELECTED PINNED-END RESULTS

Contact Section Half-Length,	ϕ_c	<u>2.0</u>	<u>4.0</u>	<u>6.0</u>
Distance to 1 st Contact Point,	ϕ_1	2.46863	2.46867	2.46867
Distance to 2 nd Contact Point,	ϕ_2	4.38232	4.38106	4.38105
Rod Half-Length	ϕ_s	6.38232	8.38106	10.38105
Angular Position at 1 st Contact Point,	θ_1	-1.18271	-1.18188	-1.18188
Rate of Change of Angular Position at 1 st Contact Point,	θ_1'	0.46652	0.46655	0.46655
	θ_2	0	0	0
Conditions on $\theta(\zeta)$ at 2 nd Contact Point,	θ_2'	0.68716	0.68721	0.68721
	θ_2''	0.02760	0.02774	0.02774
	θ_2'''	-0.03822	-0.03813	-0.03813
Rate of Change of Angular Position at Rod Half- Length Point,	$\theta'(\phi_s)$	0.70472	0.70697	0.70710
Location of Max. Bending Moment,	ζ at M_{max}	1.57080	1.57080	1.57080
Maximum Bending Moment,	M_{max}	0.59726	0.59727	0.59727

TABLE 2 SELECTED FIXED-END RESULTS

Contact Section Half-Length,	ϕ_c	<u>2.0</u>	<u>4.0</u>	<u>6.0</u>
Distance to 1 st Contact Point,	ϕ_1	3.81729	3.81729	3.81729
Distance to 2 nd Contact Point,	ϕ_2	5.24463	5.24274	5.24273
Rod Half-Length	ϕ_s	7.24463	9.24274	11.24273
Angular Position at 1 st Contact Point,	θ_1	-0.89902	-0.89780	-0.89779
Rate of Change of Angular Position at 1 st Contact Point,	θ'_1	0.53620	0.53623	0.53623
	θ_2	0	0	0
Conditions on $\theta(\zeta)$ at 2 nd Contact Point,	θ'_2	0.68148	0.68152	0.68152
	θ''_2	0.03532	0.03553	0.03553
	θ'''_2	-0.04850	-0.04843	-0.04843
Rate of Change of Angular Position at Rod Half- Length Point,	$\theta'(\phi_s)$	0.70403	0.70642	0.70710
Location of Max. Bending Moment,	ζ at $M_{\max}$	2.99232	2.99229	2.99229
Maximum Bending Moment,	$M_{\max}$	0.50489	0.50491	0.50491

**TABLE 3 COMPARISON BETWEEN CURRENT RESULTS
AND THOSE OF REFERENCE [10]: FIXED END CONDITION**

		Current Results	Reference [10] Results
Distance to 1 st Contact Point,	ϕ_1	3.82	4.49
Shear at Rod End,	$V(0)$	0.105	0.162
Bending Moment at Rod End,	$M(0)$	0.500	0.383
Rate of Change of Angular Position Far From Rod End,	$\theta'(\infty)$	0.707	0.423
Wall Contact Force Far From Rod End,	$w(\infty)$	0.250	0.147

Figure 1 Pinned-End Rod Geometry
And Nomenclature

a. Rod Centerline Dimensionless Deflection From Hole Centerline, $R = \hat{R}/r$ vs Dimensionless Distance From Rod End, $\zeta = \alpha z$

b. Rod Centerline Angular Deflection From x-Axis, θ (radians) vs Distance From Rod End, $\zeta = \alpha z$

Figure 2 Rod Displacement Results In R - θ Coordinates

a. Rod Centerline Dimensionless x-Deflection From Hole Centerline, $u_{i1} = \hat{u}_{i1}/r$, $v_1 = \hat{v}_1/r$ vs Dimensionless Distance From Rod End, $\zeta = \alpha z$

- contact points - pinned
- contact points - fixed

$$\alpha = (W/EI)^{1/2}$$

b. Rod Centerline Dimensionless y-Deflection From Hole Centerline, $u_{i2} = \hat{u}_{i2}/r$, $v_2 = \hat{v}_2/r$ vs Dimensionless Distance From Rod End, $\zeta = \alpha z$

Figure 3 Rod Displacement Results
x-y Coordinates

Figure 4 Local Helix Angle Results : Rate
 Of Change Of Angular Deflection,
 $\theta' = \gamma/\alpha r$ vs Distance From Rod
 End, $\zeta = \alpha z$

Figure 5 Bending Moment Results :
Dimensionless Bending Moment, $M = \hat{M}/Wr$
vs Dimensionless Distance From Rod
End, $\zeta = \alpha z$

a. Dimensionless Radial Bending Moment, $M_R = \hat{M}_R / Wr$ vs Dimensionless Distance From Rod End, $\zeta = \alpha z$

b. Dimensionless Tangential Bending Moment, $M_\Theta = \hat{M}_\Theta / Wr$ vs Dimensionless Distance From Rod End. $\zeta = \alpha z$

Figure 6 Radial And Tangential Bending Moment Components

Figure 7 Shear Force Results :
 Dimensionless Shear Force, $V = \hat{V} / \alpha W r$
 vs Dimensionless Distance From
 Rod End, $\zeta = \alpha z$

a. Dimensionless Radial Shear Force, $V_R = \hat{V}_R / \alpha W r$ vs Dimensionless Distance From Rod End, $\zeta = \alpha z$

b. Dimensionless Tangential Shear Force, $V_\Theta = \hat{V}_\Theta / \alpha W r$ vs Dimensionless Distance From Rod End, $\zeta = \alpha z$

Figure 8 Radial And Tangential Shear Force Components

Figure 9 Wall Contact Force Results:
 Dimensionless Contact Force, $w = \hat{w} / \alpha^2 W r$
 vs Dimensionless Distance From Rod
 End, $\zeta = \alpha z$

Figure 10 Dimensionless External Reactions
For Pinned-End Case

Figure 11 Dimensionless External Reactions For Fixed-End Case

Example #1 : Pinned-End Condition -
 Drill String Near Bit
 6-3/4"OD, 2-3/4"ID Steel Drill Collar
 8-3/4" Dia. Hole
 50,000 lb. Weight-On-Bit

Figure 12 Schematic Summary Of Results For
 Numerical Example #1 - Drill String
 At The Bit

Example #2 : Fixed-End Condition -
 Tubing Sealed At A Packer
 2-7/8", 6.5 lb/ft Steel Tubing
 7" Casing
 20,000 lb. Axial Load

Figure 13 Schematic Summary Of Results For
 Numerical Example #2 - Tubing
 At A Packer

APPENDIX I

Derivation of Governing Beam-Column Equations

The following derivation of the beam-column equations follows the work of Love [11] and Walker [12].

Figure I-1 shows a differential rod element subjected to internal and external forces and moments. Force and moment equilibrium require

$$\vec{N}' + \vec{q} = 0 \quad (I.1)$$

and

$$\vec{M}' + \vec{t} \times \vec{N} + \vec{m} = 0 \quad (I.2)$$

where $\vec{M}$ = internal bending moment

$\vec{N}$ = internal force

$\vec{q}$ = external distributed force

$\vec{m}$ = external distributed moment

$\vec{t} = \vec{r}'$ is a vector tangent to the rod's neutral axis

and a prime denotes differentiation with respect to s , the distance along the rod.

A local ξ, ζ, ρ rectangular coordinate system is introduced with ρ tangent to the rod centerline in order to express the equilibrium equations in component form. The local coordinate system orientation relative to the global coordinates is given by

$$\vec{\omega} = \kappa_{\xi} \hat{i} + \kappa_{\zeta} \hat{j} + \gamma \hat{k}$$

where $\vec{\omega}$ is the rate of rotation of the local ξ, ζ, ρ ($\hat{i}, \hat{j}, \hat{k}$) coordinate axis system as it moves along the rod, and where κ_{ξ} and κ_{ζ} represent the rod curvature in the ξ and ζ directions and γ is the rod twist per unit length.

The equilibrium equations (I.1) and (I.2) can be expressed in component form to obtain the results of Love [11].

$$\begin{aligned} N'_{\xi} - \gamma N_{\zeta} + \kappa_{\zeta} N_{\rho} + q_{\xi} &= 0 \\ N'_{\zeta} + \gamma N_{\xi} - \kappa_{\xi} N_{\rho} + q_{\zeta} &= 0 \\ N'_{\rho} + \kappa_{\zeta} N_{\xi} + \kappa_{\xi} N_{\zeta} + q_{\rho} &= 0 \end{aligned} \tag{I.3}$$

$$\begin{aligned} M'_{\xi} - \gamma M_{\zeta} + \kappa_{\zeta} M_{\rho} - N_{\zeta} + m_{\xi} &= 0 \\ M'_{\zeta} + \gamma M_{\xi} - \kappa_{\xi} M_{\rho} + N_{\xi} + m_{\zeta} &= 0 \\ M'_{\rho} - \kappa_{\zeta} M_{\xi} + \kappa_{\xi} M_{\zeta} + m_{\rho} &= 0 \end{aligned}$$

Following Love [11], the ordinary approximate theory, a generalization of Bernoulli-Eulerian theory, is used to relate the stress resultants to the beam curvature and twist. It is also assumed that the beam-column has equal principal stiffnesses. These assumptions result in the constitutive relations

$$\begin{aligned} M_{\xi} &= B \kappa_{\xi} \\ M_{\zeta} &= B \kappa_{\zeta} \\ M_{\rho} &= C \gamma \end{aligned} \tag{I.4}$$

where B and C are the principal bending and torsional stiffnesses.

The further assumption of small displacements gives the relations

$$\begin{aligned} \kappa_{\xi} &= -v'' \\ \kappa_{\zeta} &= u'' \\ \gamma &= \beta' \end{aligned} \quad (I.5)$$

where u and v are displacements in the x and y directions, β is the rod rotational position, and a prime now denotes d/dz .

Making use of the relation between the local and global coordinate systems for small displacements as given in Table I-1, it is possible to incorporate equations (I.4) and (I.5) into the equilibrium equations (I.3) to obtain

$$\begin{aligned} N'_x + q_x &= 0 \\ N'_y + q_y &= 0 \\ N'_z + q_z &= 0 \end{aligned} \quad (I.6)$$

$$\begin{aligned} Bv'''' + (B-C)\beta'u'' + N_y - \beta N_x - v'N_z - m_{\xi} &= 0 \\ Bu'''' - (B-C)\beta'v'' + N_x + \beta N_y - u'N_z + m_{\zeta} &= 0 \end{aligned} \quad (I.7)$$

$$C\beta'' + m_{\rho} = 0 \quad (I.8)$$

Equations (I.7) are now differentiated and equations (I.6) used to obtain

$$\begin{aligned}
Bv'''' + (B-C)\beta'u'' + (B-C)\beta'u'' - \beta'N_x - v''N_z - q_y + \beta q_x + v'q_z - m'_\xi &= 0 \\
Bu'''' - (B-C)\beta'v'' - (B-C)\beta'v'' + \beta'N_y - u''N_z - q_x - \beta q_y + u'q_z + m'_\zeta &= 0
\end{aligned} \tag{I.9}$$

Neglecting second order terms in the displacements gives

$$\begin{aligned}
Bv'''' - \beta'N_x - v''N_z - q_y + \beta q_x + v'q_z - m'_\xi &= 0 \\
Bu'''' + \beta'N_y - u''N_z - q_x - \beta q_y + u'q_z + m'_\zeta &= 0
\end{aligned} \tag{I.10}$$

Assuming $m_\rho = m_\xi = m_\zeta = 0$, equation (I.8) shows that C_β' is a constant equal to the applied torque. If the applied torque is also assumed zero, β can be chosen to be zero. If it is also assumed that $q_z = 0$, then the final result is

$$\begin{aligned}
N_x + q_x &= 0 \\
N_y + q_y &= 0
\end{aligned} \tag{I.11}$$

$$N_z = \text{constant} = W \tag{I.12}$$

$$\begin{aligned}
Bv'''' - Wv'' - q_y &= 0 \\
Bu'''' - Wu'' - q_x &= 0
\end{aligned} \tag{I.13}$$

Equations (I.13) are the governing differential equations which describe the deflection of a beam-column subject to the assumptions identified above. These equations are repeated in Section 2.1 as

$$EI \frac{d^4 \hat{v}_j}{dz^4} + W \frac{d^2 \hat{v}_j}{dz^2} - \hat{q}_j = 0 \tag{2.1}$$

where $EI = B =$ principal bending stiffness

$$\hat{v}_j = \begin{cases} u = \text{rod centerline displacement in the } x \\ \text{direction for } j = 1 \\ v = \text{rod centerline displacement in the } y \\ \text{direction for } j = 2 \end{cases}$$

and

$$\hat{q}_j = \begin{cases} \hat{q}_x = \text{distributed external force acting in the} \\ \text{x direction for } j = 1 \\ \hat{q}_y = \text{distributed external force acting in the} \\ \text{y direction for } j = 2 \end{cases}$$

The sign convention used for the internal bending moment and shear force is shown in Figure I-2. Equilibrium analysis of this 2-dimensional differential beam element show that

$$\frac{d\hat{v}}{dz}j + \hat{q}_j = 0 \quad (\text{I.14})$$

and

$$\frac{d\hat{M}}{dz}j + \hat{V}_j + w \frac{d\hat{v}}{dz}j = 0. \quad (\text{I.15})$$

Following equations (I.4) and I.5),

$$\hat{M}_j = EI \frac{d^2\hat{v}}{dz^2}j \quad (\text{I.16})$$

Equation (I.15) can be solved for V_j to obtain

$$\hat{V}_j = -EI \frac{d^3\hat{v}}{dz^3}j - w \frac{d\hat{v}}{dz}j \quad (\text{I.17})$$

assuming that EI is constant. Equations (I.16) and (I.17) are expressed in a non-dimensional format in Section 2.1 as equations (2.9) and (2.10). Note that equation (I.15) can be differentiated and combined with equation (I.14) to give equation (2.1) directly.

		Global Coordinates		
		x	y	z
Local Coordinates	ξ	1	β	$-u'$
	ζ	$-\beta$	1	$-v'$
	ρ	u'	v'	1

Table I-1   Local-Global Coordinate
System Transformation

Figure I-1 Differential 3-Dimensional Rod Element

Figure I-2 Sign Convention For Bending Moment And Shear Force

APPENDIX II

Derivation of Boundary Conditions at Rod Half-Length Point

Figure II-1 shows a finite length rod with deformation symmetric about the half-length point at $\zeta = \phi_s$. Since the rod behavior must be identical when viewed from either end, it is possible to describe it, as shown in Figure II-2, using either the (θ, ζ) or the $(\bar{\theta}, \bar{\zeta})$ coordinate systems. These two coordinate systems are related by

$$\begin{aligned}\bar{\theta} &= \theta_o - \theta \\ \bar{\zeta} &= 2\phi_s - \zeta\end{aligned}$$

and it follows that

$$\frac{d\bar{\theta}}{d\bar{\zeta}} = \frac{d\bar{\theta}}{d\theta} \cdot \frac{d\theta}{d\zeta} \cdot \frac{d\zeta}{d\bar{\zeta}} = (-1) \frac{d\theta}{d\zeta} (-1) = \frac{d\theta}{d\zeta}$$

$$\frac{d^2\bar{\theta}}{d\bar{\zeta}^2} = \frac{d}{d\bar{\zeta}} \left(\frac{d\bar{\theta}}{d\bar{\zeta}} \right) = \frac{d}{d\zeta} \left(\frac{d\bar{\theta}}{d\zeta} \right) \frac{d\zeta}{d\bar{\zeta}} = - \frac{d^2\theta}{d\zeta^2}$$

$$\frac{d^3\bar{\theta}}{d\bar{\zeta}^3} = \frac{d}{d\bar{\zeta}} \left(\frac{d^2\bar{\theta}}{d\bar{\zeta}^2} \right) = \frac{d}{d\zeta} \left(\frac{d^2\bar{\theta}}{d\zeta^2} \right) \frac{d\zeta}{d\bar{\zeta}} = \frac{d^3\theta}{d\zeta^3}$$

etc.

Since, because of symmetry, it must be true that

$$\frac{d\bar{\theta}}{d\bar{\zeta}} = \frac{d\theta}{d\zeta}, \quad \frac{d^2\bar{\theta}}{d\bar{\zeta}^2} = - \frac{d^2\theta}{d\zeta^2}, \quad \frac{d^3\bar{\theta}}{d\bar{\zeta}^3} = \frac{d^3\theta}{d\zeta^3}, \quad \text{etc.}$$

it is clear that

$$\frac{d^2\bar{\theta}}{d\bar{\zeta}^2} = - \frac{d^2\theta}{d\zeta^2} = 0.$$

Figure II-1 Finite-Length Rod Geometry

Figure II-2 Finite-Length Rod Mid-Point Boundary Condition

APPENDIX III

Rod Deformed Into Constant-Pitch Helix

Figure III-1 shows the helical deformation of the rod centerline for a rod constrained in a circular cylinder with a radial rod-cylinder clearance of r . The helix geometry is described parametrically as:

$$\begin{aligned}\hat{v}_1 &= r \cos \theta \\ \hat{v}_2 &= r \sin \theta \\ z &= \frac{p}{2\pi} \theta\end{aligned}\tag{III.1}$$

where p is the helix pitch, $(\hat{v}_1, \hat{v}_2)$ is the position of the rod centerline in the (x,y) coordinate system, and θ is the circumferential position as shown.

Introducing a non-dimensional notation, define

$$\begin{aligned}v_1 &= \hat{v}_1/r \\ v_2 &= \hat{v}_2/r \\ \alpha^2 &= W/EI \\ \zeta &= \alpha z \\ \lambda &= 2\pi/\alpha p.\end{aligned}$$

where

W = axial compressive force on the rod

EI = rod bending stiffness.

Differentiation of the expressions for θ , v_1 , and v_2 gives

$$\theta = \lambda \zeta$$

$$\theta' = \lambda$$

$$\theta'' = 0$$

$$\theta''' = 0$$

and

$$\begin{aligned} v_1 &= \cos \lambda \zeta \\ v_1' &= -\lambda \sin \lambda \zeta \\ v_1'' &= -\lambda^2 \cos \lambda \zeta \\ v_1''' &= \lambda^3 \sin \lambda \zeta \\ v_1'''' &= \lambda^4 \cos \lambda \zeta \end{aligned} \quad (\text{III.2})$$

and

$$\begin{aligned} v_2 &= \sin \lambda \zeta \\ v_2' &= \lambda \cos \lambda \zeta \\ v_2'' &= -\lambda^2 \sin \lambda \zeta \\ v_2''' &= -\lambda^3 \cos \lambda \zeta \\ v_2'''' &= \lambda^4 \sin \lambda \zeta \end{aligned} \quad (\text{III.3})$$

where a prime denotes a derivative with respect to ζ .

The equilibrium equations for the rod are, from Appendix I and Section 2,

$$\begin{aligned} v_1'''' + v_1'' + w \cos \lambda \zeta &= 0 \\ v_2'''' + v_2'' + w \sin \lambda \zeta &= 0 \end{aligned} \quad (\text{III.4})$$

where w is the non-dimensional distributed wall contact force acting radially inward.

These equations incorporate the standard slender beam assumptions and assume, in addition, that there is no torque, that the rod contact with the wall is frictionless, and that the rod is weightless.

Substitution of equations (III.2) and (III.3) into either of equations (III.4) gives the equilibrium requirement.

$$w = \lambda^2(1-\lambda^2) \quad (\text{III.5})$$

Reference [5] finds the relation between axial force and helix pitch for a helically buckled rod to be

$$W = \frac{8\pi^2 EI}{p^2}$$

which can be rewritten in non-dimensional form as

$$\lambda = \frac{1}{\sqrt{2}}$$

Similarly, reference [6] gives the non-dimensional result

$$\frac{1}{\sqrt{2}} \leq \lambda \leq 1.$$

Substitution of the above results into equation (III.5) gives

$$w(\lambda = \frac{1}{\sqrt{2}}) = 0.25$$

and

$$w(\lambda = 1) = 0.$$

These results are consistent with the interpretation in reference [6] that $\lambda = \frac{1}{\sqrt{2}}$ applies during loading, when the rod is pushed against the cylinder wall, and the $\lambda = 1$ applies during unloading when the rod is free to pull away from the wall.

The internal reactions are derived in Appendix I, and can be written in the non-dimensional format as

$$V_j = -(v_j^{'''} + v_j^{'}) \quad (\text{III.6})$$

$$M_j = v_j^{''} \quad (\text{III.7})$$

for $j = 1, 2$.

The shear force, V , and moment, M , magnitudes are

$$V = [v_1^2 + v_2^2]^{1/2} = 0.35355$$

and (III.8)

$$M = [M_1^2 + M_2^2]^{1/2} = 0.50000$$

everywhere along the helically buckled rod. Figure III-2 shows a developed helix and defines the helix angle γ in terms of the helix pitch, p , and the effective radius, r . For γ small,

$$\gamma = \frac{2\pi r}{p}$$

which can be written in terms of λ as

$$\gamma = \alpha \lambda r$$

This relation, along with the result $\theta' = \lambda$, allows the local helix angle to be related to θ' by

$$\gamma = \alpha r \theta'. \quad (\text{III.9})$$

It is also clear that the local helix pitch is related to θ' by

$$p = \frac{2\pi}{\alpha \theta'}, \quad (\text{III.10})$$

At $\zeta = 0$, the following conditions are present:

$$\begin{aligned} v_1(0) &= 1 \\ v_1'(0) &= 0 \\ v_1''(0) &= -\lambda^2 \\ v_1'''(0) &= 0 \end{aligned} \quad (\text{III.11})$$

$$\begin{aligned} M_1(0) &= -\lambda^2 \\ V_1(0) &= 0 \\ v_2(0) &= 0 \\ v_2'(0) &= \lambda \\ v_2''(0) &= 0 \\ v_2'''(0) &= -\lambda^3 \end{aligned} \quad (\text{III.12})$$

$$\begin{aligned} M_2(0) &= 0 \\ V_2(0) &= \lambda^2(1-\lambda) \end{aligned}$$

Figure III-1 Helically Buckled Rod

Figure III-2 Helix Angle Definition

APPENDIX IV

Pinned-End Analysis with End Section Connected Directly to Helical Section

Figure IV-1 shows a helically buckled rod with the pinned rod-end located on the cylinder axis. The rod-end can rotate about any of the three axes but cannot move radially. The rod contacts wall at $z = L$ and remains in contact for $z \geq L$ with the rod centerline deformed into a constant pitch helix. Rod contact with the circular cylinder wall is frictionless.

The equilibrium requirements and end conditions for the helical portion of the rod are presented in Appendix III. Since there is no distributed force acting on the end section, the equilibrium equations (2.1) reduce to

$$u_i'''' + u_i'' = 0 \quad i = 1, 2 \quad (\text{IV.1})$$

where $\hat{u}_1$ = rod centerline displacement in the x direction

$\hat{u}_2$ = rod centerline displacement in the y direction

r = radial clearance between the rod and cylinder

$$u_1 = \hat{u}_1/r$$

$$u_2 = \hat{u}_2/r$$

$$\zeta = \alpha z$$

$$\alpha = [W/EI]^{1/2}$$

W = axial force on rod

EI = rod bending stiffness

and a prime denotes differentiation with respect to ζ .

The solution of equation (IV.1) is

$$u_i = a_i + b_i \zeta + c_i \cos \zeta + d_i \sin \zeta. \quad (\text{IV.2})$$

Eight of the applicable boundary conditions are:

$$\begin{aligned} u_i(0) &= 0 & i &= 1, 2 \\ u_i'(0) &= 0 & i &= 1, 2 \\ u_1(\phi) &= 1 \\ u_2(\phi) &= 0 \\ u_1'(\phi) &= 0 \\ u_2'(\phi) &= 1 \end{aligned} \quad (\text{IV.3})$$

where $\phi = \alpha L$.

Application of these 8 boundary conditions gives

$$\begin{aligned} a_1 &= 0 \\ b_1 &= \frac{-\cos \phi}{\sin \phi - \phi \cos \phi} \\ c_1 &= 0 \\ d_1 &= \frac{1}{\sin \phi - \phi \cos \phi} \\ a_2 &= 0 \\ b_2 &= \frac{\lambda \sin \phi}{\sin \phi - \phi \cos \phi} \end{aligned} \quad (\text{IV.4})$$

$$c_2 = 0$$

$$d_2 = \frac{-\lambda \phi}{\sin \phi - \phi \cos \phi}$$

The two unknowns λ and ϕ remain to be evaluated, requiring two additional boundary conditions.

where p = helix pitch in the helical section and

$$\lambda = 2\pi/\alpha p,$$

There are four boundary conditions which could be applied. These are:

$$u_1''(\phi) = -\lambda^2 \quad (\text{IV.5})$$

$$u_2''(\phi) = 0 \quad (\text{IV.6})$$

$$u_1'''(\phi) = 0 \quad (\text{IV.7})$$

$$u_2'''(\phi) = -\lambda^3 \quad (\text{IV.8})$$

These require continuity of moment and shear in the x - z and y - z planes between the end and helical sections. Since a radial concentrated contact force can reasonably exist, boundary condition (IV.7) is not necessary. This leaves three boundary conditions which should be satisfied, and only two remaining unknowns to be determined. The effects of this situation are examined by considering various pairs of the remaining boundary conditions.

First consider

$$u_1''(\phi) = -\lambda^2$$

$$u_2''(\phi) = 0$$

so that moment continuity is satisfied in both planes. This is equivalent to allowing radial and tangential concentrated forces at the point of initial wall contact. Application of these additional boundary conditions gives

$$\lambda = 0$$

$$\sin \phi = 0$$

The result $\lambda = 0$ corresponds to an infinite helix pitch with the result that a helix does not form and the rod remains in the x - z plane everywhere, contacting the wall at $\zeta = \pi$. This result is identical to the two dimensional analogy analyzed in Appendix XII, and represents a degenerate solution.

Next consider

$$u_2''(\phi) = 0$$

$$u_2'''(\phi) = -\lambda^3$$

so that continuity of moment and shear in the y - z plane is satisfied. This is equivalent to allowing a concentrated external moment and external force acting in the x - z plane. Application of these additional boundary conditions gives the results

$$\lambda = 1$$

and

$$\sin \phi = 0$$

Finally, consider

$$u_1''(\phi) = -\lambda^2$$

$$u_2'''(\phi) = -\lambda^3$$

so that continuity of moment in the x - z plane and continuity of shear in the y - z plane is satisfied. This condition allows a concentrated moment in the y - z plane and a concentrated radial force at the contact point. The results of this set of boundary conditions are

$$\lambda = 1/\sqrt{2}$$

$$\phi = 2.0288$$

This value of λ is the same as that predicted for the helix in reference [5] (see Appendix III).

Of the three possible sets of boundary conditions considered, the last set is the closest to being reasonable. What this last result suggests is that the end section is not connected directly to a helix at the contact point as assumed here, but instead is connected to a perturbed helix. This perturbed helix does not have a constant pitch, but instead has a locally longer pitch at the contact point, and

also has some curvature in tangential plane at the contact point.

It is possible to demonstrate that the end section cannot be connected directly to a helical section by constructing a free body diagram in the y - z plane as shown in Figure IV-2. The end reactions on the right end of the end section are the helix internal reactions derived in Appendix III. Since a shear force is present on the right end and no distributed forces act, an equal and opposite force, shown as $V(0)$, must be present on the left end. That force will cause a non-zero moment at the right end, which is incompatible with the helix requirement that $M(L) = 0$. Therefore it is not possible for the end section to be connected directly to a helical section.

Figure IV-1 Pinned End Section Connected Directly To Helical Section

Figure IV-2 End Section Reactions

APPENDIX V

Pinned-End Analysis with End Section Connected to Perturbed Helical Section

A pinned-end rod is buckled helically within a cylinder as in Figure IV-1. Nomenclature is the same as in Section 2 except that ϕ is used in place of ϕ_1 . The rod end is located on the cylinder axis, and is free to rotate about all three axes. The rod contacts the wall at $\zeta = \phi$ and remains on the cylinder wall for $\zeta \geq \phi$. The pitch of the helical section is not restricted to being constant, but can instead be a function of ζ .

The equilibrium equations for the contact section ($\zeta > \phi$) are the same in Section 2; equations (2.7) and (2.8). Similarly, the end section equilibrium requires that equation (2.13) be satisfied for $i = 1$ and $j = 1, 2$, with the result that equation (2.14) describes the end section rod geometry.

The boundary conditions for this problem are

$$\left. \begin{aligned}
 u_{1j} (0) &= 0 \\
 u_{1j}' (0) &= 0 \\
 u_{11} (\phi) &= 1 \\
 u_{12} (\phi) &= 0 \\
 u_{11}' (\phi) &= 0 \\
 u_{12}' (\phi) &= \theta_1' \\
 u_{11}'' (\phi) &= -(\theta_1')^2 \\
 u_{12}'' (\phi) &= \theta_1'' \\
 u_{2''}'' (\phi) &= \theta_1''' - (\theta_1')^3
 \end{aligned} \right\} \quad (V.1)$$

and

$$\theta'' (\phi_s) = 0 \quad (V.2)$$

which provide 12 equations to solve for the 12 unknowns:

$\phi, \theta_1', \theta_1'', \theta_1''', a_{ij}, b_{ij}, c_{ij}, d_{ij},$ for $i = 1, 2$.

The solution procedure utilizes a shooting technique similar to that described in Section 2. A value for ϕ is assumed and equations (V.1) are solved analytically for the remaining 11 parameters. The parameter ϕ is then adjusted to satisfy equation (V.2), numerically integrating equation (2.7) using a Runge-Kutta algorithm.

Analytical solution for the eleven unknowns; in terms of ϕ gives:

$$\theta_1' = \left[\frac{\sin \phi}{\sin \phi - \phi \cos \phi} \right]^{1/2}$$

$$\theta_1'' = \phi \left[\frac{\sin \phi}{\sin \phi - \phi \cos \phi} \right]^{3/2}$$

$$\theta_1''' = \left[\frac{\sin \phi}{\sin \phi - \phi \cos \phi} \right]^{3/2} \left(1 + \frac{\phi}{\tan \phi} \right)$$

$$a_{11} = 0$$

$$b_{11} = \frac{-\cos \phi}{\sin \phi - \phi \cos \phi}$$

$$c_{11} = 0$$

$$d_{11} = \frac{1}{\sin \phi - \phi \cos \phi}$$

$$a_{12} = 0$$

$$b_{12} = \left[\frac{\sin \phi}{\sin \phi - \phi \cos \phi} \right]^{3/2}$$

$$c_{12} = 0$$

$$d_{12} = - \frac{\phi}{\sin \phi - \phi \cos \phi} \left[\frac{\sin \phi}{\sin \phi - \phi \cos \phi} \right]^{1/2}$$

The value of ϕ found to satisfy (V.2) for $\phi_s = 6$ is

$$\phi = 2.50531.$$

For this value of ϕ , the remaining parameters are:

$$\begin{aligned}
 \theta_1' &= 0.47721 & b_{11} &= 0.30825 & b_{12} &= 0.10868 \\
 \theta_1'' &= 0.27227 & d_{11} &= 0.38325 & d_{12} &= -0.45820 \\
 \theta_1''' &= -0.25986
 \end{aligned}$$

The results of this analysis are shown in Figures V-1 and V-2. These results show that the perturbation of the helical section decays with increasing ζ and that the helix has the pitch predicted in reference [5]. The results also show a negative wall contact force for $2.5053 \leq \zeta \leq 3.5863$. A negative wall contact force requires the cylinder wall to pull on the rod. This is clearly not realistic and suggests the presence of an intermediate transition section between the end and contact sections.

a. Rod Centerline Dimensionless Deflection From Hole Centerline, $R = \hat{R}/r$ vs Dimensionless Distance From Rod End, $\zeta = \alpha z$

b. Rod Centerline Angular Deflection From x-Axis, θ (radians) vs Distance From Rod End, $\zeta = \alpha z$

Figure V-1 Rod Displacement Results

Figure V-2 Local Helix Angle Results : Rate Of Change Of Angular Deflection, $\theta' = \gamma/\alpha r$ vs Distance From Rod End, $\zeta = \alpha z$

Figure V-3 Wall Contact Force Results : Dimensionless Contact Force, $w = \hat{w}/\alpha^2 W r$ vs Dimensionless Distance From Rod End, $\zeta = \alpha z$

APPENDIX VI

Fixed-End Analysis with End Section Connected Directly to Helical Section

A fixed-end rod is buckled helically as shown in Figure VI-1. Nomenclature is the same as in Appendix IV. The fixed end condition requires that the rod-end be tangent to the cylinder axis. The end section does not contact the cylinder wall and is connected to a helical section with constant pitch at the point of wall contact.

The equilibrium requirements and end conditions for the helical portion of the rod are presented in Appendix III. Since there is no distributed force acting on the end section, the equilibrium equations reduce to:

$$u_i'''' + u_i'' = 0 \quad i = 1, 2 \quad (\text{VI.1})$$

where (u_1, u_2) represents the (x, y) position of the rod. The solution of equation (VI.1) is:

$$u_i = a_i + b_i \zeta + c_i \cos \zeta + d_i \sin \zeta \quad (\text{VI.2})$$

The boundary conditions used here are the same as in reference [10]:

$$\begin{aligned} u_i(0) &= 0 \\ u_i'(0) &= 0 \end{aligned} \quad (\text{VI.3})$$

$$u_1(\phi) = 1 \quad (VI.4)$$

$$u_2(\phi) = 0$$

$$u_1'(\phi) = 0 \quad (VI.5)$$

$$u_2'(\phi) = \lambda$$

$$u_1''(\phi) = -\lambda^2 \quad (VI.6)$$

$$u_2''(\phi) = 0$$

These boundary conditions require that the rod end be on, and tangent to, the cylinder axis, and that there be continuity of position, slope and curvature at the connection of the end section to the helically buckled portion of the rod. The set of 10 boundary conditions given by equations (VI.3) through (VI.6) is used to solve for the 10 unknowns. λ , ϕ , a_i , b_i , c_i , d_i for $i = 1, 2$.

Substitution of equation (VI.2) into equation (VI.3) through (VI.6) gives

$$\lambda^2 = \frac{1 - \cos \phi}{(\cos \phi - 1)^2 - \sin \phi(\phi - \sin \phi)} \quad (VI.7)$$

$$\tan \phi = \phi \quad (VI.8)$$

$$a_1 = \frac{1 - \cos \phi}{(\cos \phi - 1)^2 - \sin \phi(\phi - \sin \phi)}$$

$$b_1 = \frac{-\sin \phi}{(\cos \phi - 1)^2 - \sin \phi(\phi - \sin \phi)} \quad (VI.9)$$

$$c_1 = -a_1$$

$$d_1 = -b_1$$

$$a_2 = \frac{-\lambda(\phi - \sin \phi)}{(\cos \phi - 1)^2 - \sin \phi(\phi - \sin \phi)}$$

$$b_2 = \frac{-\lambda(\cos \phi - 1)}{(\cos \phi - 1)^2 - \sin \phi(\phi - \sin \phi)} \quad (\text{VI.10})$$

$$c_2 = -a_2$$

$$d_c = -b_2$$

Equations (VI.7) and (VI.8) correspond to equations (53) and (52) respectively in [10]. Solution of equation (VI.8) gives $\phi = 4.4934$ and equation (VI.7) then gives $\lambda = .4225$.

Evaluation of expressions (III.5) and (III.6) for the rod end section internal reactions show that

$$V_1(0) = V_1(\phi) = 0.1431 \quad (\text{VI.11})$$

$$V_2(0) = V_2(\phi) = 0.0754$$

$$M_1(0) = 0.1785 \quad (\text{VI.12})$$

$$M_2(0) = -0.3388$$

$$M_1(\phi) = -0.1785 \quad (\text{VI.13})$$

$$M_2(\phi) = 0$$

The magnitude of the shear and moment reactions at the rod end are computed from equation (VI.11) and (VI.12) as

and

$$V = .1618$$

$$M = .3829$$

which agree with equations (56) and (57) of [10].

Evaluation of the helical section end reaction (Appendix III) gives

$$\begin{aligned} V_1(\phi) &= 0 \\ V_2(\phi) &= -.3471 \end{aligned} \tag{VI.14}$$

$$\begin{aligned} M_1(\phi) &= -.1785 \\ M_2(\phi) &= 0 \end{aligned} \tag{VI.15}$$

Comparison of equations (VI.11) and (VI.14) shows that a concentrated contact force occurs at the point of wall contact where the end and helical sections connect. This concentrated force is associated with the discontinuous shear force at this point and has a radial component, F_1 , acting inward and a tangential component, F_2 , acting in the positive y direction:

$$\begin{aligned} F_1 &= 0.1431 \\ F_2 &= 0.2717 \end{aligned} \tag{VI.16}$$

Note that the tangential component of this contact force has a magnitude of 1.9 times the radial component. This would require an unrealistically large coefficient of friction at the point of wall contact, and also contradicts

the initial assumption that rod contact with the cylinder wall is frictionless.

Note also that equation (VI.7) predicts a helix pitch corresponding to $\lambda = .4225$, even at a distance infinitely far from the rod end. This is significantly different from the result ($\lambda = .7071$) calculated in reference [5] and is clearly not a physically realistic result in that any end effects would be expected to decay with increased distance from the rod end.

Figure VI-1 Fixed End Section Connected Directly To Helical Section

APPENDIX VII

Fixed-End Analysis with End Section Connected to Perturbed Helical Section

A fixed-end rod is buckled helically as shown in Figure VI-1. Nomenclature is the same as in Appendix V. The fixed-end condition requires that the rod end be tangent to the cylinder axis. The rod contacts the wall at $\zeta = \phi$ and remains in contact for $\zeta > \phi$. The local pitch of the contact section is not required to be constant and can vary with ζ .

The equilibrium equations for the contact section ($\zeta > \phi$) are the same as Section 2.1, with the resulting requirement that equations (2.7) and (2.8) be satisfied. Similarly, end section equilibrium requires that equation (2.13) be satisfied for $i = 1$ and $j = 1, 2$. Equation (2.14) describes the end section rod geometry.

The boundary conditions for this problem are

$$\begin{aligned} u_{1j}(0) &= 0 \\ u'_{1j}(0) &= 0 \\ u_{11}(\phi) &= 1 \\ u_{12}(\phi) &= 0 \\ u'_{11}(\phi) &= 0 \\ u'_{12}(\phi) &= \theta'_1 \end{aligned} \tag{VII.1}$$

$$u_{11}'(\phi) = -(\theta_1')^2$$

$$u_{12}'(\phi) = \theta_1''$$

and

$$\begin{aligned} u_{12}'''(\phi) &= \theta_1''' - (\theta_1')^3 \\ \theta''(\phi_s) &= 0 \end{aligned} \quad (\text{VII.2})$$

These boundary conditions provide 12 equations to solve for the 12 unknowns ϕ , θ_1' , θ_1'' , θ_1''' , a_{ij} , b_{ij} , c_{ij} , d_{ij} , $j = 1, 2$.

The solution procedure is identical to that described in Appendix V. Algebraic solution for all the unknowns except ϕ in terms of ϕ is performed using equations (VII.1) to obtain:

$$\theta_1' = \left[\frac{1 - \cos \phi}{2 - 2 \cos \phi - \phi \sin \phi} \right]^{1/2}$$

$$\theta_1'' = \frac{\theta_1'(\sin \phi - \phi \cos \phi)}{2 - 2 \cos \phi - \phi \sin \phi}$$

$$\theta_1''' = \frac{\theta_1'[\phi \sin \phi - (1 - \cos \phi)]}{2 - 2 \cos \phi - \phi \sin \phi} + (\theta_1')^3$$

$$a_{11} = \frac{1 - \cos \phi}{2 - 2 \cos \phi - \phi \sin \phi}$$

$$b_{11} = \frac{-\sin \phi}{2 - 2 \cos \phi - \phi \sin \phi}$$

$$c_{11} = -a_{11}$$

$$d_{11} = -b_{11}$$

$$a_{12} = \frac{-\theta_1'(\phi - \sin \phi)}{2 - 2 \cos \phi - \phi \sin \phi}$$

$$b_{12} = \frac{\theta_1'(1 - \cos \phi)}{2 - 2 \cos \phi - \phi \sin \phi}$$

$$c_{12} = -a_{12}$$

$$d_{12} = -b_{12}$$

The value of ϕ found to satisfy (VII.2) for $\phi_c = 4$ is

$$\phi = 3.84335$$

For this value of ϕ , the remaining parameters take the values

$$\begin{array}{lll} \theta_1 & = & 0.54179 \quad a_{11} = 0.29353 \quad a_{12} = -0.40477 \\ \theta_1' & = & 0.20646 \quad b_{11} = 0.10744 \quad b_{12} = 0.15903 \\ \theta_1'' & = & -0.22372 \quad c_{11} = -0.29353 \quad c_{12} = 0.40477 \\ & & d_{11} = -0.10744 \quad d_{12} = -0.15903 \end{array}$$

The results of this analysis, shown in Figures VII-1 through VII-3 show that the perturbation of the helical section decays with increasing ζ . A negative wall contact force is present for $3.843 \leq \zeta < 4.654$. This is not realistic physically and indicates that the rod does not remain in contact with the wall after first contact as assumed.

a. Rod Centerline Dimensionless Deflection From Hole Centerline, $R = \hat{R}/r$ vs Dimensionless Distance From Rod End, $\zeta = \alpha z$

b. Rod Centerline Angular Deflection From x-Axis, θ (radians) vs Distance From Rod End, $\zeta = \alpha z$

Figure VII-1 Rod Displacement Results

Figure VII-2 Local Helix Angle Results : Rate Of Change Of Angular Deflection, $\theta' = \gamma / \alpha r$ vs Distance From Rod End, $\zeta = \alpha z$

Figure VII-3 Wall Contact Force Results : Dimensionless Contact Force, $w = \hat{w} / \alpha^2 W r$ vs Dimensionless Distance From Rod End, $\zeta = \alpha z$

APPENDIX VIII

Pinned-End Analysis Computer Listing

The FORTRAN computer program which follows applies the solution procedure described in Section 2.5 to the pinned-end problem. A Modified Quasilinearization Algorithm [16] is used to solve the non-linear system of algebraic equations (2.37). A Runge-Kutta algorithm [17] is used to numerically integrate the governing differential equation (2.7) for the contact section. An Interval-Halving procedure is used to solve for ϕ_1 in order to satisfy boundary condition (2.32).

Equations (2.37a) through (2.37n) are expressed as F(1) through F(14) respectively.

Variable definitions within the computer program are:

B11	= b_{11}	X(1)	= ϕ_2	X(8)	= b_{21}
D11	= d_{11}	X(2)	= θ_2'	X(9)	= c_{21}
B12	= b_{12}	X(3)	= θ_2''	X(10)	= d_{21}
D12	= d_{12}	X(4)	= θ_2'''	X(11)	= a_{22}
PHI	= ϕ_1	X(5)	= θ_1	X(12)	= b_{22}
PHIC	= ϕ_c	X(6)	= θ_1'	X(13)	= c_{22}
		X(7)	= a_{21}	X(14)	= d_{22}

In the contact section portion of the program,

Y(1) = θ
Y(2) = θ'
Y(3) = θ''
Y(4) = θ'''

Output variables are defined in Appendix X with the computer program results.

```

1:*
2: C$JOB
3: C      PROGRAM HELBTPIN.FORT VERSION 3/10/84
4: C
5: C      THIS PROGRAM USES AN MQA PROCEDURE TO SOLVE THE NON-LINEAR
6: C      SYSTEM OF SIMULTANEOUS EQUATIONS OBTAINED FROM THE B.C.'S
7: C      OF THE END AND TRANSITION SECTIONS TO OBTAIN THE END
8: C      CONDITIONS OF THE CONTACT SECTION. A RUNGE-KUTTA ALGORITHM
9: C      INTEGRATES THE NON-LINEAR D.E. OF THE CONTACT SECTION.
10: C
11: C
12: C      IMPLICIT REAL*8(A-H,O-Z)
13: C      DIMENSION X(15),WKAREA(300),XF(15,1),XT(15),F(15),DF(15,15),A(15)
14: C      DIMENSION Y(4),Y1(4),ZP(500),Y1P(500),Y2P(500),Y3P(500),Y4P(500)
15: C      DIMENSION WP(500),FT(15),FCHK(15)
16: C      DIMENSION XU(100),Z(100),U11(100),U12(100),R(100),THETA(100)
17: C      DIMENSION U11P(100),U12P(100),U11PP(100),U12PP(100)
18: C      DIMENSION U11PPP(100),U12PPP(100),V11(100),V12(100)
19: C      DIMENSION V(100),V1R(100),V1T(100),XM(100),XMR(100),XMT(100)
20: C      DIMENSION RP(100),TP(100)
21: C      DIMENSION XTR(100),U21(100),U22(100),U21P(100),U22P(100)
22: C      DIMENSION U21PP(100),U22PP(100),U21PPP(100),U22PPP(100)
23: C      DIMENSION V2R(100),V2T(100),V21(100),V22(100)
24: C      DIMENSION AZ(500),U1(500),U2(500),CMR(500),CM(500)
25: C      DIMENSION CVR(500),CVT(500),CV(500)
26: C      COMMON PHI,CPHI,SPHI,M
27: C
28: C.....DEFINE INPUT PARAMETERS
29: C
30: C      M=NUMBER OF SIMULTANEOUS EQUATIONS FOR MQA
31: C      NIT=NUMBER OF MQA ITERATIONS ALLOWED
32: C      NBIS=NUMBER OF MQA ALPHA BISECTIONS ALLOWED
33: C      EPS=MQA CONVERGENCE CRITERIA ON P
34: C      ALPHA1=MQA INITIAL VALUE FOR ALPHA
35: C      IPRINT=1 FOR INTERMEDIATE MQA PRINTOUT
36: C      ID6=SOLUTION ACCURACY FOR IMSL SUBROUTINE LEQT2F
37: C      IPRNT1=1 FOR PRINTOUT OF RESULTS ON EACH PHI BISECTION
38: C      NPT=NUMBER OF PRINTOUT POINTS FOR END AND TRANSITION SECTIONS
39: C      PHIL,PHIR=LEFT AND RIGHT LIMITS FOR PHI SOLUTION SEARCH
40: C      X(I)=INITIAL GUESS FOR MQA SOLUTION VECTOR
41: C      PHIC=CONTACT SECTION LENGTH
42: C      NINC=NUMBER OF CONTACT SECTION INTEGRATION STEPS
43: C      IPRT=CONTACT SECTION RESULT PRINT INTERVAL
44: C      EPSPHI=CONVERGENCE CRITERIA FOR PHI HALF-INTERVAL SEARCH
45: C      NPHI=MAX NUMBER OF PHI HALF-INTERVAL SEARCH ITERATIONS
46: C
47: C      READ (5,103) M,NIT,NBIS,EPS,ALPHA1
48: C      103 FORMAT (3I10,2E10.5)
49: C      READ (5,104) IPRINT,ID6,IPRNT1,NPT
50: C      104 FORMAT (4I10)
51: C      READ (5,100) PHIL,PHIR

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52: 100 FORMAT (2G10.5)
53:   READ (5,102) (X(I),I=1,M)
54: 102 FORMAT (5G10.5)
55:   READ(5,110) PHIC,NINC,IPRT
56: 110 FORMAT (610.4,2I10)
57:   READ (5,110) EPSPHI,NPHI
58:   PI=3.141592673
59: C
60: C.....ECHO PRINT INPUT
61:   WRITE (6,200)
62: 200 FORMAT (/15X,' HELBTPIN.FORT')
63:   WRITE (6,201) PHIL,PHIR
64: 201 FORMAT (/15X,' PHIL=',G15.9,5X,' PHIR=',G15.9)
65:   WRITE (6,230) NIT,NBIS,EPS,ALPHA1
66: 230 FORMAT (/15X,' NIT=',I3,2X,' NBIS=',I3,2X,' EPS=',G10.3,2X,
67:   *      ' ALPHA1=',G10.3)
68:   WRITE (6,231) M,IPRINT,IDG,IPRNT1,NPT
69: 231 FORMAT (/15X,' M= ',I3,2X,' IPRINT=',I3,2X,' IDG=',I3,2X,
70:   *      ' IPRNT1=',I3,2X,' NPT=',I3)
71:   WRITE (6,206)
72: 206 FORMAT (/15X,' INITIAL X VECTOR')
73:   WRITE (6,208) (X(I),I=1,M)
74: 208 FORMAT (15X,4G12.5)
75:   WRITE (6,252) PHIC,NINC,IPRT
76: 252 FORMAT (/15X,' PHIC=',G12.5,5X,' NINC=',I3,5X,' IPRT=',I3)
77:   WRITE (6,270) NPHI,EPSPHI
78: 270 FORMAT (/15X,' NPHI=',I3,5X,' EPSPHI=',G14.5)
79:   ISTOP=0
80: 87 CONTINUE
81:   DO 78 IPHI=1,NPHI
82:   PHI=.5*(PHIL+PHIR)
83:   CPHI=DCOS(PHI)
84:   SPHI=DSIN(PHI)
85:   WRITE (6,284) IPHI
86: 284 FORMAT (/ ' IPHI=',I3)
87:   WRITE (6,280) PHI,PHIL,PHIR
88: 280 FORMAT (' PHI=',G15.9,5X,' PHIL=',G15.9,3X,
89:   *      ' PHIR=',G15.9)
90: C
91: C.....BEGIN MGA PROCEDURE
92:   IB=0
93:   ALPHAB=0.
94: C
95: C.....COMPUTE F(I) AND ERROR P AT PRESENT X
96:   CALL FUNC(X,F)
97:   P=0.
98:   DO 12 I=1,M
99:   12 P=P+F(I)*F(I)
100: C
101: C.....ITERATE OVER X VECTOR
102:   DO 80 IT=1,NIT
103: C

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104: C.....PRINT RESULTS FROM LAST ITERATION
105:     ITM1=IT-1
106:     WRITE (6,234) ITM1,IB,ALPHAB,IER,IDGT,P
107: 234 FORMAT (' *IT=',I2,2X,'IB=',I2,2X,'ALPHAB=',F8.5,
108:     *      2X,'IER=',I3,2X,'IDGT=',I3,2X,'P=',G12.5)
109:     IF (IPRINT.NE.1) GO TO 17
110:     WRITE (6,236)
111: 236 FORMAT (' X VECTOR')
112:     WRITE (6,238) (X(I),I=1,M)
113: 238 FORMAT (5G13.5)
114:     WRITE (6,240)
115: 240 FORMAT (' F VECTOR')
116:     WRITE (6,238) (F(I),I=1,M)
117: 17 CONTINUE
118: C
119: C.....CHECK FOR CONVERGENCE
120:     IF (P.LE.EPS) GO TO 90
121: C
122: C.....COMPUTE DERIVATIVES OF F
123:     CALL DFUNC(X,DF)
124:     IF (IPRINT.NE.1) GO TO 25
125:     WRITE (6,258)
126: 258 FORMAT (' DF MATRIX')
127:     DO 27 I=1,M
128:     WRITE (6,260) (DF(I,J),J=1,M)
129: 260 FORMAT (5G14.5)
130: 27 CONTINUE
131: 25 CONTINUE
132: C
133: C.....SOLVE SYSTEM OF SIMULTANEOUS EQUATIONS FOR A VECTOR
134:     DO 16 I=1,M
135: 16 XF(I,1)=-F(I)
136:     IDGT=IDG
137:     CALL LEQT2F(DF,1,M,15,XF,IDGT,WKAREA,IER)
138:     DO 18 I=1,M
139: 18 A(I)=XF(I,1)
140:     IF (IPRINT.NE.1) GO TO 19
141:     DO 14 I=1,M
142:     FCHK(I)=0.
143:     DO 14 J=1,M
144: 14 FCHK(I)=FCHK(I)+DF(I,J)*A(J)
145:     PA=0.
146:     DO 15 I=1,M
147: 15 PA=PA+(FCHK(I)+F(I))**2
148: C
149: C.....PRINT A VECTOR
150:     WRITE (6,242) IER,IDGT
151: 242 FORMAT (' A VECTOR      LEQT2F IER=',I5,5X,'IDGT=',I5)
152:     WRITE (6,238) (A(I),I=1,M)
153:     WRITE (6,256) PA
154: 256 FORMAT (' ACHECK=',G14.5)
155: 19 CONTINUE

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156: C
157: C.....ITERATE OVER ALPHA -- BISECTION PROCESS
158:     ALPHAB=ALPHA1
159:     DO 22 IB=1,NBIS
160: C
161: C.....CALCULATE TEMPORARY NEW X VECTOR AND COMPUTE F(X)
162:     DO 24 I=1,M
163:     24 XT(I)=X(I)+ALPHAB*A(I)
164:     CALL FUNC(XT,FT)
165: C
166: C.....COMPUTE ERROR AND BISECT ALPHA IF NECESSARY
167:     PT=0.
168:     DO 26 I=1,M
169:     26 PT=PT+FT(I)*FT(I)
170:     IF (PT.LT.P) GO TO 70
171: C
172: C.....PRINT INTERIM RESULTS AND CONTINUE ALPHA ITERATION
173:     IF (IPRINT.NE.1) GO TO 21
174:     WRITE (6,235) IT,IB,ALPHAB,PT
175:     235 FORMAT (/// IT=',15,5X','IB=',15,5X','ALPHAB=',614.5,
176:     *          4X,'PT=',614.5)
177:     WRITE (6,236)
178:     WRITE (6,238) (XT(I),I=1,M)
179:     WRITE (6,240)
180:     WRITE (6,238) (FT(I),I=1,M)
181:     21 CONTINUE
182:     ALPHAB=ALPHAB/2.
183:     22 CONTINUE
184:     WRITE (6,244)
185:     244 FORMAT (/' MQA MAX NUMBER OF ALPHA ITERATIONS EXCEEDED')
186:     GO TO 95
187:     70 CONTINUE
188: C
189: C.....ASSIGN NEW X VECTOR
190:     DO 72 I=1,M
191:     F(I)=FT(I)
192:     72 X(I)=XT(I)
193:     P=PT
194:     80 CONTINUE
195:     WRITE (6,246)
196:     246 FORMAT (/' MQA MAX NUMBER OF IT ITERATIONS EXCEEDED')
197:     GO TO 95
198: C
199: C.....PROGRAM HAS CONVERGED -- PRINT RESULTS
200:     90 CONTINUE
201:     WRITE (6,248)
202:     248 FORMAT (' MQA SOLUTION HAS CONVERGED')
203:     95 CONTINUE
204: C
205: C.....OUTPUT OF MQA
206:     IF (IPRNT1.NE.1) GO TO 76
207:     WRITE (6,300)

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208: 300 FORMAT (///15X,'*** HELBTPIN.FORT ***')
209:   WRITE (6,202)
210: 202 FORMAT (/15X,'*** MQA RESULTS ***')
211:   DO 20 I=1,M
212:     WRITE (6,204) I,X(I),I,F(I)
213: 204 FORMAT (19X,'X(',I2,')= ',G16.9,3X,'F(',I2,')= ',G11.4)
214:   20 CONTINUE
215:     P=0.
216:     DO 23 I=1,M
217:       23 P=P+F(I)*F(I)
218:       WRITE (6,254) P
219: 254 FORMAT (/15X,'P= ',G14.5)
220:       NP1=NPT+1
221:       XNP=NPT
222:       DX=1./XNP
223:       CX5=DCOS(X(5))
224:       SX5=DSIN(X(5))
225:       D11=(CX5+PHI*SX5*X(6))/(SPHI-PHI*CPHI)
226:       B11=-D11*CPHI-SX5*X(6)
227:       D12=(SX5-PHI*CX5*X(6))/(SPHI-PHI*CPHI)
228:       B12=-D12*CPHI+CX5*X(6)
229:       WRITE (6,302) B11,D11,B12,D12
230: 302 FORMAT (/19X,'B11=',G18.9,5X,'D11=',G18.9/
231:   *      19X,'B12=',G18.9,5X,'D12=',G18.9)
232: C
233: C.....CALCULATIONS FOR END SECTION
234:   DO 30 I=1,NP1
235:     XU(I)=(I-1)*DX
236:     Z(I)=XU(I)*PHI
237:     SZ=DSIN(Z(I))
238:     CZ=DCOS(Z(I))
239:     U11(I)=B11*Z(I)+D11*SZ
240:     U12(I)=B12*Z(I)+D12*SZ
241:     R(I)=DSQRT(U11(I)**2+U12(I)**2)
242:     IF (DABS(U11(I)).LE.1.0E-6) GO TO 44
243:     THETA(I)=DATAN(U12(I)/U11(I))
244:     IF (THETA(I).GT.0.) THETA(I)=THETA(I)-PI
245:     GO TO 46
246: 44 THETA(I)=0.
247: 46 CONTINUE
248:     ST=DSIN(THETA(I))
249:     CT=DCOS(THETA(I))
250:     U11P(I)=B11+D11*CZ
251:     U12P(I)=B12+D12*CZ
252:     U11PP(I)=-D11*SZ
253:     U12PP(I)=-D12*SZ
254:     U11PPP(I)=-D11*CZ
255:     U12PPP(I)=-D12*CZ
256:     V11(I)=- (U11PPP(I)+U11P(I))
257:     V12(I)=- (U12PPP(I)+U12P(I))
258:     V(I)=DSQRT(V11(I)**2+V12(I)**2)
259:     VIR(I)=V11(I)*CT+V12(I)*ST

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260:      VIT(I)=-V11(I)*ST+V12(I)*CT
261:      XM(I)=DSQRT(U11PP(I)**2+U12PP(I)**2)
262:      XMR(I)=U11PP(I)*CT+U12PP(I)*ST
263:      XMT(I)=-U11PP(I)*ST+U12PP(I)*CT
264:      RP(I)=U11P(I)*CT+U12P(I)*ST
265:      TP(I)=-U11P(I)*ST+U12P(I)*CT
266:      30 CONTINUE
267: C
268: C.....PRINT END SECTION RESULTS
269:      WRITE (6,210) PHI
270:      210 FORMAT (//15X,'*** END SECTION RESULTS *** : PHI= ',618.9)
271:      WRITE (6,212)
272:      212 FORMAT (/T17,'Z',T24,'AZ',T34,'U1',T44,'U2',
273:      *          T54,'U1P',T64,'U2P'/)
274:      WRITE (6,214) (XU(I),Z(I),U11(I),U12(I),U11P(I),U12P(I),
275:      *I=1,NP1)
276:      214 FORMAT (15X,F6.2,5F10.5)
277:      WRITE (6,216)
278:      216 FORMAT (/T17,'Z',T24,'R',T34,'THETA',T44,'RP',T54,'THETAP'/)
279:      WRITE (6,215) (XU(I),R(I),THETA(I),RP(I),TP(I),I=1,NP1)
280:      215 FORMAT (15X,F6.2,4F10.5)
281:      WRITE (6,304)
282:      304 FORMAT (/T17,'Z',T24,'M1',T34,'M2',T44,'MR',T54,'MT',T64,'M'/)
283:      WRITE (6,214) (XU(I),U11PP(I),U12PP(I),XMR(I),XMT(I),XM(I),
284:      *I=1,NP1)
285:      WRITE (6,306)
286:      306 FORMAT (/T17,'Z',T24,'V1',T34,'V2',T44,'VR',T54,'VT',T64,'V'/)
287:      WRITE (6,214) (XU(I),V11(I),V12(I),V1R(I),V1T(I),V(I),I=1,NP1)
288: C
289: C.....CALCULATIONS FOR TRANSITION SECTION
290:      PHIH=X(1)
291:      PHIT=PHIH-PHI
292:      CX1=DCOS(X(1))
293:      SX1=DSIN(X(1))
294:      A21=X(7)
295:      B21=X(8)
296:      C21=X(9)
297:      D21=X(10)
298:      A22=X(11)
299:      B22=X(12)
300:      C22=X(13)
301:      D22=X(14)
302:      WRITE (6,220) PHIH
303:      220 FORMAT (//15X,'*** TRANS SECTION RESULTS *** : PHI2= ',618.9)
304:      DO 34 I=1,NP1
305:      XTR(I)=(I-1)*DX
306:      Z(I)=PHIT*XTR(I)+PHI
307:      SZ=DSIN(Z(I))
308:      CZ=DCOS(Z(I))
309:      U21(I)=A21+B21*Z(I)+C21*CZ+D21*SZ
310:      U22(I)=A22+B22*Z(I)+C22*CZ+D22*SZ
311:      R(I)=DSQRT(U21(I)**2+U22(I)**2)

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312:      IF (DABS(U21(I)),LE.1.0E-6) GO TO 48
313:      THETA(I)=DATAN(U22(I)/U21(I))
314:      IF (THETA(I),GT.0.) THETA(I)=THETA(I)-PI
315:      GO TO 49
316: 48 THETA(I)=0.
317: 49 CONTINUE
318:      ST=DSIN(THETA(I))
319:      CT=DCOS(THETA(I))
320:      U21P(I)=B21-C21*SZ+D21*CZ
321:      U22P(I)=B22-C22*SZ+D22*CZ
322:      U21PP(I)=-C21*CZ-D21*SZ
323:      U22PP(I)=-C22*CZ-D22*SZ
324:      U21PPP(I)=C21*SZ-D21*CZ
325:      U22PPP(I)=C22*SZ-D22*CZ
326:      V21(I)=- (U21PPP(I)+U21P(I))
327:      V22(I)=- (U22PPP(I)+U22P(I))
328:      V(I)=DSQRT(V21(I)**2+V22(I)**2)
329:      V2R(I)=V21(I)*CT+V22(I)*ST
330:      V2T(I)=-V21(I)*ST+V22(I)*CT
331:      XM(I)=DSQRT(U21PP(I)**2+U22PP(I)**2)
332:      XMR(I)=U21PP(I)*CT+U22PP(I)*ST
333:      XMT(I)=-U21PP(I)*ST+U22PP(I)*CT
334:      RP(I)=U21P(I)*CT+U22P(I)*ST
335:      TP(I)=-U21P(I)*ST+U22P(I)*CT
336: 34 CONTINUE
337: C
338: C.....PRINT TRANSITION SECTION RESULTS
339:      WRITE (6,212)
340:      WRITE (6,214) (XTR(I),Z(I),U21(I),U22(I),U21P(I),U22P(I),
341:      *I=1,NP1)
342:      WRITE (6,216)
343:      WRITE (6,215) (XTR(I),R(I),THETA(I),RP(I),TP(I),I=1,NP1)
344:      WRITE (6,304)
345:      WRITE (6,214) (XTR(I),U21PP(I),U22PP(I),XMR(I),XMT(I),XM(I),
346:      *I=1,NP1)
347:      WRITE (6,306)
348:      WRITE (6,214) (XTR(I),V21(I),V22(I),V2R(I),V2T(I),V(I),I=1,NP1)
349: 76 CONTINUE
350: C
351: C.....RUNGE-KUTTA INTEGRATION TO OBTAIN RESULTS FOR THE CONTACT SECTION
352:      DZ=PHIC/NINC
353:      ZRK=0.
354:      NZ=NINC+1
355:      Y(1)=0.
356:      Y(2)=X(2)
357:      Y(3)=X(3)
358:      Y(4)=X(4)
359:      IP=0
360:      NP=0
361: C
362: C.....BEGIN ITERATION OVER IZ
363:      DO 50 IZ=1,NZ

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364:      IF (IP.NE.0) GO TO 40
365:      IP=IPRT
366:      NP=NP+1
367:      ZP(NP)=ZRK
368:      Y1P(NP)=Y(1)
369:      Y2P(NP)=Y(2)
370:      Y3P(NP)=Y(3)
371:      Y4P(NP)=Y(4)
372:      WP(NP)=Y(2)**2-Y(2)**4+4.0*Y(2)*Y(4)+3.0*Y(3)**2
373:      40 IP=IP-1
374: C
375: C.....EXECUTE RUNGE=KUTTA
376:      CALL RUNGE(4,DZ,ZRK,Y,Y1)
377: C
378: C.....ADVANCE TO NEXT STEP
379:      ZRK=ZRK+DZ
380:      DO 42 I=1,4
381:      42 Y(I)=Y1(I)
382:      IF (DABS(Y(2)).GE.1000.) GO TO 51
383:      50 CONTINUE
384:      51 CONTINUE
385:      IF (ISTOP.EQ.1) GO TO 88
386:      WRITE (6,282) Y3P(NP)
387:      282 FORMAT (' THPP(L)=' ,619.10)
388:      YCHECK=DABS(Y3P(NP))
389:      IF (YCHECK.LE.EPSPHI) GO TO 82
390:      IF (Y3P(NP).LT.0.) GO TO 84
391:      PHIR=PHI
392:      GO TO 78
393:      84 PHIL=PHI
394:      78 CONTINUE
395:      WRITE (6,272)
396:      272 FORMAT (' *** NO CONVERGENCE *** '/')
397:      GO TO 86
398:      82 WRITE (6,274)
399:      274 FORMAT (' *** CONVERGENCE ON PHI *** '/')
400:      86 IPRNT1=1
401:      ISTOP=1
402:      GO TO 87
403:      88 CONTINUE
404: C
405: C.....PRINT CONTACT SECTION RESULTS
406:      DO 56 I=1,NP
407:      AZ(I)=ZP(I)+PHIH
408:      U1(I)=DCOS(Y1P(I))
409:      U2(I)=DSIN(Y1P(I))
410:      CMR(I)=-Y2P(I)**2
411:      CM(I)=DSQRT(Y2P(I)**4+Y3P(I)**2)
412:      CVR(I)=3.0*Y2P(I)*Y3P(I)
413:      CVT(I)=Y2P(I)**3-Y2P(I)-Y4P(I)
414:      CV(I)=DSQRT(CVR(I)**2+CVT(I)**2)
415:      56 CONTINUE

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416:      WRITE (6,226)
417: 226 FORMAT (//15X,'*** CONTACT SECTION RESULTS ***')
418:      WRITE (6,228)
419: 228 FORMAT (/T17,'Z',T24,'AZ',T34,'THETA',T44,'THETAP',
420:      *      T54,'THETAPP',T64,'THETAPPP'/)
421:      WRITE (6,214) (ZP(I),AZ(I),Y1P(I),Y2P(I),Y3P(I),Y4P(I),I=1,NP)
422:      WRITE (6,308)
423: 308 FORMAT (/T17,'Z',T24,'U1',T34,'U2',T44,'MR',T54,'M',T64,'W'/)
424:      WRITE (6,214) (ZP(I),U1(I),U2(I),CMR(I),CM(I),WP(I),I=1,NP)
425:      WRITE (6,310)
426: 310 FORMAT (/T17,'Z',T24,'VR',T34,'VT',T44,'V'/)
427:      WRITE (6,217) (ZP(I),CVR(I),CVT(I),CV(I),I=1,NP)
428: 217 FORMAT (15X,F6.2,3F10.5)
429:      STOP
430:      END
431: C
432: C
433:      SUBROUTINE RUNGE(N,DX,XJ,YJ,YJ1)
434:      IMPLICIT REAL*8(A-H,O-Z)
435:      DIMENSION YJ(4),YJ1(4),Y(4),F(4)
436:      REAL*8 K1(4),K2(4),K3(4),K4(4)
437: C
438: C.....CALCULATE K1
439:      X=XJ
440:      DO 10 I=1,N
441: 10 Y(I)=YJ(I)
442:      CALL EQUAT(N,X,Y,F)
443:      DO 20 I=1,N
444: 20 K1(I)=DX*F(I)
445: C
446: C.....CALCULATE K2
447:      X=XJ+DX/2.0
448:      DO 25 I=1,N
449: 25 Y(I)=YJ(I)+K1(I)/2.0
450:      CALL EQUAT(N,X,Y,F)
451:      DO 30 I=1,N
452: 30 K2(I)=DX*F(I)
453: C
454: C.....CALCULATE K3
455:      DO 40 I=1,N
456: 40 Y(I)=YJ(I)+K2(I)/2.0
457:      CALL EQUAT(N,X,Y,F)
458:      DO 50 I=1,N
459: 50 K3(I)=DX*F(I)
460: C
461: C.....CALCULATE K4
462:      X=XJ+DX
463:      DO 60 I=1,N
464: 60 Y(I)=YJ(I)+K3(I)
465:      CALL EQUAT(N,X,Y,F)
466:      DO 70 I=1,N
467: 70 K4(I)=DX*F(I)

```

```

468: C
469: C.....CALCULATE YJ1
470:     DO 80 I=1,N
471:     80 YJ1(I)=YJ(I)+(K1(I)+2.0*(K2(I)+K3(I))+K4(I))/6.0
472: C
473:     RETURN
474:     END
475: C
476: C
477:     SUBROUTINE EQUAT(N,X,Y,F)
478:     IMPLICIT REAL*8(A-H,O-Z)
479:     DIMENSION Y(4),F(4)
480:     F(1)=Y(2)
481:     F(2)=Y(3)
482:     F(3)=Y(4)
483:     F(4)=(6.0*Y(2)**2-1.0)*Y(3)
484:     RETURN
485:     END
486: C
487: C
488:     SUBROUTINE FUNC(X,F)
489:     IMPLICIT REAL*8(A-H,O-Z)
490:     DIMENSION X(1),F(1)
491:     COMMON PHI,CPHI,SPHI,H
492:     CX1=DCOS(X(1))
493:     SX1=DSIN(X(1))
494:     CX5=DCOS(X(5))
495:     SX5=DSIN(X(5))
496:     A=SPHI/(SPHI-PHI*CPHI)
497:     B11=(-CX5*CPHI-SX5*SPHI*X(6))/(SPHI-PHI*CPHI)
498:     B12=(-SX5*CPHI+CX5*SPHI*X(6))/(SPHI-PHI*CPHI)
499:     F(1)=-CX5+X(7)+PHI*X(8)+CPHI*X(9)+SPHI*X(10)
500:     F(2)=-SX5+X(11)+PHI*X(12)+CPHI*X(13)+SPHI*X(14)
501:     F(3)=SX5*X(6)+X(8)-SPHI*X(9)+CPHI*X(10)
502:     F(4)=-CX5*X(6)+X(12)-SPHI*X(13)+CPHI*X(14)
503:     F(5)=(CX5+PHI*SX5*X(6))*A-CPHI*X(9)-SPHI*X(10)
504:     F(6)=(SX5-PHI*CX5*X(6))*A-CPHI*X(13)-SPHI*X(14)
505:     F(7)=(B11-X(8))*SX5-(B12-X(12))*CX5
506:     F(8)=X(7)+X(8)*X(1)+CX1*X(9)+SX1*X(10)-1.0
507:     F(9)=X(11)+X(12)*X(1)+CX1*X(13)+SX1*X(14)
508:     F(10)=X(8)-SX1*X(9)+CX1*X(10)
509:     F(11)=X(12)-SX1*X(13)+CX1*X(14)-X(2)
510:     F(12)=-CX1*X(9)-SX1*X(10)+X(2)**2
511:     F(13)=CX1*X(13)+SX1*X(14)+X(3)
512:     F(14)=X(12)+X(2)**3-X(2)-X(4)
513:     RETURN
514:     END
515: C
516: C
517:     SUBROUTINE DFUNC(X,DF)
518:     IMPLICIT REAL*8(A-H,O-Z)
519:     DIMENSION X(1),DF(15,15)

```

```

520:      COMMON PHI,CPHI,SPHI,M
521: C
522: C.....DF(I,J)=PARTIAL OF F(I) WITH RESPECT TO X(J)
523: C
524:      SX1=DSIN(X(1))
525:      CX1=DCOS(X(1))
526:      SX5=DSIN(X(5))
527:      CX5=DCOS(X(5))
528:      A=SPHI/(SPHI-PHI*CPHI)
529:      B11=(-CX5*CPHI-SX5*SPHI*X(6))/(SPHI-PHI*CPHI)
530:      B12=(-SX5*CPHI+CX5*SPHI*X(6))/(SPHI-PHI*CPHI)
531:      DO 10 I=1,M
532:      DO 10 J=1,M
533:      10 DF(I,J)=0.0
534: C
535:      DF(1,5)=SX5
536:      DF(1,7)=1.
537:      DF(1,8)=PHI
538:      DF(1,9)=CPHI
539:      DF(1,10)=SPHI
540: C
541:      DF(2,5)=-CX5
542:      DF(2,11)=1.
543:      DF(2,12)=PHI
544:      DF(2,13)=CPHI
545:      DF(2,14)=SPHI
546: C
547:      DF(3,5)=CX5*X(6)
548:      DF(3,6)=SX5
549:      DF(3,8)=1.
550:      DF(3,9)=-SPHI
551:      DF(3,10)=CPHI
552: C
553:      DF(4,5)=SX5*X(6)
554:      DF(4,6)=-CX5
555:      DF(4,12)=1.
556:      DF(4,13)=-SPHI
557:      DF(4,14)=CPHI
558: C
559:      DF(5,5)=(-SX5+PHI*CX5*X(6))*A
560:      DF(5,6)=PHI*SX5*A
561:      DF(5,9)=-CPHI
562:      DF(5,10)=-SPHI
563: C
564:      DF(6,5)=(CX5+PHI*SX5*X(6))*A
565:      DF(6,6)=-PHI*CX5*A
566:      DF(6,13)=-CPHI
567:      DF(6,14)=-SPHI
568: C
569:      B11X5=(SX5*CPHI-CX5*SPHI*X(6))/(SPHI-PHI*CPHI)
570:      B12X5=(-CX5*CPHI-SX5*SPHI*X(6))/(SPHI-PHI*CPHI)
571:      DF(7,5)=B11X5*SX5+(B11-X(8))*CX5-B12X5*CX5+(B12-X(12))*SX5

```

```

572:      DF(7,6)=-A
573:      DF(7,8)=-SX5
574:      DF(7,12)=CX5
575: C
576:      DF(8,1)=CX1*X(10)+X(8)-SX1*X(9)
577:      DF(8,7)=1.
578:      DF(8,8)=X(1)
579:      DF(8,9)=CX1
580:      DF(8,10)=SX1
581: C
582:      DF(9,1)=X(12)-SX1*X(13)+CX1*X(14)
583:      DF(9,11)=1.
584:      DF(9,12)=X(1)
585:      DF(9,13)=CX1
586:      DF(9,14)=SX1
587: C
588:      DF(10,1)=-CX1*X(9)-SX1*X(10)
589:      DF(10,8)=1.
590:      DF(10,9)=-SX1
591:      DF(10,10)=CX1
592: C
593:      DF(11,1)=-CX1*X(13)-SX1*X(14)
594:      DF(11,2)=-1.
595:      DF(11,12)=1.
596:      DF(11,13)=-SX1
597:      DF(11,14)=CX1
598: C
599:      DF(12,1)=SX1*X(9)-CX1*X(10)
600:      DF(12,2)=2.*X(2)
601:      DF(12,9)=-CX1
602:      DF(12,10)=-SX1
603: C
604:      DF(13,1)=-SX1*X(13)+CX1*X(14)
605:      DF(13,3)=1.
606:      DF(13,13)=CX1
607:      DF(13,14)=SX1
608: C
609:      DF(14,2)=3.*X(2)**2-1.
610:      DF(14,4)=-1.
611:      DF(14,12)=1.
612: C
613:      RETURN
614:      END
615: END OF DATA

```

*

APPENDIX IX

Fixed-End Analysis Computer Listing

The computer listing which follows contains those portions of the FORTRAN program for the fixed-end analysis which are different from the program of Appendix VIII for the pinned-end analysis. This partial listing should be used, therefore, in conjunction with the listing of Appendix VIII.

This program applies the solution procedure of Section 2.5 to the fixed-end problem. A Modified Quasilinearization Algorithm [16] is used to solve the non-linear system of algebraic equations (3.5). A Runge-Kutta algorithm [17] is used to numerically integrate equation (2.7) for the contact section. An Interval-Halving procedure is used to solve for ϕ_1 in order to satisfy boundary condition (2.32).

Equations (3.5a) through (3.5l) are expressed as F(1) through F(12) respectively.

Variable definitions within the computer program are:

A1	= a ₁₁	X(1) = ϕ_2	X(7) = c ₂₁
B1	= b ₁₁	X(2) = θ'_2	X(8) = d ₂₁
C1	= c ₁₁	X(3) = θ_1	X(9) = a ₂₂
D1	= d ₁₁	X(4) = θ'_1	X(10) = b ₂₂
A2	= a ₁₂	X(5) = a ₂₁	X(11) = c ₂₂
B2	= b ₁₂	X(6) = b ₂₁	X(12) = d ₂₂
C2	= c ₁₂		
D2	= d ₁₂		

$$\text{PHI} = \phi_1$$

$$\text{PHIC} = \phi_c$$

In the contact section portion of the program,

$$Y(1) = \theta$$

$$Y(2) = \theta'$$

$$Y(3) = \theta''$$

$$Y(4) = \theta'''$$

Output variables are defined in Appendix XI with the computer program results.


```

C      PROGRAM HELBTFIX.FORT VERSION 3/11/84
.
.
.
.
      COMMON PHI,CPHI,SPHI,X1,X2,XD,M
.
.
.
C
C.....ECHO PRINT INPUT
      WRITE (6,200)
      200 FORMAT (///' HELBTFIX.FORT')
.
.
.
.
      DO 78 IPHI=1,NPHI
      PHI=.5*(PHIL+PHIR)
      CPHI=DCOS(PHI)
      SPHI=DSIN(PHI)
      X1=1.-CPHI
      X2=SPHI -PHI*CPHI
      X3=PHI-SPHI
      XD=2. -2.*CPHI -PHI*SPHI
.
.
.
.
.
C
C.....OUTPUT OF MQA
      IF (IPRNT1.NE.1) GO TO 76
      WRITE (6,300)
      300 FORMAT (///15X,'*** HELBTFIX.FORT ***')
.
.
.
.
      WRITE (6,254) P
      254 FORMAT (/15X,'P= ',614.5)
      NP1=NPT+1
      XNP=NPT
      DX=1./XNP
      CX3=DCOS(X(3))
      SX3=DSIN(X(3))
      A1=(CX3*X1 +X(4)*SX3*X3)/XD
      B1=(-X(4)*SX3*X1 -SPHI*CX3)/XD
      C1=-A1
      D1=-B1

```

```

      A2=(SX3*X1 -X(4)*CX3*X3)/XD
      B2=(X(4)*CX3*X1 -SPHI*SX3)/XD
      C2=-A2
      D2=-B2
      WRITE (6,302) A1,B1,C1,D1,A2,B2,C2,D2
302 FORMAT (/19X,'A11=',G18.9,5X,'B11=',G18.9/
*          19X,'C11=',G18.9,5X,'D11=',G18.9/
*          19X,'A12=',G18.9,5X,'B12=',G18.9/
*          19X,'C12=',G18.9,5X,'D12=',G18.9)
C
C.....CALCULATIONS FOR END SECTION
      DO 30 I=1,NP1
      XU(I)=(I-1)*DX
      Z(I)=XU(I)*PHI
      SZ=DSIN(Z(I))
      CZ=DCOS(Z(I))
      U11(I)=A1 +B1*Z(I) +C1*CZ +D1*SZ
      U12(I)=A2 +B2*Z(I) +C2*CZ +D2*SZ
      R(I)=DSQRT(U11(I)**2+U12(I)**2)
      IF (DABS(U11(I)).LE.1.0E-6) GO TO 44
      THETA(I)=DATAN(U12(I)/U11(I))
      IF (THETA(I).GT.0.) THETA(I)=THETA(I)-PI
      GO TO 46
44 THETA(I)=0.
46 CONTINUE
      ST=DSIN(THETA(I))
      CT=DCOS(THETA(I))
      U11P(I)=B1 -C1*SZ +D1*CZ
      U12P(I)=B2 -C2*SZ +D2*CZ
      U11PP(I)=-C1*CZ -D1*SZ
      U12PP(I)=-C2*CZ -D2*SZ
      U11PPP(I)=C1*SZ -D1*CZ
      U12PPP(I)=C2*SZ -D2*CZ
      .
      .
      .
      .
C
C.....CALCULATIONS FOR TRANSITION SECTION
      PHIH=X(1)
      PHIT=PHIH-PHI
      A21=X(5)
      B21=X(6)
      C21=X(7)
      D21=X(8)
      A22=X(9)
      B22=X(10)
      C22=X(11)
      D22=X(12)
      .
      .
      .

```

```

.
C
C.....RUNGE-KUTTA INTEGRATION TO OBTAIN RESULTS FOR THE CONTACT SECTION
      DZ=PHIC/NINC
      ZRK=0.
      NZ=NINC+1
      Y(1)=0.
      Y(2)=X(2)
      Y(3)=THPP
      Y(4)=THPPP
.
.
.
.
C
C
      SUBROUTINE FUNC(X,F)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(1),F(1)
      COMMON PHI,CPHI,SPHI,X1,X2,XD,M
      CX1=DCOS(X(1))
      SX1=DSIN(X(1))
      CX3=DCOS(X(3))
      SX3=DSIN(X(3))
      F(1)=X(5) +X(6)*PHI +X(7)*CPHI +X(8)*SPHI -CX3
      F(2)=X(9) +X(10)*PHI +X(11)*CPHI +X(12)*SPHI -SX3
      F(3)=X(6) -X(7)*SPHI +X(8)*CPHI +X(4)*SX3
      F(4)=X(10) -X(11)*SPHI +X(12)*CPHI -X(4)*CX3
      F(5)=X(7)*CPHI +X(8)*SPHI -(CX3*X1 +X(4)*SX3*X2)/XD
      F(6)=X(11)*CPHI +X(12)*SPHI -(SX3*X1 -X(4)*CX3*X2)/XD
      F(7)=-X(7)*SX3 -X(11)*CX3*SPHI +X(8)*SX3 -X(12)*CX3*CPHI
      *   +X(4)*(1.-CPHI-PHI*SPHI)/XD
      F(8)=X(5) +X(6)*X(1) +X(7)*CX1 +X(8)*SX1 -1.
      F(9)=X(9) +X(10)*X(1) +X(11)*CX1 +X(12)*SX1
      F(10)=X(6) -X(7)*SX1 +X(8)*CX1
      F(11)=X(10) -X(11)*SX1 +X(12)*CX1 -X(2)
      F(12)=X(7)*CX1 +X(8)*SX1 -X(2)**2
      RETURN
      END
C
C
      SUBROUTINE DFUNC(X,DF)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(1),DF(15,15)
      COMMON PHI,CPHI,SPHI,X1,X2,XD,M
C
C.....DF(I,J)=PARTIAL OF F(I) WITH RESPECT TO X(J)
C
      SX1=DSIN(X(1))
      CX1=DCOS(X(1))
      SX3=DSIN(X(3))
      CX3=DCOS(X(3))

```

```

      DO 10 I=1,M
      DO 10 J=1,M
10  DF(I,J)=0.0
C
      DF(1,3)=SX3
      DF(1,5)=1.
      DF(1,6)=PHI
      DF(1,7)=CPHI
      DF(1,8)=SPHI
C
      DF(2,3)=-CX3
      DF(2,9)=1.
      DF(2,10)=PHI
      DF(2,11)=CPHI
      DF(2,12)=SPHI
C
      DF(3,3)=X(4)*CX3
      DF(3,4)=SX3
      DF(3,6)=1.
      DF(3,7)=-SPHI
      DF(3,8)=CPHI
C
      DF(4,3)=X(4)*SX3
      DF(4,4)=-CX3
      DF(4,10)=1.
      DF(4,11)=-SPHI
      DF(4,12)=CPHI
C
      DF(5,3)=(SX3*X1 -X(4)*CX3*X2)/XD
      DF(5,4)=-SX3*X2/XD
      DF(5,7)=CPHI
      DF(5,8)=SPHI
C
      DF(6,3)=- (CX3*X1 +X(4)*SX3*X2)/XD
      DF(6,4)=CX3*X2/XD
      DF(6,11)=CPHI
      DF(6,12)=SPHI
C
      DF(7,3)=- (X(7)*CX3 +X(11)*SX3)*SPHI +(X(8)*CX3 +X(12)*SX3)*CPHI
      DF(7,4)=(1.-CPHI-PHI*SPHI)/XD
      DF(7,7)=-SX3*SPHI
      DF(7,8)=SX3*CPHI
      DF(7,11)=CX3*SPHI
      DF(7,12)=-CX3*CPHI
C
      DF(8,1)=X(6) -X(7)*SX1 +X(8)*CX1
      DF(8,5)=1.
      DF(8,6)=X(1)
      DF(8,7)=CX1
      DF(8,8)=SX1
C
      DF(9,1)=X(10) -X(11)*SX1 +X(12)*CX1

```

```
      DF(9,9)=1.  
      DF(9,10)=X(1)  
      DF(9,11)=CX1  
      DF(9,12)=SX1  
C  
      DF(10,1)=-X(7)*CX1 -X(8)*SX1  
      DF(10,6)=1.  
      DF(10,7)=-SX1  
      DF(10,8)=CX1  
C  
      DF(11,1)=-X(11)*CX1 -X(12)*SX1  
      DF(11,2)=-1.  
      DF(11,10)=1.  
      DF(11,11)=-SX1  
      DF(11,12)=CX1  
C  
      DF(12,1)=-X(7)*SX1 +X(8)*CX1  
      DF(12,2)=-2.*X(2)  
      DF(12,7)=CX1  
      DF(12,8)=SX1  
C  
      RETURN  
      END  
*END OF DATA
```

*

APPENDIX X

Pinned-End Analysis Results

Computer results for the pinned-end analysis of Section 2 and Appendix VIII are presented here. Definitions of the input parameters are given in the program listing of Appendix VIII. Output variables used here are related to those of the main text by the following definitions.

The variables $X(1)$ through $X(14)$, $B11$, $B12$, $D11$, $D12$, and the functions $F(1)$ through $F(14)$, are defined in Appendix VIII. The Modified Quasilinearization Algorithm convergence error parameter P is defined as

$$P = F(1)^2 + F(2)^2 + \dots + F(14)^2.$$

Variable definitions in END SECTION RESULTS are

$$PHI1 = \phi_1$$

$$Z = z/L_1$$

$$AZ = \zeta = \alpha z$$

$$U1 = u_{11} = \hat{u}_{11}/r$$

$$U2 = u_{12} = \hat{u}_{12}/r$$

$$U1P = u'_{11}$$

$$U2P = u'_{12}$$

$$R = R$$

$$THETA = \theta$$

$$RP = R'$$

$$THETAP = \theta'$$

$$M1 = M_1 = \hat{M}_1 / Wr$$

$$M2 = M_2 = \hat{M}_2 / Wr$$

$$MR = M_R = \hat{M}_R / Wr$$

$$MT = M_\theta = \hat{M}_\theta / Wr$$

$$M = M = [M_1^2 + M_2^2]^{1/2}$$

$$V1 = V_1 = \hat{V}_1 / \alpha Wr$$

$$V2 = V_2 = \hat{V}_2 / \alpha Wr$$

$$VR = V_R = \hat{V}_R / \alpha Wr$$

$$VT = V_\theta = \hat{V}_\theta / \alpha Wr$$

$$V = V = [V_1^2 + V_2^2]^{1/2}$$

Transition section output variable definitions in
TRANS SECTION RESULTS are the same as for the end section
except

$$Z = (z - L_1) / (L_2 - L_1).$$

Contact section output variable definitions are the
same as above except that

$$Z = \zeta - \phi_2 = \alpha(z - L_2)$$

$$THETAPP = \theta''$$

$$THETAPPP = \theta'''$$

$$W = w = \hat{w} / \alpha^2 Wr.$$

HELBTPIN.FORT

PHIL= 2.46800000   PHIR= 2.46880000
 NIT= 50   NBIS= 20   EPS= .100D-19   ALPHA1= 1.00
 M= 14   IPRINT= 0   IDS= 5   IPRNT1= 0   NPT= 20
 INITIAL X VECTOR
 4.5933   .67809   .36600D-02   -.54510D-01
 -1.3214   .46275   .36381   .38400D-01
 -.92740D-01   -.45199   -1.4285   .31179
 .36414   -.39870D-01
 PHIC= 6.0000   NINC=600   IPRT= 10
 NPHI= 30   EPSPHI= .10000D-05

*** HELBTPIN.FORT ***

*** MQA RESULTS ***

X(1)=	4.38105187	F(1)=	-.8327D-16
X(2)=	.687208686	F(2)=	-.6158D-16
X(3)=	.277442199D-01	F(3)=	.9714D-16
X(4)=	-.381321413D-01	F(4)=	-.4163D-16
X(5)=	-1.18188083	F(5)=	.8327D-16
X(6)=	.466547261	F(6)=	-.3556D-16
X(7)=	.571274742	F(7)=	.9714D-16
X(8)=	-.993608857D-02	F(8)=	.2220D-15
X(9)=	-.144232795	F(9)=	.1388D-16
X(10)=	-.449801229	F(10)=	.8327D-16
X(11)=	-1.39407478	F(11)=	.4163D-16
X(12)=	.324538272	F(12)=	.1388D-16
X(13)=	.351969531	F(13)=	-.2515D-16
X(14)=	-.917443170D-01	F(14)=	-.1388D-16

P= .98712D-31

B11=	.221474215	D11=	-.268838168
B12=	-.240169470	D12=	-.533346268

*** END SECTION RESULTS *** : PHI1= 2.46866597

Z	AZ	U1	U2	U1P	U2P
0.0	0.0	0.0	0.0	-0.04736	-0.77352
0.05	0.12343	-0.00576	-0.09531	-0.04532	-0.76946
0.10	0.24687	-0.01102	-0.18962	-0.03921	-0.75735
0.15	0.37030	-0.01528	-0.28195	-0.02914	-0.73736
0.20	0.49373	-0.01806	-0.37134	-0.01526	-0.70982
0.25	0.61717	-0.01890	-0.45689	0.00223	-0.67512
0.30	0.74060	-0.01737	-0.53773	0.02305	-0.63381
0.35	0.86403	-0.01308	-0.61311	0.04690	-0.58651
0.40	0.98747	-0.00568	-0.68231	0.07340	-0.53394
0.45	1.11090	0.00513	-0.74473	0.10215	-0.47690
0.50	1.23433	0.01961	-0.79989	0.13272	-0.41625
0.55	1.35777	0.03795	-0.84738	0.16464	-0.35293
0.60	1.48120	0.06029	-0.88695	0.19742	-0.28789
0.65	1.60463	0.08670	-0.91842	0.23057	-0.22213
0.70	1.72807	0.11720	-0.94179	0.26358	-0.15664
0.75	1.85150	0.15174	-0.95715	0.29595	-0.09242
0.80	1.97493	0.19022	-0.96470	0.32719	-0.03044
0.85	2.09837	0.23245	-0.96479	0.35682	0.02834
0.90	2.22180	0.27822	-0.95787	0.38439	0.08303
0.95	2.34523	0.32724	-0.94450	0.40948	0.13281
1.00	2.46867	0.37919	-0.92532	0.43171	0.17691

Z	R	THETA	RP	THETAP
0.0	0.0	0.0	-0.04736	-0.77352
0.05	0.09548	-1.63118	0.77079	0.00120
0.10	0.18994	-1.62885	0.75835	0.00479
0.15	0.28236	-1.62494	0.73786	0.01080
0.20	0.37178	-1.61939	0.70972	0.01924
0.25	0.45728	-1.61213	0.67446	0.03013
0.30	0.53802	-1.60309	0.63274	0.04351
0.35	0.61325	-1.59213	0.58538	0.05940
0.40	0.68233	-1.57913	0.53331	0.07784
0.45	0.74475	-1.56391	0.47759	0.09886
0.50	0.80013	-1.54629	0.41938	0.12248
0.55	0.84823	-1.52604	0.35994	0.14868
0.60	0.88899	-1.50293	0.30062	0.17744
0.65	0.92251	-1.47667	0.24281	0.20867
0.70	0.94906	-1.44699	0.18799	0.24222
0.75	0.96910	-1.41357	0.13762	0.27783
0.80	0.98327	-1.37612	0.09316	0.31512
0.85	0.99240	-1.33437	0.05603	0.35353
0.90	0.99746	-1.28812	0.02748	0.39229
0.95	0.99958	-1.23727	0.00856	0.43039
1.00	1.00000	-1.18188	0.00000	0.46655

Z	M1	M2	MR	MT	M
0.0	0.0	0.0	0.0	0.0	0.0
0.05	0.03310	0.06567	-0.06754	0.02908	0.07354

0.10	0.06570	0.13033	-0.13392	0.05802	0.14595
0.15	0.09729	0.19302	-0.19800	0.08670	0.21615
0.20	0.12741	0.25276	-0.25865	0.11498	0.28306
0.25	0.15558	0.30866	-0.31483	0.14270	0.34566
0.30	0.18139	0.35987	-0.36553	0.16968	0.40300
0.35	0.20444	0.40559	-0.40986	0.19574	0.45421
0.40	0.22438	0.44515	-0.44700	0.22067	0.49850
0.45	0.24091	0.47793	-0.47626	0.24419	0.53521
0.50	0.25376	0.50344	-0.49707	0.26603	0.56378
0.55	0.26276	0.52129	-0.50901	0.28582	0.58377
0.60	0.26776	0.53121	-0.51183	0.30317	0.59487
0.65	0.26868	0.53304	-0.50543	0.31759	0.59693
0.70	0.26552	0.52676	-0.48994	0.32854	0.58990
0.75	0.25832	0.51247	-0.46570	0.33537	0.57389
0.80	0.24718	0.49038	-0.43330	0.33738	0.54916
0.85	0.23229	0.46083	-0.39360	0.33376	0.51606
0.90	0.21385	0.42426	-0.34778	0.32371	0.47511
0.95	0.19217	0.38124	-0.29732	0.30639	0.42694
1.00	0.16756	0.33242	-0.24406	0.28110	0.37227

Z	V1	V2	VR	VT	V
0.0	-0.22147	0.24017	-0.22147	0.24017	0.32670
0.05	-0.22147	0.24017	-0.22637	-0.23556	0.32670
0.10	-0.22147	0.24017	-0.22691	-0.23504	0.32670
0.15	-0.22147	0.24017	-0.22783	-0.23415	0.32670
0.20	-0.22147	0.24017	-0.22913	-0.23288	0.32670
0.25	-0.22147	0.24017	-0.23081	-0.23121	0.32670
0.30	-0.22147	0.24017	-0.23289	-0.22911	0.32670
0.35	-0.22147	0.24017	-0.23539	-0.22655	0.32670
0.40	-0.22147	0.24017	-0.23832	-0.22347	0.32670
0.45	-0.22147	0.24017	-0.24169	-0.21981	0.32670
0.50	-0.22147	0.24017	-0.24553	-0.21552	0.32670
0.55	-0.22147	0.24017	-0.24984	-0.21051	0.32670
0.60	-0.22147	0.24017	-0.25464	-0.20468	0.32670
0.65	-0.22147	0.24017	-0.25992	-0.19792	0.32670
0.70	-0.22147	0.24017	-0.26568	-0.19012	0.32670
0.75	-0.22147	0.24017	-0.27189	-0.18114	0.32670
0.80	-0.22147	0.24017	-0.27848	-0.17083	0.32670
0.85	-0.22147	0.24017	-0.28536	-0.15906	0.32670
0.90	-0.22147	0.24017	-0.29241	-0.14570	0.32670
0.95	-0.22147	0.24017	-0.29944	-0.13064	0.32670
1.00	-0.22147	0.24017	-0.30621	-0.11387	0.32670

*** TRANS SECTION RESULTS *** : PHI2 = 4.38105187

Z	AZ	U1	U2	U1P	U2P
0.0	2.46867	0.37919	-0.92532	0.43171	0.17691
0.05	2.56429	0.42117	-0.90687	0.44569	0.20932
0.10	2.65990	0.46428	-0.88526	0.45550	0.24278

0.15	2.75552	0.50814	-0.86041	0.46107	0.27700
0.20	2.85114	0.55232	-0.83227	0.46233	0.31164
0.25	2.94676	0.59642	-0.80081	0.45928	0.34641
0.30	3.04238	0.64002	-0.76603	0.45194	0.38097
0.35	3.13800	0.68271	-0.72797	0.44038	0.41502
0.40	3.23362	0.72410	-0.68669	0.42471	0.44824
0.45	3.32924	0.76381	-0.64229	0.40506	0.48033
0.50	3.42486	0.80145	-0.59488	0.38163	0.51100
0.55	3.52048	0.83667	-0.54462	0.35461	0.53996
0.60	3.61610	0.86916	-0.49168	0.32427	0.56696
0.65	3.71172	0.89859	-0.43626	0.29087	0.59174
0.70	3.80734	0.92469	-0.37859	0.25473	0.61408
0.75	3.90296	0.94723	-0.31891	0.21617	0.63378
0.80	3.99857	0.96597	-0.25748	0.17554	0.65065
0.85	4.09419	0.98074	-0.19458	0.13321	0.66454
0.90	4.18981	0.99140	-0.13049	0.08958	0.67532
0.95	4.28543	0.99784	-0.06553	0.04504	0.68290
1.00	4.38105	1.00000	-0.00000	0.00000	0.68721

Z	R	THETA	RP	THETAP
0.0	1.00000	-1.18188	0.00000	0.46655
0.05	0.99989	-1.13602	-0.00212	0.49239
0.10	0.99962	-1.08777	-0.00344	0.51616
0.15	0.99926	-1.03733	-0.00405	0.53786
0.20	0.99887	-0.98490	-0.00402	0.55754
0.25	0.99850	-0.93065	-0.00349	0.57526
0.30	0.99821	-0.87478	-0.00259	0.59108
0.35	0.99802	-0.81747	-0.00147	0.60512
0.40	0.99793	-0.75889	-0.00027	0.61749
0.45	0.99796	-0.69919	0.00088	0.62833
0.50	0.99810	-0.63853	0.00188	0.63777
0.55	0.99831	-0.57703	0.00263	0.64599
0.60	0.99859	-0.51481	0.00308	0.65314
0.65	0.99889	-0.45198	0.00322	0.65936
0.70	0.99920	-0.38860	0.00306	0.66481
0.75	0.99947	-0.32476	0.00264	0.66962
0.80	0.99970	-0.26049	0.00204	0.67391
0.85	0.99986	-0.19585	0.00134	0.67776
0.90	0.99995	-0.13087	0.00069	0.68124
0.95	0.99999	-0.06558	0.00019	0.68439
1.00	1.00000	-0.00000	0.00000	0.68721

Z	M1	M2	MR	MT	M
0.0	0.16756	0.33242	-0.24406	0.28110	0.37227
0.05	0.12463	0.34500	-0.26041	0.25835	0.36682
0.10	0.08056	0.35442	-0.27646	0.23596	0.36346
0.15	0.03576	0.36061	-0.29232	0.21416	0.36238
0.20	-0.00938	0.36350	-0.30806	0.19319	0.36362
0.25	-0.05442	0.36307	-0.32369	0.17322	0.36713
0.30	-0.09897	0.35933	-0.33921	0.15444	0.37271

0.35	-0.14262	0.35230	-0.35453	0.13697	0.38007
0.40	-0.18496	0.34205	-0.36957	0.12092	0.38885
0.45	-0.22561	0.32868	-0.38421	0.10636	0.39866
0.50	-0.26420	0.31230	-0.39828	0.09330	0.40906
0.55	-0.30038	0.29307	-0.41162	0.08175	0.41966
0.60	-0.33381	0.27117	-0.42406	0.07166	0.43007
0.65	-0.36419	0.24678	-0.43540	0.06294	0.43993
0.70	-0.39125	0.22014	-0.44549	0.05549	0.44893
0.75	-0.41473	0.19149	-0.45415	0.04915	0.45681
0.80	-0.43442	0.16110	-0.46126	0.04377	0.46333
0.85	-0.45015	0.12922	-0.46669	0.03915	0.46833
0.90	-0.46176	0.09617	-0.47036	0.03509	0.47167
0.95	-0.46915	0.06224	-0.47222	0.03137	0.47326
1.00	-0.47226	0.02774	-0.47226	0.02774	0.47307

Z	V1	V2	VR	VT	V
0.0	0.00994	-0.32454	0.30407	-0.11387	0.32469
0.05	0.00994	-0.32454	0.29853	-0.12769	0.32469
0.10	0.00994	-0.32454	0.29202	-0.14194	0.32469
0.15	0.00994	-0.32454	0.28450	-0.15648	0.32469
0.20	0.00994	-0.32454	0.27590	-0.17117	0.32469
0.25	0.00994	-0.32454	0.26622	-0.18588	0.32469
0.30	0.00994	-0.32454	0.25542	-0.20046	0.32469
0.35	0.00994	-0.32454	0.24352	-0.21476	0.32469
0.40	0.00994	-0.32454	0.23053	-0.22865	0.32469
0.45	0.00994	-0.32454	0.21648	-0.24200	0.32469
0.50	0.00994	-0.32454	0.20141	-0.25467	0.32469
0.55	0.00994	-0.32454	0.18537	-0.26657	0.32469
0.60	0.00994	-0.32454	0.16844	-0.27758	0.32469
0.65	0.00994	-0.32454	0.15068	-0.28761	0.32469
0.70	0.00994	-0.32454	0.13216	-0.29658	0.32469
0.75	0.00994	-0.32454	0.11297	-0.30440	0.32469
0.80	0.00994	-0.32454	0.09319	-0.31103	0.32469
0.85	0.00994	-0.32454	0.07290	-0.31640	0.32469
0.90	0.00994	-0.32454	0.05220	-0.32047	0.32469
0.95	0.00994	-0.32454	0.03118	-0.32319	0.32469
1.00	0.00994	-0.32454	0.00994	-0.32454	0.32469

*** CONTACT SECTION RESULTS ***

Z	AZ	THETA	THETAP	THETAPP	THETAPPP
0.0	4.38105	0.0	0.68721	0.02774	-0.03813
0.10	4.48105	0.06885	0.68980	0.02418	-0.03335
0.20	4.58105	0.13795	0.69206	0.02105	-0.02914
0.30	4.68105	0.20726	0.69402	0.01833	-0.02544
0.40	4.78105	0.27675	0.69574	0.01595	-0.02220
0.50	4.88105	0.34640	0.69723	0.01388	-0.01935
0.60	4.98105	0.41618	0.69852	0.01207	-0.01686
0.70	5.08105	0.48609	0.69965	0.01049	-0.01468

0.80	5.18105	0.55611	0.70063	0.00912	-0.01278
0.90	5.28105	0.62621	0.70148	0.00793	-0.01112
1.00	5.38105	0.69640	0.70222	0.00689	-0.00968
1.10	5.48105	0.76665	0.70286	0.00599	-0.00842
1.20	5.58105	0.83697	0.70342	0.00520	-0.00732
1.30	5.68105	0.90734	0.70390	0.00452	-0.00636
1.40	5.78105	0.97775	0.70433	0.00393	-0.00553
1.50	5.88105	1.04820	0.70469	0.00341	-0.00481
1.60	5.98105	1.11868	0.70501	0.00296	-0.00418
1.70	6.08105	1.18920	0.70529	0.00257	-0.00363
1.80	6.18105	1.25974	0.70553	0.00223	-0.00315
1.90	6.28105	1.33030	0.70573	0.00194	-0.00274
2.00	6.38105	1.40089	0.70591	0.00168	-0.00238
2.10	6.48105	1.47149	0.70607	0.00146	-0.00207
2.20	6.58105	1.54210	0.70621	0.00127	-0.00179
2.30	6.68105	1.61273	0.70633	0.00110	-0.00156
2.40	6.78105	1.68337	0.70643	0.00096	-0.00135
2.50	6.88105	1.75401	0.70652	0.00083	-0.00117
2.60	6.98105	1.82467	0.70660	0.00072	-0.00102
2.70	7.08105	1.89533	0.70666	0.00063	-0.00089
2.80	7.18105	1.96600	0.70672	0.00054	-0.00077
2.90	7.28105	2.03668	0.70677	0.00047	-0.00067
3.00	7.38105	2.10736	0.70682	0.00041	-0.00058
3.10	7.48105	2.17804	0.70685	0.00036	-0.00050
3.20	7.58105	2.24873	0.70689	0.00031	-0.00044
3.30	7.68105	2.31942	0.70692	0.00027	-0.00038
3.40	7.78105	2.39011	0.70694	0.00023	-0.00033
3.50	7.88105	2.46080	0.70696	0.00020	-0.00029
3.60	7.98105	2.53150	0.70698	0.00018	-0.00025
3.70	8.08105	2.60220	0.70700	0.00015	-0.00022
3.80	8.18105	2.67290	0.70701	0.00013	-0.00019
3.90	8.28105	2.74360	0.70703	0.00011	-0.00016
4.00	8.38105	2.81431	0.70704	0.00010	-0.00014
4.10	8.48105	2.88501	0.70705	0.00009	-0.00012
4.20	8.58105	2.95572	0.70705	0.00007	-0.00011
4.30	8.68105	3.02642	0.70706	0.00006	-0.00009
4.40	8.78105	3.09713	0.70707	0.00006	-0.00008
4.50	8.88105	3.16783	0.70707	0.00005	-0.00007
4.60	8.98105	3.23854	0.70708	0.00004	-0.00006
4.70	9.08105	3.30925	0.70708	0.00004	-0.00005
4.80	9.18105	3.37996	0.70708	0.00003	-0.00005
4.90	9.28105	3.45067	0.70709	0.00003	-0.00004
5.00	9.38105	3.52138	0.70709	0.00002	-0.00004
5.10	9.48105	3.59208	0.70709	0.00002	-0.00003
5.20	9.58105	3.66279	0.70709	0.00002	-0.00003
5.30	9.68105	3.73350	0.70709	0.00001	-0.00003
5.40	9.78105	3.80421	0.70710	0.00001	-0.00002
5.50	9.88105	3.87492	0.70710	0.00001	-0.00002
5.60	9.98105	3.94563	0.70710	0.00001	-0.00002
5.70	10.08105	4.01634	0.70710	0.00001	-0.00002
5.80	10.18105	4.08705	0.70710	0.00000	-0.00002
5.90	10.28105	4.15776	0.70710	0.00000	-0.00002

6.00	10.38105	4.22847	0.70710	-0.00000	-0.00002
Z	U1	U2	MR	M	W
0.0	1.00000	0.0	-0.47226	0.47307	0.14672
0.10	0.99763	0.06880	-0.47582	0.47644	0.15914
0.20	0.99050	0.13751	-0.47895	0.47941	0.17021
0.30	0.97860	0.20577	-0.48167	0.48202	0.18004
0.40	0.96195	0.27323	-0.48405	0.48431	0.18874
0.50	0.94060	0.33951	-0.48612	0.48632	0.19642
0.60	0.91464	0.40427	-0.48793	0.48808	0.20318
0.70	0.88416	0.46718	-0.48951	0.48962	0.20912
0.80	0.84932	0.52789	-0.49088	0.49096	0.21434
0.90	0.81025	0.58608	-0.49207	0.49213	0.21891
1.00	0.76716	0.64146	-0.49311	0.49316	0.22291
1.10	0.72024	0.69373	-0.49401	0.49405	0.22641
1.20	0.66972	0.74262	-0.49480	0.49483	0.22946
1.30	0.61585	0.78787	-0.49548	0.49550	0.23213
1.40	0.55889	0.82924	-0.49607	0.49609	0.23445
1.50	0.49913	0.86653	-0.49659	0.49660	0.23648
1.60	0.43687	0.89953	-0.49704	0.49705	0.23824
1.70	0.37240	0.92807	-0.49743	0.49743	0.23978
1.80	0.30606	0.95201	-0.49777	0.49777	0.24112
1.90	0.23818	0.97122	-0.49806	0.49806	0.24228
2.00	0.16909	0.98560	-0.49832	0.49832	0.24329
2.10	0.09915	0.99507	-0.49854	0.49854	0.24417
2.20	0.02869	0.99959	-0.49873	0.49873	0.24494
2.30	-0.04192	0.99912	-0.49890	0.49890	0.24560
2.40	-0.11233	0.99367	-0.49904	0.49904	0.24618
2.50	-0.18219	0.98326	-0.49917	0.49917	0.24668
2.60	-0.25115	0.96795	-0.49928	0.49928	0.24712
2.70	-0.31887	0.94780	-0.49937	0.49937	0.24750
2.80	-0.38500	0.92292	-0.49946	0.49946	0.24783
2.90	-0.44921	0.89343	-0.49953	0.49953	0.24811
3.00	-0.51118	0.85947	-0.49959	0.49959	0.24836
3.10	-0.57061	0.82122	-0.49964	0.49964	0.24858
3.20	-0.62718	0.77887	-0.49969	0.49969	0.24876
3.30	-0.68063	0.73263	-0.49973	0.49973	0.24893
3.40	-0.73068	0.68272	-0.49977	0.49977	0.24907
3.50	-0.77708	0.62941	-0.49980	0.49980	0.24919
3.60	-0.81960	0.57294	-0.49982	0.49982	0.24930
3.70	-0.85802	0.51361	-0.49985	0.49985	0.24939
3.80	-0.89216	0.45172	-0.49987	0.49987	0.24947
3.90	-0.92184	0.38757	-0.49988	0.49988	0.24954
4.00	-0.94692	0.32147	-0.49990	0.49990	0.24960
4.10	-0.96726	0.25378	-0.49991	0.49991	0.24965
4.20	-0.98277	0.18481	-0.49992	0.49992	0.24970
4.30	-0.99338	0.11492	-0.49993	0.49993	0.24974
4.40	-0.99901	0.04445	-0.49994	0.49994	0.24977
4.50	-0.99966	-0.02624	-0.49995	0.49995	0.24980
4.60	-0.99930	-0.09680	-0.49996	0.49996	0.24983
4.70	-0.98598	-0.16687	-0.49996	0.49996	0.24985

4.80	-0.97173	-0.23611	-0.49997	0.49997	0.24987
4.90	-0.95262	-0.30418	-0.49997	0.49997	0.24988
5.00	-0.92875	-0.37072	-0.49997	0.49997	0.24990
5.10	-0.90023	-0.43541	-0.49998	0.49998	0.24991
5.20	-0.86722	-0.49792	-0.49998	0.49998	0.24992
5.30	-0.82988	-0.55795	-0.49998	0.49998	0.24993
5.40	-0.78838	-0.61518	-0.49998	0.49998	0.24993
5.50	-0.74295	-0.66935	-0.49999	0.49999	0.24994
5.60	-0.69380	-0.72016	-0.49999	0.49999	0.24994
5.70	-0.64119	-0.76738	-0.49999	0.49999	0.24995
5.80	-0.58537	-0.81077	-0.49999	0.49999	0.24995
5.90	-0.52663	-0.85010	-0.49999	0.49999	0.24995
6.00	-0.46525	-0.88518	-0.49999	0.49999	0.24995

Z	VR	VT	V
0.0	0.05720	-0.32454	0.32954
0.10	0.05003	-0.32822	0.33202
0.20	0.04371	-0.33146	0.33433
0.30	0.03816	-0.33429	0.33646
0.40	0.03329	-0.33677	0.33841
0.50	0.02903	-0.33894	0.34018
0.60	0.02529	-0.34083	0.34177
0.70	0.02203	-0.34248	0.34319
0.80	0.01918	-0.34392	0.34446
0.90	0.01669	-0.34518	0.34558
1.00	0.01452	-0.34627	0.34657
1.10	0.01263	-0.34722	0.34745
1.20	0.01098	-0.34805	0.34822
1.30	0.00954	-0.34877	0.34890
1.40	0.00829	-0.34940	0.34950
1.50	0.00721	-0.34994	0.35002
1.60	0.00626	-0.35042	0.35047
1.70	0.00544	-0.35083	0.35087
1.80	0.00473	-0.35119	0.35122
1.90	0.00411	-0.35150	0.35152
2.00	0.00357	-0.35177	0.35179
2.10	0.00310	-0.35200	0.35202
2.20	0.00269	-0.35221	0.35222
2.30	0.00234	-0.35238	0.35239
2.40	0.00203	-0.35254	0.35254
2.50	0.00176	-0.35267	0.35268
2.60	0.00153	-0.35279	0.35279
2.70	0.00133	-0.35289	0.35289
2.80	0.00115	-0.35298	0.35298
2.90	0.00100	-0.35305	0.35305
3.00	0.00087	-0.35312	0.35312
3.10	0.00075	-0.35318	0.35318
3.20	0.00066	-0.35323	0.35323
3.30	0.00057	-0.35327	0.35327
3.40	0.00049	-0.35331	0.35331
3.50	0.00043	-0.35334	0.35334

3.60	0.00037	-0.35337	0.35337
3.70	0.00032	-0.35339	0.35339
3.80	0.00028	-0.35341	0.35341
3.90	0.00024	-0.35343	0.35343
4.00	0.00021	-0.35345	0.35345
4.10	0.00018	-0.35346	0.35346
4.20	0.00016	-0.35347	0.35347
4.30	0.00014	-0.35348	0.35348
4.40	0.00012	-0.35349	0.35349
4.50	0.00010	-0.35350	0.35350
4.60	0.00009	-0.35351	0.35351
4.70	0.00008	-0.35351	0.35351
4.80	0.00007	-0.35352	0.35352
4.90	0.00006	-0.35352	0.35352
5.00	0.00005	-0.35353	0.35353
5.10	0.00004	-0.35353	0.35353
5.20	0.00003	-0.35353	0.35353
5.30	0.00003	-0.35353	0.35353
5.40	0.00002	-0.35354	0.35354
5.50	0.00002	-0.35354	0.35354
5.60	0.00001	-0.35354	0.35354
5.70	0.00001	-0.35354	0.35354
5.80	0.00001	-0.35354	0.35354
5.90	0.00000	-0.35354	0.35354
6.00	-0.00000	-0.35354	0.35354

END OF DATA

*

APPENDIX XI

Fixed-End Analysis Results

Computer results for the fixed-end analysis of Section 3 and Appendix IX are presented here. Definitions of the input parameters are given in the program listing of Appendix VIII. Output parameters used here are related to those of the main text by the definitions of Appendix X with the following exceptions.

The variables $X(1)$ through $X(12)$ and the functions $F(1)$ through $F(12)$ are defined as in Appendix IX. In addition,

$$A_{11} = a_{11}$$

$$A_{12} = a_{12}$$

$$B_{11} = b_{11}$$

$$B_{12} = b_{12}$$

$$C_{11} = c_{11}$$

$$C_{12} = c_{12}$$

$$D_{11} = d_{11}$$

$$D_{12} = d_{12}.$$

HELBTFIX.FORT

PHIL= 3.81729000   PHIR= 3.81730000
 NIT= 50   NBIS= 20   EPS= .100D-19   ALPHA1= 1.00
 M= 12   IPRINT= 0   IDG= 5   IPRNT1= 0   NPT= 20
 INITIAL X VECTOR
 5.2427   .68152   -.89780   .53623
 .78238   -.47083D-01   .27556   -.37685
 -1.6240   .31654   .29687   .21527
 PHIC= 6.0000   NINC=600   IPRT= 10
 NPFI= 25   EPSPHI= .10000D-05

*** HELBTFIX.FORT ***

*** MQA RESULTS ***

X(1)=	5.24273130	F(1)=	.7936D-11
X(2)=	.681518427	F(2)=	.6324D-11
X(3)=	-.897794527	F(3)=	-.3906D-12
X(4)=	.536226981	F(4)=	.6932D-11
X(5)=	.782383221	F(5)=	.7036D-11
X(6)=	-.470843484D-01	F(6)=	-.6503D-11
X(7)=	.275557467	F(7)=	-.1746D-11
X(8)=	-.376848713	F(8)=	.2220D-14
X(9)=	-1.62401762	F(9)=	-.1943D-15
X(10)=	.316543069	F(10)=	.4302D-15
X(11)=	.296866716	F(11)=	.6523D-15
X(12)=	.215266655	F(12)=	-.8327D-16
THPP=	.355326329D-01		
THPPP=	-.484322897D-01		

P= .24603D-21

A11=	-.126622017	B11=	.191044058
C11=	.126622017	D11=	-.191044058
A12=	-.483701214	B12=	.178190158D-01
C12=	.483701214	D12=	-.178190158D-01

*** END SECTION RESULTS *** : PH11= 3.81729022

Z	AZ	U1	U2	U1P	U2P
0.0	0.0	0.0	0.0	0.0	0.0
0.05	0.19086	-0.00208	-0.00876	-0.02055	-0.09144
0.10	0.38173	-0.00736	-0.03465	-0.03342	-0.17891
0.15	0.57259	-0.01432	-0.07660	-0.03813	-0.25923
0.20	0.76346	-0.02138	-0.13297	-0.03453	-0.32950
0.25	0.95432	-0.02697	-0.20158	-0.02272	-0.38715
0.30	1.14519	-0.02956	-0.27982	-0.00316	-0.43009
0.35	1.33605	-0.02773	-0.36472	0.02346	-0.45676
0.40	1.52692	-0.02022	-0.45308	0.05616	-0.46620
0.45	1.71778	-0.00598	-0.54156	0.09377	-0.45806
0.50	1.90865	0.01580	-0.62683	0.13490	-0.43263
0.55	2.09951	0.04565	-0.70567	0.17808	-0.39085
0.60	2.29037	0.08381	-0.77508	0.22172	-0.33422
0.65	2.48124	0.13022	-0.83243	0.26426	-0.26481
0.70	2.67210	0.18451	-0.87551	0.30413	-0.18513
0.75	2.86297	0.24605	-0.90263	0.33990	-0.09808
0.80	3.05383	0.31392	-0.91269	0.37025	-0.00683
0.85	3.24470	0.38697	-0.90518	0.39411	0.08533
0.90	3.43556	0.46389	-0.88027	0.41058	0.17503
0.95	3.62643	0.54319	-0.83873	0.41908	0.25902
1.00	3.81729	0.62334	-0.78195	0.41930	0.33425

Z	R	THETA	RP	THETAP
0.0	0.0	0.0	0.0	0.0
0.05	0.00901	-1.80367	0.09371	0.00110
0.10	0.03542	-1.77997	0.18195	0.00446
0.15	0.07793	-1.75555	0.26183	0.01014
0.20	0.13468	-1.73024	0.33080	0.01823
0.25	0.20337	-1.70382	0.38674	0.02882
0.30	0.28137	-1.67605	0.42804	0.04205
0.35	0.36577	-1.64668	0.45367	0.05802
0.40	0.45353	-1.61539	0.46323	0.07689
0.45	0.54159	-1.58184	0.45699	0.09882
0.50	0.62703	-1.54560	0.43589	0.12396
0.55	0.70714	-1.50620	0.40153	0.15248
0.60	0.77960	-1.46309	0.35612	0.18451
0.65	0.84256	-1.41563	0.30247	0.22016
0.70	0.89474	-1.36309	0.24387	0.25942
0.75	0.93557	-1.30467	0.18402	0.30213
0.80	0.96516	-1.23952	0.12688	0.34790
0.85	0.98443	-1.16680	0.07646	0.39592
0.90	0.99502	-1.08580	0.03657	0.44483
0.95	0.99927	-0.99608	0.01040	0.49256
1.00	1.00000	-0.89779	0.00000	0.53623

Z	M1	M2	MR	MT	M
0.0	-0.12662	-0.48370	-0.12662	-0.48370	0.50000

0.05	-0.08808	-0.47154	0.47914	0.02312	0.47969
0.10	-0.04634	-0.44225	0.44223	0.04650	0.44467
0.15	-0.00292	-0.39690	0.39068	0.07005	0.39691
0.20	0.04061	-0.33713	0.32640	0.09362	0.33957
0.25	0.08267	-0.26512	0.25181	0.11710	0.27771
0.30	0.12172	-0.18348	0.16968	0.14032	0.22018
0.35	0.15635	-0.09518	0.08305	0.16312	0.18304
0.40	0.18531	-0.00342	-0.00485	0.18527	0.18534
0.45	0.20753	0.08847	-0.09075	0.20654	0.22560
0.50	0.22221	0.17714	-0.17148	0.22661	0.28418
0.55	0.22883	0.25938	-0.24406	0.24510	0.34589
0.60	0.22713	0.33219	-0.30585	0.26153	0.40242
0.65	0.21719	0.39295	-0.35466	0.27531	0.44897
0.70	0.19936	0.43943	-0.38887	0.28569	0.48253
0.75	0.17428	0.46995	-0.40757	0.29174	0.50122
0.80	0.14288	0.48340	-0.41065	0.29234	0.50407
0.85	0.10629	0.47930	-0.39893	0.28614	0.49094
0.90	0.06583	0.45779	-0.37430	0.27167	0.46250
0.95	0.02299	0.41965	-0.33974	0.24742	0.42028
1.00	-0.02069	0.36627	-0.29930	0.21213	0.36686

Z	V1	V2	VR	VT	V
0.0	-0.19104	-0.01782	-0.19104	-0.01782	0.19187
0.05	-0.19104	-0.01782	0.06143	-0.18178	0.19187
0.10	-0.19104	-0.01782	0.05710	-0.18318	0.19187
0.15	-0.19104	-0.01782	0.05261	-0.18452	0.19187
0.20	-0.19104	-0.01782	0.04792	-0.18579	0.19187
0.25	-0.19104	-0.01782	0.04300	-0.18699	0.19187
0.30	-0.19104	-0.01782	0.03779	-0.18811	0.19187
0.35	-0.19104	-0.01782	0.03225	-0.18914	0.19187
0.40	-0.19104	-0.01782	0.02632	-0.19006	0.19187
0.45	-0.19104	-0.01782	0.01993	-0.19084	0.19187
0.50	-0.19104	-0.01782	0.01300	-0.19143	0.19187
0.55	-0.19104	-0.01782	0.00545	-0.19180	0.19187
0.60	-0.19104	-0.01782	-0.00282	-0.19185	0.19187
0.65	-0.19104	-0.01782	-0.01192	-0.19150	0.19187
0.70	-0.19104	-0.01782	-0.02196	-0.19061	0.19187
0.75	-0.19104	-0.01782	-0.03305	-0.18901	0.19187
0.80	-0.19104	-0.01782	-0.04529	-0.18645	0.19187
0.85	-0.19104	-0.01782	-0.05871	-0.18267	0.19187
0.90	-0.19104	-0.01782	-0.07330	-0.17732	0.19187
0.95	-0.19104	-0.01782	-0.08889	-0.17004	0.19187
1.00	-0.19104	-0.01782	-0.10515	-0.16049	0.19187

*** TRANS SECTION RESULTS *** : PHI2 = 5.24273130

Z	AZ	U1	U2	U1P	U2P
0.0	3.81729	0.62334	-0.78195	0.41930	0.33425
0.05	3.88856	0.65314	-0.75720	0.41665	0.36029

0.10	3.95983	0.68267	-0.73060	0.41164	0.38610
0.15	4.03111	0.71176	-0.70217	0.40429	0.41157
0.20	4.10238	0.74025	-0.67195	0.39466	0.43655
0.25	4.17365	0.76796	-0.63996	0.38279	0.46092
0.30	4.24492	0.79476	-0.60626	0.36873	0.48456
0.35	4.31619	0.82047	-0.57091	0.35256	0.50734
0.40	4.38747	0.84496	-0.53397	0.33436	0.52916
0.45	4.45874	0.86809	-0.49551	0.31422	0.54989
0.50	4.53001	0.88971	-0.45561	0.29225	0.56944
0.55	4.60128	0.90971	-0.41437	0.26856	0.58771
0.60	4.67255	0.92796	-0.37187	0.24326	0.60460
0.65	4.74383	0.94435	-0.32822	0.21649	0.62003
0.70	4.81510	0.95878	-0.28353	0.18838	0.63392
0.75	4.88637	0.97117	-0.23790	0.15908	0.64619
0.80	4.95764	0.98143	-0.19145	0.12873	0.65679
0.85	5.02892	0.98950	-0.14432	0.09748	0.66567
0.90	5.10019	0.99531	-0.09661	0.06551	0.67277
0.95	5.17146	0.99882	-0.04846	0.03296	0.67806
1.00	5.24273	1.00000	-0.00000	0.00000	0.68152

Z	R	THETA	RP	THETAP
0.0	1.00000	-0.89779	-0.00000	0.53623
0.05	0.99997	-0.85905	-0.00068	0.55082
0.10	0.99991	-0.81930	-0.00108	0.56437
0.15	0.99983	-0.77862	-0.00123	0.57692
0.20	0.99974	-0.73707	-0.00119	0.58850
0.25	0.99966	-0.69473	-0.00100	0.59914
0.30	0.99960	-0.65166	-0.00072	0.60889
0.35	0.99956	-0.60792	-0.00038	0.61781
0.40	0.99954	-0.56358	-0.00003	0.62594
0.45	0.99955	-0.51868	0.00030	0.63334
0.50	0.99959	-0.47327	0.00057	0.64006
0.55	0.99963	-0.42742	0.00078	0.64616
0.60	0.99969	-0.38115	0.00090	0.65170
0.65	0.99976	-0.33450	0.00094	0.65674
0.70	0.99983	-0.28752	0.00089	0.66131
0.75	0.99989	-0.24023	0.00076	0.66548
0.80	0.99993	-0.19266	0.00059	0.66929
0.85	0.99997	-0.14483	0.00039	0.67277
0.90	0.99999	-0.09676	0.00020	0.67595
0.95	1.00000	-0.04848	0.00006	0.67886
1.00	1.00000	-0.00000	0.00000	0.68152

Z	M1	M2	MR	MT	M
0.0	-0.02069	0.36627	-0.29930	0.21213	0.36686
0.05	-0.05385	0.36408	-0.31086	0.19703	0.36804
0.10	-0.08673	0.36004	-0.32229	0.18244	0.37034
0.15	-0.11918	0.35418	-0.33358	0.16843	0.37369
0.20	-0.15102	0.34651	-0.34472	0.15506	0.37799
0.25	-0.18209	0.33708	-0.35568	0.14238	0.38312

0.30	-0.21224	0.32595	-0.36644	0.13043	0.38896
0.35	-0.24132	0.31316	-0.37694	0.11922	0.39535
0.40	-0.26916	0.29877	-0.38715	0.10878	0.40214
0.45	-0.29564	0.28287	-0.39699	0.09911	0.40917
0.50	-0.32062	0.26554	-0.40641	0.09021	0.41630
0.55	-0.34397	0.24686	-0.41535	0.08206	0.42338
0.60	-0.36558	0.22692	-0.42375	0.07465	0.43028
0.65	-0.38532	0.20583	-0.43154	0.06792	0.43685
0.70	-0.40312	0.18369	-0.43866	0.06184	0.44300
0.75	-0.41886	0.16063	-0.44505	0.05636	0.44860
0.80	-0.43248	0.13674	-0.45066	0.05141	0.45358
0.85	-0.44390	0.11217	-0.45544	0.04693	0.45785
0.90	-0.45307	0.08702	-0.45936	0.04284	0.46135
0.95	-0.45994	0.06143	-0.46237	0.03907	0.46402
1.00	-0.46447	0.03553	-0.46447	0.03553	0.46582

Z	V1	V2	VR	VT	V
0.0	0.04708	-0.31654	0.27687	-0.16049	0.32003
0.05	0.04708	-0.31654	0.27045	-0.17110	0.32003
0.10	0.04708	-0.31654	0.26343	-0.18171	0.32003
0.15	0.04708	-0.31654	0.25583	-0.19228	0.32003
0.20	0.04708	-0.31654	0.24762	-0.20273	0.32003
0.25	0.04708	-0.31654	0.23882	-0.21303	0.32003
0.30	0.04708	-0.31654	0.22942	-0.22312	0.32003
0.35	0.04708	-0.31654	0.21945	-0.23294	0.32003
0.40	0.04708	-0.31654	0.20890	-0.24244	0.32003
0.45	0.04708	-0.31654	0.19781	-0.25157	0.32003
0.50	0.04708	-0.31654	0.18619	-0.26029	0.32003
0.55	0.04708	-0.31654	0.17406	-0.26855	0.32003
0.60	0.04708	-0.31654	0.16145	-0.27631	0.32003
0.65	0.04708	-0.31654	0.14840	-0.28354	0.32003
0.70	0.04708	-0.31654	0.13492	-0.29020	0.32003
0.75	0.04708	-0.31654	0.12105	-0.29625	0.32003
0.80	0.04708	-0.31654	0.10682	-0.30167	0.32003
0.85	0.04708	-0.31654	0.09228	-0.30643	0.32003
0.90	0.04708	-0.31654	0.07745	-0.31051	0.32003
0.95	0.04708	-0.31654	0.06237	-0.31389	0.32003
1.00	0.04708	-0.31654	0.04708	-0.31654	0.32003

*** CONTACT SECTION RESULTS ***

Z	AZ	THETA	THETAP	THETAPP	THETAPP
0.0	5.24273	0.0	0.68152	0.03553	-0.04843
0.10	5.34273	0.06832	0.68484	0.03099	-0.04245
0.20	5.44273	0.13695	0.68774	0.02702	-0.03716
0.30	5.54273	0.20586	0.69026	0.02354	-0.03250
0.40	5.64273	0.27499	0.69246	0.02050	-0.02839
0.50	5.74273	0.34434	0.69437	0.01785	-0.02478
0.60	5.84273	0.41386	0.69604	0.01553	-0.02162

0.70	5.94273	0.48354	0.69749	0.01751	-0.01885
0.80	6.04273	0.55335	0.69875	0.01175	-0.01642
0.90	6.14273	0.62328	0.69985	0.01022	-0.01430
1.00	6.24273	0.69332	0.70080	0.00888	-0.01245
1.10	6.34273	0.76344	0.70163	0.00772	-0.01083
1.20	6.44273	0.83364	0.70235	0.00671	-0.00942
1.30	6.54273	0.90391	0.70297	0.00583	-0.00819
1.40	6.64273	0.97423	0.70352	0.00506	-0.00712
1.50	6.74273	1.04461	0.70399	0.00440	-0.00619
1.60	6.84273	1.11503	0.70440	0.00382	-0.00538
1.70	6.94273	1.18549	0.70476	0.00332	-0.00468
1.80	7.04273	1.25598	0.70507	0.00288	-0.00406
1.90	7.14273	1.32650	0.70533	0.00250	-0.00353
2.00	7.24273	1.39704	0.70557	0.00217	-0.00307
2.10	7.34273	1.46761	0.70577	0.00189	-0.00266
2.20	7.44273	1.53820	0.70595	0.00164	-0.00231
2.30	7.54273	1.60880	0.70610	0.00142	-0.00201
2.40	7.64273	1.67942	0.70623	0.00124	-0.00175
2.50	7.74273	1.75004	0.70635	0.00107	-0.00152
2.60	7.84273	1.82068	0.70645	0.00093	-0.00132
2.70	7.94273	1.89133	0.70653	0.00081	-0.00114
2.80	8.04273	1.96199	0.70661	0.00070	-0.00099
2.90	8.14273	2.03266	0.70668	0.00061	-0.00086
3.00	8.24273	2.10333	0.70673	0.00053	-0.00075
3.10	8.34273	2.17400	0.70678	0.00046	-0.00065
3.20	8.44273	2.24468	0.70682	0.00040	-0.00056
3.30	8.54273	2.31537	0.70686	0.00035	-0.00049
3.40	8.64273	2.38605	0.70689	0.00030	-0.00043
3.50	8.74273	2.45674	0.70692	0.00026	-0.00037
3.60	8.84273	2.52744	0.70695	0.00023	-0.00032
3.70	8.94273	2.59813	0.70697	0.00020	-0.00028
3.80	9.04273	2.66883	0.70699	0.00017	-0.00024
3.90	9.14273	2.73953	0.70700	0.00015	-0.00021
4.00	9.24273	2.81023	0.70702	0.00013	-0.00018
4.10	9.34273	2.88093	0.70703	0.00011	-0.00016
4.20	9.44273	2.95164	0.70704	0.00010	-0.00014
4.30	9.54273	3.02234	0.70705	0.00008	-0.00012
4.40	9.64273	3.09305	0.70705	0.00007	-0.00010
4.50	9.74273	3.16375	0.70706	0.00006	-0.00009
4.60	9.84273	3.23446	0.70707	0.00005	-0.00008
4.70	9.94273	3.30517	0.70707	0.00005	-0.00007
4.80	10.04273	3.37587	0.70708	0.00004	-0.00006
4.90	10.14273	3.44658	0.70708	0.00004	-0.00005
5.00	10.24273	3.51729	0.70708	0.00003	-0.00005
5.10	10.34273	3.58800	0.70709	0.00003	-0.00004
5.20	10.44273	3.65871	0.70709	0.00002	-0.00003
5.30	10.54273	3.72942	0.70709	0.00002	-0.00003
5.40	10.64273	3.80013	0.70709	0.00002	-0.00003
5.50	10.74273	3.87083	0.70709	0.00001	-0.00002
5.60	10.84273	3.94154	0.70710	0.00001	-0.00002
5.70	10.94273	4.01225	0.70710	0.00001	-0.00002
5.80	11.04273	4.08296	0.70710	0.00001	-0.00002

5.90	11.14273	4.15367	0.70710	0.00001	-0.00002
6.00	11.24273	4.22438	0.70710	0.00000	-0.00001

Z	U1	U2	MR	M	W
0.0	1.00000	0.0	-0.46447	0.46582	0.12050
0.10	0.99767	0.06827	-0.46901	0.47003	0.13563
0.20	0.99064	0.13653	-0.47298	0.47375	0.14922
0.30	0.97889	0.20441	-0.47646	0.47704	0.16138
0.40	0.96243	0.27154	-0.47950	0.47994	0.17220
0.50	0.94130	0.33757	-0.48215	0.48248	0.18180
0.60	0.91557	0.40215	-0.48447	0.48472	0.19029
0.70	0.88535	0.46492	-0.48649	0.48668	0.19779
0.80	0.85077	0.52554	-0.48825	0.48839	0.20439
0.90	0.81197	0.58370	-0.48978	0.48989	0.21018
1.00	0.76913	0.63909	-0.49112	0.49120	0.21527
1.10	0.72246	0.69141	-0.49228	0.49234	0.21972
1.20	0.67219	0.74038	-0.49329	0.49334	0.22362
1.30	0.61855	0.78575	-0.49417	0.49421	0.22703
1.40	0.56180	0.82727	-0.49494	0.49496	0.23000
1.50	0.50224	0.86473	-0.49560	0.49562	0.23260
1.60	0.44015	0.89792	-0.49618	0.49619	0.23486
1.70	0.37585	0.92668	-0.49668	0.49669	0.23684
1.80	0.30964	0.95085	-0.49712	0.49713	0.23855
1.90	0.24188	0.97031	-0.49750	0.49750	0.24005
2.00	0.17288	0.98494	-0.49783	0.49783	0.24135
2.10	0.10300	0.99468	-0.49811	0.49812	0.24249
2.20	0.03259	0.99947	-0.49836	0.49836	0.24347
2.30	-0.03799	0.99928	-0.49858	0.49858	0.24433
2.40	-0.10841	0.99411	-0.49876	0.49877	0.24507
2.50	-0.17829	0.98398	-0.49893	0.49893	0.24572
2.60	-0.24730	0.96894	-0.49907	0.49907	0.24628
2.70	-0.31508	0.94907	-0.49919	0.49919	0.24677
2.80	-0.38129	0.92445	-0.49930	0.49930	0.24720
2.90	-0.44561	0.89523	-0.49939	0.49939	0.24756
3.00	-0.50771	0.86153	-0.49947	0.49947	0.24789
3.10	-0.56728	0.82352	-0.49954	0.49954	0.24816
3.20	-0.62403	0.78140	-0.49960	0.49960	0.24841
3.30	-0.67766	0.73538	-0.49965	0.49965	0.24862
3.40	-0.72790	0.68568	-0.49970	0.49970	0.24880
3.50	-0.77452	0.63256	-0.49974	0.49974	0.24896
3.60	-0.81726	0.57627	-0.49977	0.49977	0.24909
3.70	-0.85593	0.51710	-0.49980	0.49980	0.24921
3.80	-0.89031	0.45535	-0.49983	0.49983	0.24932
3.90	-0.92026	0.39132	-0.49985	0.49985	0.24941
4.00	-0.94560	0.32533	-0.49987	0.49987	0.24948
4.10	-0.96622	0.25772	-0.49989	0.49989	0.24955
4.20	-0.98201	0.18881	-0.49990	0.49990	0.24961
4.30	-0.99290	0.11897	-0.49992	0.49992	0.24966
4.40	-0.99882	0.04853	-0.49993	0.49993	0.24971
4.50	-0.99975	-0.02216	-0.49994	0.49994	0.24974
4.60	-0.99569	-0.09273	-0.49994	0.49994	0.24978

4.70	-0.98665	-0.16285	-0.49995	0.49995	0.24981
4.80	-0.97268	-0.23214	-0.49996	0.49996	0.24983
4.90	-0.95385	-0.30028	-0.49996	0.49996	0.24985
5.00	-0.93025	-0.36692	-0.49997	0.49997	0.24987
5.10	-0.90200	-0.43173	-0.49997	0.49997	0.24989
5.20	-0.86925	-0.49437	-0.49998	0.49998	0.24990
5.30	-0.83215	-0.55455	-0.49998	0.49998	0.24991
5.40	-0.79089	-0.61196	-0.49998	0.49998	0.24992
5.50	-0.74568	-0.66630	-0.49998	0.49998	0.24993
5.60	-0.69674	-0.71732	-0.49999	0.49999	0.24994
5.70	-0.64432	-0.76476	-0.49999	0.49999	0.24995
5.80	-0.58868	-0.80837	-0.49999	0.49999	0.24995
5.90	-0.53010	-0.84794	-0.49999	0.49999	0.24996
6.00	-0.46886	-0.88327	-0.49999	0.49999	0.24996

Z	VR	VF	V
0.0	0.07265	-0.31654	0.32477
0.10	0.06368	-0.32119	0.32745
0.20	0.05575	-0.32529	0.33003
0.30	0.04875	-0.32888	0.33247
0.40	0.04259	-0.33203	0.33475
0.50	0.03718	-0.33479	0.33685
0.60	0.03243	-0.33721	0.33877
0.70	0.02827	-0.33932	0.34050
0.80	0.02463	-0.34117	0.34205
0.90	0.02145	-0.34277	0.34344
1.00	0.01867	-0.34418	0.34468
1.10	0.01625	-0.34540	0.34578
1.20	0.01413	-0.34646	0.34675
1.30	0.01229	-0.34739	0.34761
1.40	0.01069	-0.34820	0.34836
1.50	0.00929	-0.34890	0.34902
1.60	0.00807	-0.34951	0.34960
1.70	0.00702	-0.35004	0.35011
1.80	0.00610	-0.35050	0.35055
1.90	0.00530	-0.35090	0.35094
2.00	0.00460	-0.35125	0.35128
2.10	0.00400	-0.35155	0.35158
2.20	0.00347	-0.35182	0.35183
2.30	0.00301	-0.35204	0.35206
2.40	0.00262	-0.35224	0.35225
2.50	0.00227	-0.35242	0.35242
2.60	0.00197	-0.35257	0.35257
2.70	0.00171	-0.35270	0.35270
2.80	0.00149	-0.35281	0.35281
2.90	0.00129	-0.35291	0.35291
3.00	0.00112	-0.35299	0.35299
3.10	0.00097	-0.35307	0.35307
3.20	0.00085	-0.35313	0.35313
3.30	0.00073	-0.35319	0.35319
3.40	0.00064	-0.35323	0.35324

3.50	0.00055	-0.35328	0.35328
3.60	0.00048	-0.35331	0.35331
3.70	0.00042	-0.35334	0.35334
3.80	0.00036	-0.35337	0.35337
3.90	0.00031	-0.35340	0.35340
4.00	0.00027	-0.35342	0.35342
4.10	0.00024	-0.35343	0.35343
4.20	0.00021	-0.35345	0.35345
4.30	0.00018	-0.35346	0.35346
4.40	0.00015	-0.35348	0.35348
4.50	0.00013	-0.35349	0.35349
4.60	0.00012	-0.35349	0.35349
4.70	0.00010	-0.35350	0.35350
4.80	0.00009	-0.35351	0.35351
4.90	0.00008	-0.35351	0.35351
5.00	0.00007	-0.35352	0.35352
5.10	0.00006	-0.35352	0.35352
5.20	0.00005	-0.35353	0.35353
5.30	0.00004	-0.35353	0.35353
5.40	0.00004	-0.35353	0.35353
5.50	0.00003	-0.35354	0.35354
5.60	0.00003	-0.35354	0.35354
5.70	0.00002	-0.35354	0.35354
5.80	0.00002	-0.35354	0.35354
5.90	0.00001	-0.35354	0.35354
6.00	0.00001	-0.35354	0.35354

*

APPENDIX XII

Two-Dimensional Analogy

The 2-dimensional analogy of the problem discussed in Section 2 is shown in Figure XII-1. Nomenclature is the same as in Section 2 except that u_{ij} is replaced by u , ϕ_1 is replaced by ϕ , V_1 is replaced by V , and M_1 is replaced by M .

The governing differential equation is

$$u'''' + u'' = 0,$$

which has as its solution

$$u = a + b\zeta + c \cos \zeta + d \sin \zeta.$$

The boundary conditions are

$$\begin{aligned} u(0) &= 0 & u(\phi) &= 1 \\ u'(0) &= 0 & u'(\phi) &= 0 \\ u''(0) &= 0 \end{aligned}$$

Application of these boundary conditions gives

$$\begin{aligned} \phi &= \pi \\ a &= 0 \\ b &= 1/\pi \\ c &= 0 \\ d &= -1/\pi. \end{aligned}$$

Figures XII-2 and XII-3 show the deflection, u , and moment, $M = u''$, curves. A discontinuity of the shear force, $V = -(u''' + u')$, occurs at $\zeta = \phi$ since $V = 0$ for $\zeta > \phi$. The magnitude of the associated concentrated force is $V = 1/\pi = .3183$.

Other end conditions at $\zeta = \phi$ can be investigated by applying the boundary conditions

$$\begin{aligned} u(0) &= 0 & u(\phi) &= 1 \\ u'(0) &= 0 \\ u''(0) &= 0 & u'(\phi) &= s \end{aligned}$$

where s is a specified slope at the end. Setting $s = 0$ results in a fixed end condition. The end slope for the pinned-end case is $s = 2/\pi$. By selecting values of s such that

$$0 \leq s \leq 2/\pi,$$

the effect of end-fixity on rod behavior can be investigated in two dimensions.

Application of the above boundary conditions shows that ϕ is determined from the solution of

$$f(\phi) = s(\phi - \sin \phi) + \cos \phi - 1 = 0$$

and that

$$a = c = 0$$

$$b = -d = \frac{1}{\phi - \sin \phi}.$$

Results for

$$s = 2/\pi = 0.63662 \quad (\text{pinned-end case}),$$

$$s = 1/\pi = 0.31831,$$

$$s = 0.05,$$

$$s = 0 \quad (\text{fixed-end case}),$$

are presented in Figures XII-2 and XII-3. The maximum bending moment always occurs at $\zeta = \pi/2$ and is shown in Figure XII-4 as a function of end slope. It is apparent that the largest maximum moment occurs for the pinned-end case. Any increased end fixity reduces the maximum bending moment.

Figure XII-1 Two-Dimensional Analogy

Figure XII-2 Two-Dimensional Rod Displacement Results

Figure XII-3 Two-Dimensional Internal Bending Moment Results

Figure XII-4 Maximum Bending Moment For Various End Fixity Conditions