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MARK DUANE STEWART

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Irregularities of Uniform Distribution

§1. Introduction. Let X be the unit interval with endpoints identified (a group via addition mod 1). Quantities greater than 1 will be considered as lying in X upon reducing them mod 1. Now fix θ irrational in X and $I \subseteq X$ any subinterval, and define $S_n(\theta, I) =$

$$\sum_{j=0}^{n-1} \chi_I(j\theta). \quad \text{By the Kronecker-Weyl theorem, } \lim_{n \rightarrow \infty} n^{-1}(S_n(\theta, I-x)) = |I|,$$

where $|\cdot|$ denotes the length of an interval, and the limit is uniform in x . We say $\{j\theta\}_{j \geq 1}$ is uniformly distributed mod 1 (u.d. mod 1). The Weyl Criterion (effectively an approximation argument) then shows that for any f in $C(X)$ and x in X , $\{n^{-1} \sum_{j=0}^{n-1} f(x+j\theta)\}$ converges to

$\int_X f(y) \mu(dy)$, where μ denotes Haar measure on X .

We point out a consequence of the u.d. mod 1 of $\{j\theta\}$. Define the rotation $R_\theta: X \rightarrow X$ by $R_\theta(x) = x + \theta$, and let $\mathcal{B}$ be the Borel field on X . Another approximation argument reveals that the only R_θ -invariant functions in $L^1(X, \mathcal{B}, \mu)$ are the constants. Therefore if $A \in \mathcal{B}$, then $R_\theta A = A$ forces $\mu(A) = 0$ or $\mu(A^c) = 0$. We then say that R_θ is ergodic with respect to μ . R_θ is then irreducible in the sense that X does not decompose into measurable R_θ -invariant subsets in a nontrivial way. The reason that the ergodicity of R_θ is weaker than the u.d. mod 1 of $\{j\theta\}$ is that the u.d. mod 1 forces μ to be the only R_θ -invariant Borel probability measure on X . For let ν also be R_θ -invariant, and pick $f \in C(X)$. Then $\nu(f) = \int f(x) d\nu = \int f(x+j\theta) d\nu$ for all $j \in \mathbb{N}$. Therefore

$\nu(f) = \int \lim_{n \rightarrow \infty} (n^{-1} \sum_{j=0}^{n-1} f(x+j\theta)) d\nu = \int (\mu(f)) d\nu = \mu(f)$. As a result, R_θ is ergodic with respect to μ , and therefore R_θ is termed uniquely ergodic.

The results of this paper have to do with the u.d. mod 1 of sequences arising from $\{S_n(\theta, I-x)\}$ and the ergodicity with respect to Haar measure (but not the unique ergodicity) of corresponding transformations called skew products, of which we shall give examples later.

To start, take α and β in $\mathbb{R}$ and consider $\{S_n \alpha - n\beta\}$, where, from now on, S_n denotes $S_n(\theta, I-x)$. Let $\exp(2\pi i y)$ be denoted $e(y)$. If we define $f: X \rightarrow \mathbb{C}$ by $f(x) = e(\alpha \chi_I(x) - \beta)$ and $f^{(n)}(x) = \prod_{i=0}^{n-1} f(x+i\theta)$, then $f^{(n)}(x) = e(S_n \alpha - n\beta)$. Choosing, say, $\beta = 0$, we have a function $f_0(x) = e(\alpha \chi_I(x))$ with $f_0^{(n)}(x) = e(S_n \alpha)$. Finally, let $\alpha_n = \mu(f^{(n)})$, $\alpha_{n,0} = \mu(f_0^{(n)})$, and define

$$(1.1) \quad \Gamma_\alpha(\theta) = \{t = |I| : \lim_{|n\theta| \rightarrow 0} |\alpha_{n,0}| = 1\}.$$

Clearly any fixed β would have given rise to the same $\Gamma_\alpha(\theta)$. Also, the location of I in X is immaterial in (1.1); only $t = |I|$ matters.

The motivation for this definition is the following: Suppose for some x in X that

$$(1.2) \quad \lim_{N \rightarrow \infty} N^{-1} \sum_{n=1}^N e(S_n \alpha - n\beta) = 0$$

fails. Then two things are true: i) $\{S_n \alpha - n\beta\}$ is not u.d. mod 1 -- this follows from the Weyl Criterion; ii) there is a nontrivial measurable function h on X with $h(x) = f(x)h(x+\theta)$. (Part ii) actually is equivalent to the failure of (1.2) for some x ; see, e.g., [12, p.779].) Since $|h(x)| = |h(x+\theta)|$, the ergodicity of R_θ implies that $|h|$ is constant; let that constant be 1. If $*$ denotes complex conjugation, we

have $h(x)h^*(x+\theta) = f(x)$ and $h(x)h^*(x+n\theta) = f^{(n)}(x)$ for $n > 0$. Then by the continuity of L^2 translation

$$(1.3) \quad \lim_{||n\theta|| \rightarrow 0} \alpha_n = 1,$$

with the consequence that

$$(1.4) \quad \lim_{||n\theta|| \rightarrow 0} |\alpha_n| = 1,$$

which is equivalent to (1.1). That (1.3) is equivalent to $h(x) = f(x)h(x+\theta)$ will be shown in Section 4.

It has long been known that the analysis of θ as a continued fraction could unlock the mod 1 distribution properties of $\{j\theta\}$ and sequences related to $\{S_n\}$; e.g., see [3]. Such analysis was furthered in [10], helping discover, for instance, values of θ and t and the existence of intervals I of length t for which $\lim_{N \rightarrow \infty} \{N^{-1} \sum_{n=0}^{N-1} (S_n \bmod 2)\}$ fails even to exist. (Note that this is equivalent with the non-existence of the limit in (1.2) for $(\alpha, \beta) = (\frac{1}{2}, 0)$.) The first result herein describes the arithmetic nature of numbers t satisfying (1.1) using this expansion of θ .

Therefore, let $\theta = [\bar{a}_1, a_2, \dots]$ as a continued fraction, with convergents $[\bar{a}_1, a_2, \dots, a_k] = p_k/q_k$ in lowest terms. The set of partial quotients of θ is the set $\{a_i\}$, $i \geq 1$. Also let $||x|| = \min_n |n-x|$; $||\cdot||$ gives rise to the standard Euclidean metric on X . Then define

$$(1.5) \quad \Gamma^0(\theta) = \{t \in X: t = m\theta + \sum_{i=1}^{\infty} b_i q_i \theta, b_i \in \mathbb{Z}, \text{ and } \lim_{i \rightarrow \infty} b_i q_i ||q_i \theta|| = 0\},$$

$$(1.6) \quad \Gamma_{\alpha}^0(\theta) = \{t \in \Gamma^0(\theta): \lim_{i \rightarrow \infty} (b_i \alpha) \bmod 1 = 0\},$$

$$(1.7) \quad \Gamma_{\alpha}^1(\theta) = \{t \in \Gamma_{\alpha}^0(\theta) : \sum_{i=1}^{\infty} |b_i| q_i ||q_i \theta|| < \infty \text{ and } \sum_{i=1}^{\infty} ||b_i \alpha|| < \infty\}.$$

Convergence of the sum of (1.5) is understood with respect to $||\cdot||$.

This brings us to the first two results, proved for $\alpha = \frac{1}{2}$ in [10] and sketched for general α in [11].

Theorem 1. If θ is irrational, then $\Gamma_{\alpha}^1(\theta) \subseteq \Gamma_{\alpha}(\theta) \subseteq \Gamma_{\alpha}^0(\theta)$ for all $\alpha \in \mathbb{R} - \mathbb{Z}$.

Theorem 2. Suppose α is rational, $\alpha = p/q$ in lowest terms ($q \neq 1$), and let Δ be the group of complex q^{th} roots of unity. Also, define $f(x) = e(\alpha \chi_{[0,t)}(x))$. If $t \in \Gamma_{\alpha}^1(q\theta)$, then for some $\delta \in \Delta$ there is a measurable solution g (Δ -valued a.e.) to $g(x) = \delta f(x)g(x+\theta)$.

In particular, if we put $\alpha = 1/q$ and $\delta = e(+j/q)$ for a fixed j between 1 and q , then we have from (1.2) that $\{(S_n - nj) \bmod q\}$ does not assume all values between 1 and q with the relative frequency $1/q$ as n tends to ∞ . (Indeed, let θ have unbounded partial quotients, and suppose $t \in \Gamma_{\alpha}^1(q\theta) - \mathbb{Z}\theta$, $I = [0, t)$. Then there are an uncountable number of x such that the average value $\lim_{N \rightarrow \infty} N^{-1} \sum_{n=1}^N [(S_n(\theta, I-x) - nj) \bmod q]$ need not even exist; the proof is that of [10, Theorem 1]. It follows that, under these assumptions, $\lim_{N \rightarrow \infty} N^{-1} \sum_{n=1}^N (S_n \bmod q)$ does not exist either.)

Theorem 2 also has an ergodic-theoretic consequence. Form the space $Y = X \times \Delta$ and let $T: Y \rightarrow Y$ be the skew product $T(x, \gamma) = (x+\theta, \gamma \delta f(x))$, with δ and f as above; we have $T^{(n)}(x, \gamma) = (x+n\theta, \gamma(\delta f)^{(n)}(x))$. Then X has q ergodic components; that is, X decomposes into the q disjoint, measurable sets $\{(x, \eta g^*(x))\}_{x \in X}$, where η runs

through Δ , and on each of these sets T is ergodic. But again, if θ has unbounded partial quotients and $t \notin \mathbb{Z}\theta$, then the orbits under T are dense in X [10, p.6]. Thus we see that such minimal (every orbit dense) skew products can fail to be ergodic. See also [7] for this result.

We now state some corollaries of the right-hand inclusion of Theorem 1; these were noted previously in [12].

Theorem: For every irrational θ and $t = |I|$ with $t \notin \mathbb{Z}\theta$ it is true for almost all α that (1.2) holds for all β .

Proof. Let $A = \{\alpha: \text{for some } \beta \in \mathbb{R}, (1.2) \text{ fails}\}$. Then, if $\alpha \in A$, we have $\{(b_i \alpha) \bmod 1\}$ approaches 0, where the b_i come from (1.5) for t . But the set of α causing $\{(b_i \alpha) \bmod 1\}$ to tend to 0 has measure 0 by the Riemann-Lebesgue lemma.

Note that for almost all α the previous result is true for all $n\alpha$, $n \in \mathbb{Z} - \{0\}$. Using the Weyl Criterion, one obtains

Corollary: If $t \notin \mathbb{Z}\theta$, then for almost all α the sequence $\{S_n \alpha - n\beta\}$ is u.d. mod 1 for all β .

Setting $\beta = t\alpha$, we obtain

Theorem: If θ is irrational and $t \notin \mathbb{Z}\theta$, then for almost all α the sequence $\{(S_n - nt)\alpha\}$ is u.d. mod 1.

This result implies that for θ irrational and $t \notin \mathbb{Z}\theta$, the sequence $\{S_n - nt\}$ is unbounded. For if not, then there could not exist numbers α arbitrarily close to 0 for which $\{(S_n - nt)\alpha\}$ would be dense, much less u.d. mod 1.

Therefore we obtain a new proof of a result of Kesten:

Theorem [3]: Let θ be irrational, $I \subseteq X$ any interval. If there exists x in X such that $\{S_n(\theta, I-x) - n|I|\}$ is bounded, then $|I| \in \mathbb{Z}\theta$.

Kesten's theorem raises some ergodic-theoretic questions concerning certain skew products called cylinder flows, one of which questions is answered here. Again fix θ irrational and $I \subseteq X$ a subinterval of length t . Let E be the closed subgroup of $\mathbb{R}$ (not X) generated by $-t$ and $1-t$. Put $Y = X \times E$ and define the skew product $T: Y \rightarrow Y$ by $T(x, y) = (x + \theta, y + (\chi_I(x) - t))$. Then $T^n(x, y) = (x + n\theta, y + (S_n - nt))$, so that if $t \notin \mathbb{Z}\theta$, Kesten's theorem says T has no bounded orbits. Since T clearly preserves Haar measure on Y , it is therefore natural to ask whether T is ergodic. In general, this is false, but we do have the following result.

Theorem 4. Let θ have bounded partial quotients. i) If $t \notin \Gamma^0(\theta)$ and if $1, t$, and θ are rationally independent, then T is ergodic with respect to Haar measure on Y ; ii) if t is rational, then T is ergodic with respect to Haar measure on Y .

The proof follows that of [1, Theorem 4] in the main, but it also draws upon Schmidt's notion [8, Sects. 2, 3] of an essential value of a cocycle associated to an ergodic transformation. (In our context the cocycle is the "skewing function" $\chi_I(x) - t$, and the ergodic transformation is R_θ .) Also of importance in the proof is the following number-theoretic result. Recall that $\{q_k\}$ is the sequence of denominators of the convergents of θ .

Theorem 3. Let θ have bounded partial quotients. Then $t \notin \Gamma^0(\theta)$ if and only if $\limsup_{i \rightarrow \infty} (||q_i t|| - \frac{1}{2} q_i ||q_i \theta||) > 0$.

This result is itself of interest, for it comes as a bit of a surprise when compared with the following fact:

Proposition: Given a sequence $\{r_i\}$ of positive integers with

$r_{i+1}/r_i \geq 2/\lambda > 2$ for all i , there are an uncountable number of t in X with $||r_i t|| < \lambda/2$ for all i .

The proof, due essentially to T. Vijayaraghavan, is short and will be given. Let $\mu = \lambda/2$. First, every t in $[-\mu/r_1, \mu/r_1] \subseteq X$ satisfies $||r_1 t|| \leq \mu$. Now dilate $[-\mu/r_1, \mu/r_1]$ by r_2 , and note that its length (in $\mathbb{R}$, not in X) is greater than 2. Thus $[-\mu/r_1, \mu/r_1]$ contains two disjoint sets, each of which is either an interval or two intervals, and each of measure λ/r_2 , which are dilated onto $[-\mu, \mu]$ in X . Every t in these two sets satisfies $||r_i t|| \leq \mu$ for $i = 1, 2$. Next dilate the two sets by r_3 to obtain four sets of numbers t satisfying $||r_i t|| \leq \mu$ for $i = 1, 2, 3$ and continue indefinitely. As a result, the set of t satisfying $||r_i t|| \leq \mu$ for all i is uncountable.

Now from Theorem 3, suppose that θ has bounded partial quotients, so that $\Gamma^0(\theta) = \mathbb{Z}\theta$, and let denominators $\{q_{i,j}\}$ be chosen with $\lim_{j \rightarrow \infty} (||q_{i,j} t|| - \frac{1}{2} q_{i,j} ||q_{i,j} \theta||) = c > 0$. The $q_{i,j}$ can be chosen to satisfy $q_{i,j+1}/q_{i,j} \geq 2/\lambda > 2$ by taking every other one of them if necessary, and the quantities $\frac{1}{2} q_{i,j} ||q_{i,j} \theta||$ are bounded below by a positive number -- see Proposition 2.1 i) and vi). As a result, $q_{i,j} t$ must lie in $[-\lambda/2, -\varepsilon] \cup [\varepsilon, \lambda/2]$, all j , for some $\varepsilon > 0$ depending only on θ , and not on t .

Let us give the right-hand side of the equivalence in Theorem 3 the name condition A. It is motivated by conditions A and A' of [1]; there it is simply noted that for each θ the conditions are satisfied for almost all t .

Theorem 4 was previously known only in some special cases. For example, Schmidt dealt with $t = \frac{1}{2}$ and θ with bounded partial quotients in [9], while Conze in [1] treated $t = \frac{1}{2}$, θ irrational (and other cases via his conditions A, A').

§2. Continued Fractions I. X and θ are as in the introduction, and we have $\theta < 1$. Since $\theta^{-1} > 1$, we may extract the greatest integer a_1 in θ^{-1} . Then $\theta^{-1} - a_1$ is irrational and less than 1, so we may repeat the process of inverting and drawing out the greatest integer. Since θ is irrational, this process is infinite, and we write $\theta = [a_1, a_2, \dots]$ or $\theta = 1/a_1 + 1/a_2 + \dots$. Call a_n the n^{th} partial quotient of θ , and let the rational number $[a_1, a_2, \dots, a_n]$ be called the n^{th} convergent of θ , equal to p_n/q_n in lowest terms. For uniformity we adopt the following conventions: $p_0 = 0$ and $q_0 = 1$, $p_{-1} = 1$ and $q_{-1} = 0$; $||q_0\theta|| = \theta$ and $||q_{-1}\theta|| = 1$.

We then have the following relations for all $i \geq 1$; for proofs, see, e.g., [4].

Proposition 2.1. i) $p_i = a_i p_{i-1} + p_{i-2}$; $q_i = a_i q_{i-1} + q_{i-2}$.

ii) $\theta < p_i/q_i$, i odd; $\theta > p_i/q_i$, i even.

iii) $q_i\theta \in (0, \frac{1}{2})$, i even; $q_i\theta \in (\frac{1}{2}, 1)$, i odd. (We

will express this relationship by saying that $q_i\theta$ and $q_{i+1}\theta$ have opposite senses.)

$$\text{iv) } ||q_i\theta|| = |q_i\theta - p_i| = \min_{1 \leq q < q_{i+1}} ||q\theta||.$$

$$\text{v) } |\theta - p_i/q_i| < (q_i q_{i+1})^{-1}.$$

$$\text{vi) } q_i ||q_i\theta|| \in (1/(a_{i+1}+2), 1/a_{i+1});$$

$$q_{i+1} ||q_i\theta|| \in (\frac{1}{2}, 1).$$

vii) If $|q\theta - p| < 1/2q$ and $(p, q) = 1$, then

$$p = p_j \text{ and } q = q_j, \text{ some } j.$$

viii) If $v \in \{1, 2, \dots, a_i\}$, then $||vq_{i-1}\theta|| =$

$$v ||q_{i-1}\theta||.$$

$$\text{ix) If } v \in \{1, 2, \dots, a_i\}, \text{ then } ||(vq_{i-1} + q_{i-2})\theta|| = ||q_{i-2}\theta|| - v||q_{i-1}\theta||.$$

Since the following find use often in the sequel, we present their proofs.

$$\begin{aligned} \text{Proposition 2.2.} \quad \text{i) } \sum_{j=0}^{\infty} a_{i+2j+1} q_{i+2j} \theta &= -q_{i-1} \theta. \\ \text{ii) } \sum_{j=0}^{\infty} a_{i+2j+1} ||q_{i+2j} \theta|| &= ||q_{i-1} \theta||. \end{aligned}$$

Proof. From Proposition 2.1i) we have $a_{k+1}q_k = q_{k+1} - q_{k-1}$, so that $a_{i+2j+1}q_{i+2j}\theta = q_{i+2j+1}\theta - q_{i+2j-1}\theta$. Therefore the sum in i) telescopes to $-q_{i-1}\theta$, as desired. For ii) Proposition 2.1ix) implies that $||q_{k+1}\theta|| = ||(a_{k+1}q_k + q_{k-1})\theta|| = ||q_{k-1}\theta|| - a_{k+1}||q_k\theta||$. (Compare this with Proposition 2.1i).) Therefore $a_{i+2j+1}||q_{i+2j}\theta|| = ||q_{i+2j-1}\theta|| - ||q_{i+2j+1}\theta||$, so that the sum in ii) telescopes to $||q_{i-1}\theta||$.

$$\begin{aligned} \text{Proposition 2.3.} \quad \text{i) If } N = \sum_{i=0}^m c_i q_i \text{ is a positive integer with } c_i \text{ nonnegative integers for all } i, \text{ then } N\theta &= \sum_{\substack{i=0 \\ i \text{ even}}}^m c_i ||q_i \theta|| - \sum_{\substack{i=0 \\ i \text{ odd}}}^m c_i ||q_i \theta||. \end{aligned}$$

$$\text{ii) If } t = \sum_{i=r+1}^{\infty} b_i q_i \theta \text{ with } b_i \in \{0, 1, \dots, a_{i+1}\} \text{ for all } i, \text{ then } ||t|| \leq ||q_r \theta||.$$

Proof. From Proposition 2.1iii) and the definitions it follows

$$\begin{aligned} \text{that for all } i \geq 0, q_i \theta &= (-1)^i ||q_i \theta||. \text{ Therefore } N\theta = \sum_{i=0}^m c_i q_i \theta = \sum_{i=0}^m (-1)^i c_i ||q_i \theta||, \text{ which yields i). As for ii), we note that Proposition 2.1iii) also says that the contribution of successive terms in} \end{aligned}$$

$$\begin{aligned}
& \sum_{i=r}^{\infty} b_i q_i \theta, \text{ if nonzero, have opposite senses. Therefore } \left| \sum_{i=r}^{\infty} b_i q_i \theta \right| \leq \\
& \max \left(\sum_{j=1}^{\infty} b_{r+2j-1} |q_{r+2j-1} \theta|, \sum_{j=1}^{\infty} b_{r+2j} |q_{r+2j} \theta| \right) \\
& \leq \max \left(\sum_{j=1}^{\infty} a_{r+2j} |q_{r+2j-1} \theta|, \sum_{j=1}^{\infty} a_{r+2j+1} |q_{r+2j} \theta| \right) \\
& = \max (|q_r \theta|, |q_{r+1} \theta|) = |q_r \theta|.
\end{aligned}$$

Proposition 2.4. Each interval $[r/q_k, (r+1)/q_k)$, $0 \leq r < q_k$, contains an element of $\{i\theta\}$, $0 \leq i < q_k$.

Proof. From Proposition 2.1iv) we have that $|i\theta - (ip_k)/q_k| = (i/q_k) |q_k \theta|$, $0 \leq i < q_k$. Thus if k is even (resp. odd), the first q_k multiples of θ all sit just to the right (resp. left) of $\{i/q_k\}$. (This uses Proposition 2.1iii) and v); each of the multiples of θ are within $1/q_{k+1}$ of the nearest i/q_k .)

Corollary 2.5. i) The element of $\{i\theta\}$, $0 \leq i < q_k$, closest to j/q_k , $0 \leq j < q_k$, is $i_j \theta$, where $j \equiv i_j p_k \pmod{q_k}$.

$$\text{ii) } \sum_{i=0}^{q_k-1} |i\theta - (ip_k/q_k)| = \frac{1}{2}(q_k-1) |q_k \theta|.$$

Proposition 2.6. For $n \in \mathbb{Z}$, $||n\alpha|| = ||n^*|| |\alpha|$.

Proof. $||n^*|| |\alpha| = ||n(\min_{m \in \mathbb{Z}} |m-\alpha|)| = ||n(m_0-\alpha)|| =$

$$||\pm(nm_0 - n\alpha)|| = ||n\alpha||.$$

§3. $\Gamma_\alpha(\theta)$, $\Gamma^0(\theta)$, and the Canonical Representation

In this section we produce two infinite series representations for certain elements t of X . The first series appears in (1.5) and is valid under the assumption that t satisfies (1.4). The second series is valid for $t \notin \mathbb{Z}\theta$, and it is of a more elementary nature. Its primary use is in Theorem 3, but we shall note some similarities it bears with the former expansion.

Therefore let us assume first that t satisfies (1.4). The following was developed in [10], but the reader is warned that the notation there is quite different from ours.

Let $\{m_j\}$ be the sequence of (not necessarily positive) multiples of θ that best approximate $t \bmod 1$; i.e., $||m_j\theta - t|| = \min_{|m| < |m_{j+1}|} ||m\theta - t||$.

It follows from [10, Lemma 11] that for large j $m_{j+1} - m_j = \varepsilon_j q_{k_j}$, $\varepsilon_j = \pm 1$. Then if k is also large and q_k occurs in this fashion b_k times, the occurrences must be successive, say $q_k = q_{k_j} = q_{k_{j+1}} = \dots = q_{k_{j+b_k-1}}$, and $\varepsilon_j = \varepsilon_{j+1} = \dots = \varepsilon_{j+b_k-1}$.

Therefore we may write $t = m\theta + \sum_{j=j_0}^{\infty} \varepsilon_j b_j q_j \theta$. For $k \geq j_0$ let n_k denote

the quantity $m + \sum_{j=j_0}^k \varepsilon_j b_j q_j$. In particular, each n_k is some m_j . It also follows that if δ is preassigned and positive, then there is an index j_1 such that if $j > j_1$ and $b_{j+1} \neq 0$, then $|n_j| < \delta q_{j+1}$. Let $k > j$ be the next index for which $b_i \neq 0$. Then also $q_k < |n_k| < \delta q_{k+1}$, and $|n_k|/b_k q_k = |1 + n_j/b_k q_k|$, where $|n_j/b_k q_k| < \delta(q_{j+1}/q_k) < \delta$.

Therefore $\lim_{k \rightarrow \infty, b_k \neq 0} |n_k/b_k q_k| = 1$. We also have $\{|n_k|/q_k \theta\} \rightarrow 0$; as a

result,

$$(3.1) \quad \lim_{k \rightarrow \infty} b_k q_k ||q_k \theta|| = 0.$$

Therefore, to prove the right-hand inclusion of Theorem 1, it is sufficient to prove that $\{(b_k \alpha) \bmod 1\}$ tends to 0.

We have that $\Gamma_\alpha(\theta) \subseteq \Gamma^0(\theta)$. In [10] it was also noted that i) $\Gamma_\alpha(\theta)$ is a subgroup of X (Lemma 13; the proof is for $\alpha = \frac{1}{2}$, but works in general), and ii) any two expansions for t , say, $t = m_1 \theta + \sum_{i=1}^{\infty} b_i q_i \theta$ and $t = m_2 \theta + \sum_{i=1}^{\infty} c_i q_i \theta$ with $\lim_{i \rightarrow \infty} b_i q_i ||q_i \theta|| = \lim_{i \rightarrow \infty} c_i q_i ||q_i \theta|| = 0$ satisfy $b_i = c_i$ for all $i \geq i_0$, some i_0 (Lemma 14).

This series representation for $t \in \Gamma^0(\theta)$ enables us to prove

Lemma 3.2. If $t \in \Gamma^0(\theta)$, then $\lim_{k \rightarrow \infty} ||q_k t|| = 0$.

Proof. Write $t = m\theta + \sum_{i=1}^{\infty} b_i q_i \theta$ as in (1.5), and pick $\delta > 0$. Let k_0 be so large that $k \geq k_0$ forces $|b_k| q_k ||q_k \theta|| < \delta$ and $|n_k| < \delta q_{k+1}$. Then $||q_k t|| = ||n_k q_k \theta + q_k \sum_{i=k}^{\infty} b_i q_i \theta|| \leq ||n_k q_k \theta|| + ||q_k \sum_{i=k}^{\infty} b_i q_i \theta|| \leq || |n_k| \cdot ||q_k \theta|| || + q_k (\sum_{i=k}^{\infty} |b_i| \cdot ||q_i \theta||) \leq \delta + \delta(1 + q_k/q_{k+1} + \dots)$. Now since $q_{i+2} > 2q_i$ by Proposition 2.1i), the last series in parenthesis splits into the series $\sum_{i=1}^{\infty} q_k/q_{k+2i-1}$ and $\sum_{i=0}^{\infty} q_k/q_{k+2i}$, each of which is dominated by 2. Therefore $||q_k t|| \leq 5\delta$, and the lemma is proved.

Note that this lemma proves the "if" statement of Theorem 3. We are left, then, with proving that $t \notin \Gamma^0(\theta)$ satisfies condition A. To this end we introduce another infinite series representation. Now, instead of assuming $t \in \Gamma_\alpha(\theta)$, we assume $t \notin \mathbb{Z}\theta$.

Lemma 3.3. One may expand any positive integer N as

$$(3.4) \quad N = \sum_{i=0}^m c_i q_i,$$

where m depends on N , and for all j we have i) $0 \leq c_j \leq a_{j+1}$ (we also require $0 \leq c_0 < a_1$) and ii) if $c_{j+1} = a_{j+2}$, then $c_j = 0$. If we also require $c_m > 0$, the expansion (3.4) is unique.

Proof. $N = 1$ may be written as $1q_0$, and any $N < q_1$ may be written as Nq_0 . $N = q_1$ may be written as $1q_1$, and these are all obviously unique.

Now we claim that for all j no number $N \geq q_j$ can be expanded as

$$\sum_{i=0}^{j-1} c_i q_i \text{ subject to the given conditions. By the preceding paragraph,}$$

we are done for $j=1$. Assume now the claim is true for all $j \leq k$. Then

$$\begin{aligned} \sum_{i=0}^k c_i q_i &\leq \max(a_{k+1} q_k + \max \sum_{i=0}^{k-2} c_i q_i, (a_{k+1}-1) q_k + \max \sum_{i=0}^{k-1} c_i q_i) \\ &= \max(a_{k+1} q_k + (q_{k-1}-1), a_{k+1} q_k - 1) \\ &= q_{k+1} - 1. \end{aligned}$$

With the claim established, we show that the expansion (3.4) exists and is unique for $q_j \leq N < q_{j+1}$, all $j \geq 0$. We are done for $j=0$.

Suppose we know the result for $j \leq k$, and let $q_k < N \leq q_{k+1}$. If $N = q_{k+1}$, then in view of the above claim, the unique expansion is $1q_{k+1}$, so we may assume that $q_k < N < q_{k+1}$. Write $N = N_1 q_k + N_2$ with N_1 and N_2 nonnegative, and N_1 maximal. Then $N_2 < q_k$, so let its unique expansion

$$\text{be } N_2 = \sum_{i=0}^{k-1} c_i q_i. \text{ Then a valid expansion for } N \text{ is } N_1 q_k + \sum_{i=0}^{k-1} c_i q_i. \text{ It}$$

$$\text{is also unique, since if } N = N_0 q_k + \sum_{i=0}^{k-1} d_i q_i \text{ with } N_0 < N_1, \text{ then } \sum_{i=0}^{k-1} d_i q_i$$

$= M$ is a valid expansion for $M \geq q_k$, which is a contradiction. The lemma is proved.

In (3.4) let $c_i = c_i(N)$, and extend to all nonnegative integers by defining $c_i(N) = 0$, $i > m$. Then for every positive integer N we have a well-defined infinite sequence of nonnegative integers, of which only a finite number are positive. If $N \neq M$, the sequences $\{c_i(N)\}$, $\{c_i(M)\}$ are distinct. Now define functions b_i on $\mathbb{Z}^+\theta$ by setting $b_i(N\theta) = c_i(N)$.

Lemma 3.5. If $t \notin \mathbb{Z}\theta$, then $\lim_{N\theta \rightarrow t} b_i(N\theta) = b_i(t)$ exists. If $b_{i+1}(t) = a_{i+2}$, then $b_i(t) = 0$.

Proof. We induct on i . First define $M(c) = \{N\theta : c_0(N) = c\}$, $0 \leq c \leq a_1 - 1$. Then

$$M(0) \subseteq I(0) = \left(\sum_{i=1}^{\infty} a_{2i} q_{2i-1} \theta, \sum_{i=1}^{\infty} a_{2i+1} q_{2i} \theta \right) = (-q_0 \theta, -q_1 \theta)$$

$$M(c) \subseteq I(c) = \left(|cq_0 + (a_2 - 1)q_1| \theta + \sum_{i=2}^{\infty} a_{2i} q_{2i-1} \theta, cq_0 \theta + \sum_{i=1}^{\infty} a_{2i+1} q_{2i} \theta \right)$$

$$= ((-q_1 + (c-1)q_0) \theta, (-q_1 + cq_0) \theta), \quad 1 \leq c \leq a_1 - 1,$$

where $I(0) = X$ if $a_1 = 1$. The estimates for the endpoints of $I(c)$ are greatest possible in view of Proposition 2.3i), and the infinite sums are evaluated by Proposition 2.2i). $I(0)$ contains 0, and moving from left to right across $[0, 1)$ we encounter $I(1), I(2), \dots, I(a_1 - 1)$ in order. That the right endpoint of $I(a_1 - 1)$ is the left endpoint of $I(0)$ is clear from Proposition 2.1i). Thus the $I(c)$ are disjoint for $0 \leq c \leq$

$a_1 - 1$, and $\bigcup_{i=0}^{a_1-1} \overline{I(c)} = X$. Finally the common endpoints of the $I(c)$ are all in $\mathbb{Z}\theta$. Therefore $\lim_{N\theta \rightarrow t} b_0(N\theta)$ must exist, for if $\{N_i\theta\}$ and $\{M_i\theta\}$

converge to t and $b_0(M_i) \equiv m$, $b_0(N_i) \equiv n$ with $m \neq n$, then t must be an endpoint of an $I(c)$, contradicting $t \notin \mathbb{Z}\theta$.

To see that $\lim_{N\theta \rightarrow t} b_1(N\theta)$ must exist, define $M(c^{(0)}, c^{(1)}) =$

$\{N\theta : c_i(N) = c^{(i)}, i = 0, 1\}$. Immediately we have $M(c^{(0)}) =$

$$\frac{\lfloor a_2 \rfloor}{c^{(1)}=0} M(c^{(0)}, c^{(1)}) \text{ for } c^{(0)} = 0, \text{ and } M(c^{(0)}) = \frac{a_2 - 1}{c^{(1)}=0} M(c^{(0)}, c^{(1)}) \text{ for}$$

$c^{(0)} > 0$. Using estimates analogous to those above, one may find disjoint intervals $I(c^{(0)}, c^{(1)})$ containing $M(c^{(0)}, c^{(1)})$ such that

$$\bigcup_{c^{(1)}=0}^a \overline{I(c^{(0)}, c^{(1)})} = \overline{I(c^{(0)})}, \text{ where } a = a_2 \text{ or } a_2 - 1 \text{ according to } c^{(0)} =$$

0 or $c^{(0)} > 0$. The endpoints of each $I(c^{(0)}, c^{(1)})$ are again in $\mathbb{Z}\theta$, so that, as before, $\lim_{N\theta \rightarrow t} (b_1(N\theta))$ must exist for $t \notin \mathbb{Z}\theta$, and $b_1(t) = a_2$

forces $b_0(t) = 0$. The inductive step is now obvious, and we omit it.

The lemma is done.

Lemma 3.6. If $b_i(N\theta) = b_i(\hat{N}\theta)$, $0 \leq i \leq R$, then $||N\theta - \hat{N}\theta|| \leq 2/q_R$.

Proof. $N\theta = \sum_{i=0}^{\infty} b_i(N\theta)q_i\theta$ and $\hat{N}\theta = \sum_{i=0}^{\infty} b_i(\hat{N}\theta)q_i\theta$, so that

$$||N\theta - \hat{N}\theta|| = ||\sum_{i=R+1}^{\infty} b_i(N\theta)q_i\theta - \sum_{i=R+1}^{\infty} b_i(\hat{N}\theta)q_i\theta|| <$$

$$2 \max ||\sum_{i=R+1}^{\infty} c_i q_i\theta|| < 2/q_R, \text{ by Proposition 2.3ii).}$$

Lemma 3.7. i) For all $t \notin \mathbb{Z}\theta$ we have

$$(3.8) \quad t = \sum_{i=0}^{\infty} b_i(t)q_i\theta,$$

with $b_i(t) = a_{i+2}$ forcing $b_i(t) = 0$, and $b_i(t) \in \{0, 1, \dots, a_{i+1}\}$ for all i .

ii) This expansion is unique with respect to these properties.

Proof. Given $\varepsilon > 0$, choose an integer $R > 0$ with $2/q_R < \varepsilon/2$. Then choose $N > 0$ with $||N\theta - t|| < \varepsilon/2$ and $b_i(t) = b_i(N\theta)$, $0 \leq i \leq R$. We have

$$\begin{aligned} ||t - \sum_{i=0}^R b_i(t) q_i \theta|| &\leq ||t - N\theta|| + ||N\theta - \sum_{i=0}^R b_i(t) q_i \theta|| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Letting ε tend to 0, i) obtains. As for ii), suppose $t = \sum_{i=0}^{\infty} \tilde{b}_i q_i \theta$ with

the same properties. Let $N_R = \sum_{i=0}^R \tilde{b}_i q_i$. Uniqueness in Lemma 3.3

implies that $\tilde{b}_i = c_i(N_R)$, $0 \leq i \leq R$. Thus $b_i(N_R \theta) = \tilde{b}_i$. As $R \rightarrow \infty$ and $N_R \theta \rightarrow t$, continuity of b_i at t forces $\tilde{b}_i \rightarrow b_i(t)$, all i . That is, $\tilde{b}_i = b_i(t)$, all i .

Definition. The expansion (3.8) is called the canonical representation for $t \notin \mathbb{Z}\theta$. (See [6] also for another application of this representation.)

Suppose θ has unbounded partial quotients and $t \in \Gamma^0(\theta) - \mathbb{Z}\theta$, so that t has both the representation $m\theta + \sum_{i=1}^{\infty} b_i q_i \theta$ from (1.5) and the canonical representation $\sum_{i=0}^{\infty} c_i q_i \theta$. Note that $\{b_i q_i / |q_i \theta|\} \rightarrow 0$ is equivalent to $\{b_i / a_{i+1}\} \rightarrow 0$, by Proposition 2.1vi). Therefore, by

the uniqueness of each of these representations, they are eventually the same if $b_i \geq 0$ for all large i . One might ask more generally if there is a simple criterion for $t = \sum_{i=0}^{\infty} c_i q_i^{\theta}$ (canonically) to belong to $\Gamma^0(\theta)$. Lemma 3.2 is a beginning to the answer. Further, the cases into which Theorem 3 falls for θ with unbounded partial quotients all restrict the behavior of $\{c_i\}$ slightly. However, it is hard to imagine a concise criterion for the solution. For example, by a modification of Case A of Section 10 for θ with unbounded partial quotients, it follows that $\{c_i/a_{i+1}\}$ can only cluster at 0 and 1, but one easily constructs counterexamples to the converse.

§ 4. Compact Abelian Group Extensions of R_θ . In this section we prove Theorem 2, assuming Theorem 1. (The proof of Theorem 1 is contained in the two following sections.) Theorem 2 gives conditions under which a certain functional equation has a measurable solution; the consequential failing of ergodicity of a corresponding skew product was noted in the introduction. (Henceforth, ergodicity with respect to Haar measure is implied.) Before the proof we present two results of a similar nature which show that Theorem 2 fits into a rather broad scheme. In the following, G is a compact abelian group and $f: X \rightarrow G$ is a measurable function. On the product space $Y = X \times G$ define the skew product $T: Y \rightarrow Y$ by $T(x, \zeta) = (x + \theta, \zeta f(x))$. T preserves Haar measure, so one may ask if T is ergodic. A classic result in this direction is the following (see, e.g., [2, Lemma 2.1]):

Proposition 4.1. With notations as above, let $G = X$, and let f be any measurable function. Then T is ergodic if and only if there is no nontrivial measurable solution g to

$$(4.2) \quad g(x) = f^k(x)g(x+\theta)$$

for any $k \in \mathbb{Z} - \{0\}$.

Proof. The characters $\{\zeta \mapsto \zeta^k\}$, $k \in \mathbb{Z}$, span $L^2(G)$. Therefore, suppose T is not ergodic. Then there is a function $h \in L^2(X \times G)$ with $Th = h$ everywhere, h not constant a.e. (x, ζ) . (Here $Th(x) = h(Tx)$.) Then we write $h(x, \zeta) = \sum_{\mathbb{Z}} c_k(x) \zeta^k$, with $c_k \in L^2(X)$ for all k , so that $\sum_{\mathbb{Z}} c_k(x) \zeta^k = \sum_{\mathbb{Z}} c_k(x+\theta) \zeta^k f^k(x)$. Necessarily $c_k(x) = c_k(x+\theta) f^k(x)$ for some nonzero k and nontrivial $c_k(x)$, so that (4.2) is satisfied.

Conversely, if (4.2) is satisfied, then $h(x, \zeta) = g(x)\zeta^k$ is nonconstant, measurable, and T -invariant.

Virtually the same result is true, and the same proof works, if $G = \Delta =$ the complex q^{th} roots of unity. Here $\hat{G}$ the character group of G equals $\{\zeta \mapsto \zeta^k\}$, $1 \leq k \leq q$, and the requirement that (4.2) fail for $k \in \mathbb{Z} - \{0\}$ is replaced by the failure of (4.2) for $k \in \{1, \dots, q-1\}$. In turn, if q is prime, it is equivalent to require (4.2) to fail for $k=1$ alone. Specializing $f(x) = e(p/q\chi_{[0,t)}(x))$ for $(p, q) = 1$, we obtain

Proposition [10]. Let α be rational, $\alpha = p/q$ in lowest terms, and let $G = \Delta =$ the group of complex q^{th} roots of unity. Set $f(x) = e(\alpha\chi_{[0,t)}(x))$. Then i) T is ergodic if and only if (4.2) has no nontrivial solution for $k \in \{1, 2, \dots, q-1\}$; ii) if q is prime and T is ergodic, then there is a nontrivial solution to (4.2) for $k = 1$; iii) for any q , a nontrivial solution to (4.2) exists for $k = 1$ if and only if t satisfies (1.3).

In iii) above, (1.3) obviously holds if there is a solution to (4.2) for $k = 1$. That the converse is true will follow from the line of proof of Theorem 2.

For Theorem 2 we continue with all the notations of the above proposition, as well as $t \in \Gamma_{\alpha}^1(q\theta)$. Set $\zeta = q\theta$, and let $f^{(n)}(x) = \prod_{i=0}^{n-1} f(x + i\zeta)$, $\alpha_n = \mu(f^{(n)})$. With α_n so defined, (1.4) is true for t .

(This uses Theorem 1, which we are assuming.) Recall that Theorem 2 states that if $t \in \Gamma_{\alpha}^1(\zeta)$, then there exist $\delta \in \Delta$ and $g: X \rightarrow \Delta$ measurable

with

$$(4.2') \quad \delta f(x)g(x+\theta) = g(x).$$

Lemma 4.3. If t satisfies (1.4), then there is a continuous function $h(x, \cdot)$ from X to $L^2(X)$ (with a measurable representation $h(\cdot, \cdot)$ on $X \times X$) such that for all x and w it is true for almost all y that $h(x, y)$ is in Δ and

$$(4.4) \quad h(x+w, y) = h(x, y+qw)h(w, y).$$

Proof. Define $h_0: \{n\zeta\}_{n \geq 1} \rightarrow L^2(X)$ by $h_0(n\zeta, \cdot) = f^{(qn)}(\cdot)$. Then

$$(4.5) \quad h_0((m+n)\zeta, y) = h_0(m\zeta, y+nq\zeta) h_0(n\zeta, y).$$

Now given $\varepsilon > 0$, let W be a neighborhood of 0 in X such that $|\alpha_m| > 1 - \varepsilon$ whenever $m\zeta \in W$. Then a compactness argument shows that there exist $\gamma \in \Delta$ and $\eta > 0$, η depending on q and ε , and η tending to 0 as ε

tends to 0, with $\mu\{x: f^{(m)}(x) \neq \gamma\} < \eta$. Then, by (4.5), $\mu\{x: f^{(qm)}(x) = 1\} > 1 - q\eta$. This implies $\|1 - h_0(m\zeta, \cdot)\|_2 < 2\sqrt{q\eta}$, so that h_0 is uniformly continuous from $\{n\zeta\}$ to $L^2(X)$. Therefore there is a continuous extension $\tilde{h}$ from X to $L^2(X)$. However, as defined, $\tilde{h}$ may not be measurable in $X \times X$. To remedy this, let $\{P^n\}$, $n \geq 1$, be a sequence of partitions of X satisfying $P^1 \supset P^2 \supset \dots$ and $\lim_{n \rightarrow \infty} (\text{mesh } P^n) = 0$. Let

$\{P_i^n\}$, $1 \leq i \leq d_n$, be the distinct elements of P^n , and pick $m_i^n \zeta \in P_i^n$ for $1 \leq i \leq d_n$. Next, set $h_n(x, y) = \sum_{i=1}^{d_n} \chi_{P_i^n}(x) h_0(m_i^n \zeta, y)$. Finally, arrange

that $\|\tilde{h}(x) - \tilde{h}(x')\|_2 < 1/n$ whenever x and x' lie in the same P_i^n .

Then $\{h_n(\cdot, \cdot)\}$ is a Cauchy sequence in $L^2(X \times X)$ with limit h , say. By

altering it on a set of measure 0 if necessary, the function h is now seen to verify all the requirements of the lemma.

Remark: The converse of Lemma 4.3 is true, as the proof of [10, Proposition 6] shows.

Theorem 2 now follows quickly. We have a.e. (x)

$$h(\zeta, x) = \prod_{i=0}^{q-1} f(x+i\zeta) = \prod_{i=0}^{q-1} h(\theta, x+i\zeta)$$

$$h(\zeta, x+\zeta) = \prod_{i=1}^q f(x+i\zeta) = \prod_{i=1}^q h(\theta, x+i\zeta),$$

so that $f(x)/f(x+q\zeta) = h(\theta, x)/h(\theta, x+q\zeta)$. This says $f(x)/h(\theta, x)$ is invariant under $R_{q\zeta}$, which is ergodic on X since $q\zeta$ is irrational. Therefore, for some $\delta \in \Delta$, $\delta f(x) = h(\theta, x)$ a.e. (x). Now pick λ in X with $q\lambda = \theta$. Putting

$$(4.6) \quad g(x) = \prod_{i=0}^{q-1} h^{-(i+1)}(\lambda, x+i\theta),$$

one immediately has g measurable and $g(x)/g(x+\theta) = h(\theta, x) = \delta f(x)$ a.e. (x), which is (4.2'). Theorem 2 is proved.

Remark. That (1.3) implies the existence of a solution to (4.2) for $k = 1$ in the proposition from [10] previously mentioned now follows by proving the existing of $h: X \rightarrow L^2(X)$ satisfying

$$(4.7) \quad h(x+w, y) = h(x, y+w)h(w, y)$$

a.e. (y) for fixed x and w . To do so, define $h_0: \{n\theta\}_{n \geq 1} \rightarrow L^2(X)$ by

$$h_0^{(n)}(x, y) = f^{(n)}(y) = \prod_{i=0}^{n-1} f(x+i\theta), \text{ and argue as in Lemma 4.3. Then set}$$

$x = \theta$ in (4.7). This formula must hold a.e. (w) for almost all y by the Fubini theorem. Fixing such a y , $h(w+\theta, y) = f(y+w)h(w, y)$. Then

(4.2) holds for $g(w) = h^*(w-y, y)$, and g is nontrivial.

Now we offer an explicit construction of both g and δ in Theorem

2. Assume $t \in \Gamma_\alpha^1(\zeta)$. Since (4.2') holds, we have $t \in \Gamma_\alpha^0(\theta)$. (Here

$$f^{(n)}(x) = \prod_{i=0}^{n-1} f(x+i\theta).$$

By Theorem 1, which we are assuming, $t \in \Gamma_\alpha^0(\theta)$,

so that $t = k\theta + \sum_{j=1}^{\infty} \varepsilon_j b_j q_j \theta$, as in (1.5). Let Q_k be the denominators

of the convergents of ζ , and recall that $||q_j \theta|| < 1/(2q_j)$ for large j by (3.1). For all j , one of the following applies.

Case I. $q_j = rh_j$ with $r|q$ and r maximally so. Then $|h_j \zeta - q/r(p_j)| = q/r ||q_j \theta|| < q/(2h_j)$. Then $(h_j, q/r) = 1$ forces h_j to be a denominator of a convergent of ζ , by 2.1vii).

Case II. $(q_j, q) = 1$. Then $|q_j \zeta - qp_j| < 1/(2q_j)$. In this case, q_j is a denominator for ζ .

Remark: Clearly Case II fits into Case I with $r = 1$, but the distinction between $r = 1$ and $r > 1$ will prove notationally convenient.

Since $t \in \Gamma_\alpha^1(\zeta)$, $t = h\zeta + \sum_{j=1}^{\infty} \delta_j c_j Q_j \zeta$. But from the above we also

have $t = k\theta + \sum_I \varepsilon_j (rb_j/q) (h_j \zeta) + \sum_{II} \varepsilon_j (b_j/q) (q_j \zeta)$. By the remark

preceding Lemma 3.2, these last two series for t eventually agree. In particular, if Case II occurs, then $q^2 |b_j|$, large j . If this happens, we write $b_j = qd_j$, $d_j \equiv 0(q)$. Now

$$(4.8) \quad t = m\theta + \sum_{i=1}^{\infty} \varepsilon_i b_i q_i \theta,$$

where $q^2 |b_j|$, large j , whenever q_j is a denominator for ζ . As in

Section 3, we set $n_j = gm + \sum_{i=1}^j \varepsilon_i b_i q_i$.

Notation: $\phi_x(x)$ will denote $e(\alpha \chi_{[0, r\theta]}(x))$. $A_0 = qm\theta$, $A_j = n_j\theta$ for $j \geq 1$.

We have the following relationship, which is straightforward to check:

$$(4.9) \quad \phi_{a+b}(x) = (\phi_a(x) \phi_b(x-a)) / \phi_b(-a).$$

Suppose $\varepsilon = \pm 1$. If we let $w(qn\varepsilon)$ denote $\prod_{i=0}^{qn-1} \phi_\varepsilon^*(-i\varepsilon\theta) = \prod_{i=1}^{qn-1} \phi_\varepsilon^*(-i\varepsilon\theta)$, then by (4.9) we have

$$\phi_{qn\varepsilon}(x) = w(qn\varepsilon) \cdot \prod_{i=0}^{qn-1} \phi_\varepsilon(x-i\varepsilon\theta).$$

Now put

$$(4.10) \quad H_{qn\varepsilon}(x) = \prod_{i=0}^{qn-1} \phi_\varepsilon^i(x-i\varepsilon\theta)$$

$$(4.11) \quad g_{qn\varepsilon}(x) = H_{qn\varepsilon}^\varepsilon(x + \frac{1}{2}(\varepsilon-1)\theta).$$

Then by (4.9) for an infinite sum (namely the sum (4.8)), we have

$$f(x) = \phi_{qm}(x) \cdot \prod_{j=1}^{\infty} [\phi_{\varepsilon_j b_j q_j}(x-A_{j-1}) / \phi_{\varepsilon_j b_j q_j}(-A_{j-1})].$$

(This product converges for all $x \neq 0, t$, because ϕ_{qm} multiplied by the k^{th} partial product yields ϕ_{n_k} by (4.9), and clearly $\{\phi_{n_k}(x)\} \rightarrow f(x)$

for $x \neq 0, t$.) Next, let $m = n\varepsilon$, with $n > 0$, $\varepsilon = \pm 1$. One may also immediately verify

$$\text{Lemma 4.12.} \quad \phi_{qn\varepsilon}(x) = w(qn\varepsilon) g_{qn\varepsilon}(x+\theta) / g_{qn\varepsilon}(x).$$

Now write $g_j = g_{\varepsilon_j b_j q_j}$, $H_j = H_{\varepsilon_j b_j q_j}$, $w_j(n) = \prod_{i=0}^{n-1} \phi_{\varepsilon_j}^*(-A_{j-1}-i\varepsilon_j\theta)$.

Repeatedly using Lemma 4.12, we obtain

$$(4.13) \quad f(x) = w(qn\epsilon) g_{qn\epsilon}(x+\theta)/g_{qn\epsilon}(x) \cdot \prod_{j=1}^{\infty} [g_j(x-A_{j-1}+\theta)/g_j(x-A_{j-1})] \cdot (w_j(b_j q_j)).$$

Therefore, formally defining

$$(4.14) \quad g^*(x) = g_{qn\epsilon}(x) \cdot \prod_{j=1}^{\infty} g_j(x-A_{j-1})$$

$$(4.15) \quad \delta = w^*(qn\epsilon) \cdot \prod_{j=1}^{\infty} w_j^*(b_j q_j),$$

we may formally write (4.2') from (4.13). To validate this construction, we will show that (4.14) converges a.e.(x). Then, since (4.13) and (4.14) converge for a common value of x such that (4.14) also converges for $x+\theta$, it will follow immediately that (4.15) converges.

That (4.14) converges in L^1 and a.e.(x) is a consequence of

Lemma 4.16. If $t \in \Gamma_{\alpha}^1(\zeta)$, then for all large j with $b_j \neq 0$

$$||H_j - 1||_1 \leq 2b_j q_j ||q_j \theta||.$$

Proof. For large j, if $b_j \neq 0$, then either Case I or Case II holds. Assume Case II holds; Case I is treated similarly upon replacing q_j with h_j and d_j with b_j .

Group the factors of (4.10) q at a time as follows: if $0 \leq i \leq q_j - 1$, the factors in a group have subscripts $\{i, i+q_j, \dots, i+(q-1)q_j\}, \dots, \{i+(d_j-q)q_j, \dots, i+(d_j-1)q_j\}$. The corresponding products will be equal to 1 except on a set of measure at most $q ||q_j \theta||$. There being $d_j q_j$ such groups, the lemma follows.

Since $t \in \Gamma_{\alpha}^1(\zeta)$, we have $\sum_{j=1}^{\infty} b_j q_j ||q_j \theta|| < \infty$. But then, using

Lemma 4.16, $||1 - \prod_{j=1}^{\infty} H_j||_1 \leq 2 \sum_{j=1}^{\infty} b_j q_j ||q_j \theta|| < \infty$. Therefore (4.14)

converges in L^1 and a.e.(x). As noted previously, the expansion for δ must be valid. (There is a straightforward proof of this fact as well.) The construction of g and δ is complete.

§5. $\Gamma_\alpha(\theta) \subseteq \Gamma_\alpha^0(\theta)$. Choose $t \in \Gamma_\alpha(\theta)$; that is, t satisfies (1.4).

We assume $\alpha \in (0,1)$, since the inclusion is true simultaneously for all

$\alpha + n$, $n \in \mathbb{Z}$. Also, by (3.1), we have $\Gamma_\alpha(\theta) \subseteq \Gamma^0(\theta)$, so that already any

t satisfying (1.4) has an expansion $t = m\theta + \sum_{j=1}^{\infty} \varepsilon_j b_j q_j \theta$ with

$$\lim_{j \rightarrow \infty} b_j q_j ||q_j \theta|| = 0. \text{ It is enough to prove } \lim_{j \rightarrow \infty} (b_j \alpha) \bmod 1 = 0.$$

To this end define ϕ_a and ψ_a (a in X) functions from X to $\mathbb{C}$ by
 $\phi_a(x) = e(\alpha \chi_{[0,a)}(x))$, $\psi_a(x) = \phi_a^*(x) = e(\alpha \chi_{[a,1)}(x))$. By convention,

$\phi_0(x) \equiv 1 \equiv \psi_0(x)$. We have $f(x) = \phi_t(x)$. Also define $\phi^{(n)}(x) = \prod_{i=0}^{n-1} \phi_{-i\theta}(x) \psi_{t-i\theta}(x)$. Since $\phi^{(n)}$ and $f^{(n)}$ have the same discontinuities

and the same jumps at these discontinuities, we have that $\phi^{(n)} =$

$e(-\alpha)(f(0)) * f^{(n)}$ by comparing values at $x = 0$. It follows that if

$\beta_n = \mu(\phi^{(n)})$, then

$$(5.1) \quad \lim_{n \rightarrow 0} |\beta_n| = 1.$$

For $n > 0$ let δ_n denote the length of the shortest interval of constancy for $f^{(n)}$, and define z_n and j_n ($|j_n| < n$) by $||j_n \theta - t|| = z_n = \min_{|i| < n} ||i\theta - t||$. It is proved in [10] that as $n\theta \rightarrow 0$, $n\delta_n \rightarrow 0$. Letting

$\lambda = \text{sgn}(j_n)$, define $\psi^{(n)} = \prod_{i=1}^{|j_n|} \phi_{i\lambda\theta} \psi_{(i\lambda-n)\theta}$ and $\Lambda = \{n: \delta_n = z_n\}$. It

is also proved in [10] that

$$(5.2) \quad ||\phi^{(n)} - \psi_{t-j_n\theta} \psi^{(n)}||_1 \leq 2(n-1)z_n,$$

which tends to 0 as $n \rightarrow \infty$, $n \in \Lambda$. Defining $\gamma_n = \mu(\psi^{(n)})$, we obtain

from (5.2)

$$(5.3) \quad \lim_{n\theta \rightarrow 0, n \in \Lambda} |\gamma_n| = 1.$$

The following lemma is stated without proof.

Lemma 5.4 [10, Lemma 2]. Given $\delta > 0$ and $0 < \sigma < \tau < 1$, there exists an index j_1 such that if $j \geq j_1$ and $n \in (\sigma q_{j+1}, \tau q_{j+1})$, then $n \in \Lambda$ and $|j_n| \cdot ||q_j \theta|| < \delta$.

Now let $b_j \neq 0$. Note that if j is large, only one integer r can satisfy $|rq_j - n_j| < \delta q_j$, where $n_k = m + \sum_{j=1}^k \varepsilon_j b_j$ was defined in

Section 3. That integer r must equal $\varepsilon_j b_j$, large j , since $|\varepsilon_j b_j q_j - n_j| = |n_{j-1}| < \delta q_j$. Therefore, the fact that $\{(b_j \alpha) \bmod 1\}$ tends to 0 will be a corollary of the following lemma. Recall that $\{m_k\}$ is the sequence of multiples of θ that best approximate t .

Lemma 5.5. If $0 < \alpha < 1$ and $\delta > 0$, there exists an index j_2 such that if $j \geq j_2$, $q_j < |m_k| < \alpha q_{j+1}$, and $||m_k \theta - t|| = \min_{|h| < \alpha q_{j+1}} ||h\theta - t||$,

then the sequence of integers d_j satisfying

$$(5.6) \quad |m_k - d_j q_j| < 2\delta q_j$$

also satisfy $\lim_{j \rightarrow \infty} \{(d_j \alpha) \bmod 1\} = 0$.

Proof. Suppose $\delta < \min(\frac{1}{2}, \alpha)$. In Lemma 5.4 let $\tau = \alpha$, $\sigma = \delta$, and use $\delta/2$ for δ . Then, if $[\cdot]$ denotes the greatest integer function,

$j_{[\alpha q_{j+1}]} = |m_k|$, so that either $|m_k| < \delta q_{j+1}$ or $|m_k| \cdot ||q_j \theta|| < \delta/2$.

The latter is impossible, however, if $|m_k| \geq \delta q_{j+1}$, so we may assume $q_j < |m_k| < \delta q_{j+1}$. Let $\varepsilon > 0$ be such that $||n\theta|| < \varepsilon$ forces $n\delta_n < \delta$, and $n \in \Lambda$ forces $|\gamma_n| > 1 - \delta$. For large enough j the interval

$(\delta q_{j+1}, \alpha q_{j+1})$ contains an integer n with $||n\theta|| < \varepsilon$. By Lemma 5.4, $n \in \Lambda$ and $||m_k \theta - t|| = \delta_n = z_n$. Since $n + |m_k| < q_{j+1}$, we see that

$||q_j \theta||$ is the shortest interval of constancy for $\psi^{(n)}$. Define $a \geq 1$ by $|m_k| = a q_j + r$, where $0 \leq r < q_j$. Clearly $a = b_j$ or $b_j - 1$ for large j . The proof now breaks into two cases.

Case I. α rational, $\alpha = p/q$ in lowest terms.

Subcase A. $a \equiv 0(q)$. Let $\eta_k = \text{sgn } m_k$; then $\eta_k = \text{sgn } j_n$, n

having been chosen above. Define $\psi_0^{(n)} = \prod_{h=1}^r \phi_{h\eta_k} \psi_{h\eta_k - n}$, where ϕ_s and

ψ_s denote $\phi_{s\theta}$ and $\psi_{s\theta}$, respectively. Also let $a = bq$.

The $|m_k|$ ϕ -discontinuities of $\psi^{(n)}$ gather themselves into r bunches with $(bq+1)$ in a bunch and (q_j-r) bunches with bq in a bunch. The first r bunches of discontinuities are located at $\eta_k(i+hq_j)\theta$, $h = 0, 1, \dots, bq-1$, $i = 1, 2, \dots, r$. The other bunches are located at $\eta_k(i+hq_j)\theta$, $h = 0, 1, \dots, bq-1$; $i = r+1, \dots, q_j$. If $r = 0$, then there simply are no bunches of the former kind. Now assume j is odd; j even is treated similarly, by switching the words left and right in what follows. Immediately to the right of a ϕ -bunch is the corresponding ψ -bunch; between this bunch and the next ϕ -bunch on the

right the value of $\psi^{(n)}$ is 1. Therefore, any x with $\psi^{(n)}(x) \neq \psi_0^{(n)}(x)$ lies between the shorter are joining the endpoints of one of the $2r$ bunches of $\psi_0^{(n)}$. As a result $\{x: \psi^{(n)} \neq \psi_0^{(n)}\}$ has measure less than $2|m_k| \cdot ||q_j \theta|| < 2\delta$. If we let $\delta_0^{(k)} = \mu(\psi_0^{(k)})$, then for the n

previously chosen we have

$$(5.7) \quad |\delta_0^{(n)}| > 1 - 5\delta.$$

Now the number of factors of $\psi_0^{(n)}$ is $2r = 2|m_k - \eta_k a q_j|$. Were $||q_{j-1}\theta||$ the shortest distance between discontinuities of $\psi_0^{(n)}$, we would have $|m_k - \eta_k a q_j| \cdot ||q_{j-1}\theta|| < 5\delta/2$, or $|m_k - \eta_k a q_j| < 5\delta q_j$, which implies (5.6). While $||q_{j-1}\theta||$ is not the shortest length, $\frac{1}{4}||q_{j-1}\theta||$ bounds the shortest length from below; hence (5.6) follows.

To compute the shortest length of constancy for $\psi_0^{(n)}$ first note that anything shorter than $||q_{j-1}\theta||$ must come from a term such as $\phi_c \psi_{d-n}$. The range of $c - d + n$ is $M = \{n-m+1, \dots, n+m-1\}$. The cardinality of M is less than $2q_j$, so that in M there are at most two numbers of the form uq_j and two of the form $vq_j + q_{j-1}$.

We may further require (raising j_2 if necessary) that $n \in [\frac{1}{4}q_{j+1}, \frac{1}{3}q_{j+1}]$ and similarly for all $m \in M$ (also lowering δ if necessary). Note that $|m_k| \cdot ||q_j\theta|| < \delta$ forces $q_j ||q_j\theta|| < \delta$, or $q_j < 2\delta q_{j+1}$. From $uq_j > \frac{1}{4}q_{j+1}$ we obtain $||uq_j\theta|| > \frac{1}{4}(q_{j+1}/q_j) ||q_j\theta|| > 1/(8q_j) \geq \frac{1}{4}||q_{j-1}\theta||$. From $vq_j < \frac{1}{3}q_{j+1}$ we obtain $|(vq_j + q_{j-1})\theta| = ||q_{j-1}\theta|| - v||q_j\theta|| > 1/(2q_j) - 1/(3q_j) = 1/(6q_j) > 1/3 ||q_{j-1}\theta||$. All of this implies (5.6).

Subcase B. $a \equiv -1(q)$. The ϕ -discontinuities of $\psi^{(n)}$ bunch together at $\eta_k(i+hq_j)\theta$, $h = 0, 1, \dots, a$; $i = 1, \dots, r$, and at $\eta_k(i+hq_j)\theta$, $h = 0, 1, \dots, a-1$; $i = r+1, \dots, q_j$. (The same convention is adopted as

before in case $r = 0$. If we now define $\psi_1^{(n)} = \prod_{h=r+1}^{q_j} \psi_{h\eta_k} \phi_{h\eta_k-n}$ and

$\delta_1^{(n)} = \mu(\psi_1^{(n)})$, we similarly arrive at $|\delta_1^{(n)}| > 1 - 5\delta$. The rest of

the argument proceeds as at the end of Subcase A.

Subcase C. $a \equiv c(q)$, $c \in \{0, -1\}$. We show this subcase to be impossible. The discontinuities of $\psi^{(n)}$ are just as in Subcase B.

Therefore, defining $\psi_2^{(n)} = \left(\prod_{h=1}^r \phi_{h\eta_k} \psi_{h\eta_k-n} \right)^{a+1} \left(\prod_{h=r+1}^{q_j} \phi_{h\eta_k} \psi_{h\eta_k-n} \right)^a$, we

again have $\psi_2^{(n)}$ and $\psi^{(n)}$ agreeing outside the bunches. Letting

$\delta_2^{(n)} = \mu(\psi_2^{(n)})$, we obtain

$$(5.8) \quad |\delta_2^{(n)}| > 1 - 5\delta.$$

Now from Subcase A we have that the $2q_j$ intervals between bunches of discontinuities of $\psi_2^{(n)}$ have measure greater than $\frac{1}{4}||q_{j-1}\theta||$ each. Let the union of these intervals be divided into two sets U and V , U containing the q_j intervals between each of the ϕ_k bunches and the corresponding ψ_{k-n} bunches, and V containing the other intervals. Now $\mu(U) > \frac{1}{4}q_j||q_{j-1}\theta|| > 1/8$, and $\mu(V) > 1/8$ as well. But $\psi_2^{(n)}|_V = \psi^{(n)}|_V = 1$ and $\psi_2^{(n)}|_U = \psi^{(n)}|_U$ assumes two values ($e(a\alpha)$ and $e((a+1)\alpha)$), neither of which are equal to 1, by assumption. This contradicts (5.8), however, for small enough δ .

Case II. α irrational. Suppose we fix $\rho > 0$ and an integer $j_3 > 0$ so that $j \geq j_3$ forces $|\epsilon_j b_j q_j - n_j| < \rho q_j$. Then, writing $|m_k| = a q_j + r$ with $0 \leq r < q_j$, it follows that $r/q_j < \rho$ if $a = b_j$ and $r/q_j > 1 - \rho$ if $a = b_j - 1$. Define $s = q_j - r$ (resp. r) if $a = b_j$ (resp. $b_j - 1$). We have $s/q_j > 1 - \rho$. Recall from Subcase C above the function $\psi_2^{(n)}$ and the sets U, V . Then $\psi_2^{(n)}$ assumes the value $e(b_j \alpha)$ on s of the q_j intervals of U (each of length greater than $1/(8q_j)$ from Subcase A), the value $e(B\alpha)$, $B = b_j \pm 1$, on the other $q_j - s$ intervals

of U (each of length less than $7/(8q_j)$), and the value 1 on V (a set of measure at least $1/8$). Also, from Subcase A, it is clear that $\mu(U \cup V)^c \leq 2\rho$. On $U \cup V$, therefore, $\delta_2^{(n)}$ is a convex combination $\lambda_1 e(b_j \alpha) + \lambda_2 e(B\alpha) + (1-\lambda_1-\lambda_2)1$, where the middle term is negligible, and the other two terms are not negligible. As a result, (5.8) forces $\lim_{j \rightarrow \infty} (b_j \alpha) = 0 \pmod{1}$. This completes the lemma, which in turn proves that $\Gamma_\alpha(\theta) \subseteq \Gamma_\alpha^0(\theta)$.

§6. $\Gamma_{\alpha}^1(\theta) \subseteq \Gamma_{\alpha}(\theta)$. Since $f^{(n)}(x) = \prod_{i=0}^{n-1} f(x+i\theta)$, we have that

$$(6.1) \quad f^{(m+n)}(x) = f^{(m)}(x+n\theta) f^{(n)}(x).$$

Clearly if two of $\{|\alpha_{m+n}|, |\alpha_m|, |\alpha_n|\}$ are near 1, then so must be the third. We will establish (1.4) for $t \in \Gamma_{\alpha}^1(\theta)$ by proving first that

$$(6.2) \quad \lim_{\substack{|n\theta| \rightarrow 0 \\ |n_k| < n < q_{k+1} - |n_k|}} |\alpha_n| = 1.$$

A general value of n will not lie in an interval as in (6.2). However for large h depending on n , $n + (b_h + 1)q_h$ does, because $|n_h| < (b_h + 1)q_h$ for large h . Moreover, $|(b_h + 1)q_h \theta| = |m\theta|$ is small for large h , so that if $|n\theta|$ is small, (6.1) and (6.2) will prove (1.4).

Therefore, suppose $|n_k| < n < q_{k+1} - |n_k|$, in which case $n \in \Lambda$ and $\delta_n = ||n_k \theta - t||$. If k is large, the difference in (5.2) is small in L^1 -norm. Therefore it is sufficient to prove (6.2) for $|\gamma_n|$ rather than for $|\alpha_n|$.

We introduce notation, the significance of which will become clear. Define $\xi_j, \eta_j, j \geq 2$ by $\xi_j = \frac{1}{2}(\varepsilon_j + \varepsilon_{j-1})$ and $\eta_j = \frac{1}{2}(\varepsilon_j - \varepsilon_{j-1})$. Then if $k \geq h > 2$, define $\xi_{kh} = \frac{1}{2}(\varepsilon_k + \varepsilon_{h-1})$ and $\eta_{kh} = \frac{1}{2}(\varepsilon_k - \varepsilon_{h-1})$. Also set

$$\begin{aligned} \mu_k &= \varepsilon_k & \sigma_k &= \varepsilon_k b_k q_k \\ \mu_{k-1} &= \sigma_k + \xi_k & \sigma_{k-1} &= \sigma_k + \varepsilon_{k-1} b_{k-1} q_{k-1} + \eta_k \\ &\vdots & & \\ \mu_h &= \sigma_{h+1} + \xi_{h+1} & \sigma_h &= \sigma_{h+1} + \varepsilon_h b_h q_h + \eta_{h+1} \end{aligned}$$

$$\mu_{h-1} = \sigma_h + \xi_{kh}$$

$$\sigma_{h-1} = \alpha_k + \eta_{kh}$$

and

$$\psi_k = \phi_{\mu_k} \cdot \dots \cdot \phi_{\sigma_k}$$

$\vdots$

$$\psi_h = \phi_{\mu_h} \cdot \dots \cdot \phi_{\sigma_h}$$

$$\psi_0 = \phi_{\mu_{h-1}} \cdot \dots \cdot \phi_{\sigma_{h-1}},$$

where all subscripts in the definition of the ψ_i are understood to be multiplied by θ . Note that if $\varepsilon_j = \text{sgn}(n_j)$ for $j > h-1$, then

$$\phi_{\varepsilon_k} \dots \phi_{\alpha_k} = \psi_k \dots \psi_h \psi_0, \text{ and } \psi_j \text{ contains } 1 + |\sigma_j - \mu_j| = b_j \alpha_j \text{ terms.}$$

Also, $\sigma_h = n_k - n_{h-1} + \eta_{h-1} + \dots + \eta_k$. If $\eta_j \neq 0$ and $\eta_i = 0$ for

$j_0 < i < j$, $\eta_{j_0} \neq 0$, then $\eta_{j_0} = -\eta_j$. Therefore ψ_0 has $|\sigma_{h-1} - \mu_{h-1}| + 1 \leq |\eta_{h-1}| + 1$ terms.

If $|n_k| < n < \alpha_{k+1} - |n_k|$, let $\psi_{j;n}$ (for $j = 0$ and $h \leq j \leq k$) have

a similar definition with ψ_{a-n} replacing ϕ_a for each a . The relation

$$||\alpha_h \theta|| < ||n\theta|| < ||\alpha_{h-1} \theta|| \text{ defines } h; h \geq 2 \text{ if } ||n\theta|| \text{ is small.}$$

Define $\zeta_n = \mu(\psi_0 \psi_{0;n})$. Pairing terms $\phi_a \psi_{a-n}$ with integrals of norm $\geq$

$1 - 2||n\theta||$, we conclude that $\zeta_n \geq 1 - 2(|n_{h-1}| + 1) \cdot ||n\theta|| \geq$

$1 - 2(|n_{h-1}| + 1) \cdot ||\alpha_{h-1} \theta||$, so that as $||n\theta|| \rightarrow 0$, $\zeta_n \rightarrow 1$. This

This reduces (6.2) further to proving

$$(6.3) \quad \lim_{\substack{||n\theta|| \rightarrow 0 \\ |n_k| < n < q_{k+1} - |n_k|}} ||\Psi^{(n)} \pm \Psi_0 \Psi_{0;n}||_1 = 0,$$

where the $\pm$ sign varies with n , and h is as above. Assume $t \in \Gamma_\alpha^1(\theta)$,

$$t = m\theta + \sum_{j=1}^{\infty} \varepsilon_j b_j q_j \theta. \text{ We consider two cases.}$$

Case I. α rational, $\alpha = p/q$ in lowest terms. Recall that

$$||\Psi^{(n)} - \Psi_0 \Psi_{0;n}||_1 \leq \sum_{j=h}^k ||1 - \Psi_j \Psi_{j;n}||, \text{ that } \Psi_j \text{ has } b_j q_j \text{ factors, and}$$

that $b_j \equiv 0 \pmod{q}$ for large j satisfying $h \leq j \leq k$. Grouping terms as

in Lemma 5.5, we obtain $|\mu(\psi_j)| \geq 1 - b_j q_j ||q_j \theta||$, and similarly for

$$\Psi_{j;n}. \text{ Therefore, the error in (6.3) is less than } 2 \sum_{j=h}^k b_j q_j ||q_j \theta||,$$

which tends to 0 as $h \rightarrow \infty$ (and this happens as $||n\theta|| \rightarrow 0$).

Case II. α irrational. This time do not break up Ψ_j and $\Psi_{j;n}$. Simply notice that $\Psi_j \Psi_{j;n}$ (takes the values 1 and $e(b_j \alpha)$ except on a

set of measure $b_j q_j ||q_j \theta||$. The error in (6.3) is therefore

$$\sum_{j=h}^k ||1 - \Psi_j \Psi_{j;n}|| \leq \sum_{j=h}^k 2b_j q_j ||q_j \theta|| + \sum_{j=h}^k |1 - e(b_j \alpha)|, \text{ which tends}$$

to 0 as $||n\theta|| \rightarrow 0$. The inclusion is proved, finishing the proof of

Theorem 1.

§7. Noncompact Abelian Group Extensions of R_θ . Again let $I \subseteq X$ be an interval of length t , and let E be the closed subgroup of $\mathbb{R}$ (not X) generated by $-t$ and $1-t$. Set $Y = X \times E$ and $\phi(x) = \chi_I(x) - t$. Then define the skew product $T: Y \rightarrow Y$ by $T(x, y) = (x + \theta, y + \phi(x))$. If for

$n > 0$ we define $\phi^{(n)}(x) = \sum_{i=0}^{n-1} \phi(x + i\theta)$, then $T^n(x, y) = (x + n\theta, y + \phi^{(n)}(x))$.

In fact, $\phi^{(n)}$ can be defined for $n \leq 0$ by setting $\phi^{(0)}(x) \equiv 0$ and $\phi^{(n)}(x)$

$= - \sum_{i=1}^n \phi(x - i\theta)$ for $n < 0$. Then the preceding formula for $T^n(x, y)$ is

valid for all $n \in \mathbb{Z}$.

In the language of [8], ϕ is a cocycle for R_θ on X . That is, ϕ is a Borel function from X to E such that for all $m, n \in \mathbb{Z}$ and $x \in X$,

$\phi^{(m+n)}(x) = \phi^{(m)}(x + n\theta) + \phi^{(n)}(x)$. We say $\alpha \in E$ is an essential value of ϕ if given any Borel set A in X of positive measure and $\varepsilon > 0$, there exists

an integer n with $\mu(A \cap R_\theta^{-n}A \cap \{x: |\phi^{(n)}(x) - \alpha| < \varepsilon\}) > 0$. Trivially 0 is an essential value for ϕ . Schmidt proves the following in [8], and we refer to them in the next section in the proof of Theorem 4.

EV 1: The set of essential values of ϕ is a closed subgroup of E , denoted $E(\phi)$.

EV 2: If Z is the set of measurable T -invariant sets in Y , then $E(\phi)$ may be characterized as $\{\alpha \in E: \rho_\alpha B = B \text{ for all } B \in Z\}$, where $\rho_\alpha(x, y) = (x, y + \alpha)$.

EV 3: T is ergodic if and only if $E(\phi) = E$. (This is immediate from EV 2.)

EV 4: If $\tilde{\phi}$ denotes the cocycle $\tilde{\phi}^{(n)}(x) = \phi^{(n)}(x) + E(\phi)$ from X to

$E/E(\phi)$, then 0 is the only essential value of $\tilde{\phi}$.

Relation EV 4 says that if we can find $\alpha \in E(\phi)$ with α nonzero, then we can effectively reduce T to a "compact skew product" $\tilde{T}$: $X \times E/\alpha\mathbb{Z} \rightarrow X \times E/\alpha\mathbb{Z}$, and T will be ergodic if and only if $\tilde{T}$ is ergodic. (See the method of proof of [9, Theorem 2.8].)

As in the introduction, we say t satisfies Condition A if

$$\limsup_{i \rightarrow \infty} (||q_i t|| - \frac{1}{2} q_i ||q_i \theta||) > 0.$$

We now give three lemmas to be used in the proof of Theorem 4.

Lemma 7.1. For every integer q with qt not an integer, there exist sets B and B' in X such that i) $\mu(B) \geq ||qt||, \mu(B') \geq ||qt||$;

ii) the function $\psi(x) = \sum_{j=0}^{q-1} \phi(x+(j/q))$ is constant on B and on B' ;

iii) for every $x \in B$ and $x' \in B'$, $|\psi(x) - \psi(x')| = 1$.

Proof. $\psi(x)$ is invariant under translation of $1/q$ and is constant on the elements of the partition $\mathcal{D}$ defined by the points $\{-j/q, t-j/q\}$, $j = 0, 1, \dots, q-1$. Define $t - h/q$ as the left endpoint of the element of $\mathcal{D}$ whose right endpoint is 1. Then put

$$B = \bigcup_{j=0}^{q-1} [t - (h+j)/q, -j/q), B' = B^c.$$

Since each element of $\mathcal{D}$ has length either $||qt||/q$ or $(1-||qt||)/q$, i) checks. Since $\psi(x) = \psi(x + 1/q)$, ii) checks and reveals that for every x in B and x' in B' , $|\psi(x) - \psi(x')|$ is constant. To compute the constant, set $x = t$ and $x' =$ any element s in the element of $\mathcal{D}$ just to the left of the one containing t . $\psi(t)$ samples the function ϕ at the points $\{t + j/q\}$, $0 \leq j < q$. For some r , then, $1 \leq r \leq q-1$, r of these points lie in $[0, t)$ and $q-r$ of them lie in

$[t, 1)$. Thus $|\psi(t) - \psi(s)| = |r(1-t) + (q-r)(-t) - [r(1-t) + (q-r-1)(-t)]| = 1$, and the lemma is done.

Lemma 7.2. Under condition A there exists $\varepsilon_0 > 0$ such that for infinitely many k there exist sets C_k, C'_k in X with i) $\mu(C_k) > \varepsilon_0$,

$\mu(C'_k) > \varepsilon_0$; ii) $\phi^{(q_k)}(x)$ is constant on C_k and on C'_k ; iii) for every

$x \in C_k$ and $x' \in C'_k$, $|\phi^{(q_k)}(x) - \phi^{(q_k)}(x')| = 1$.

Proof. Define $\psi_k(x) = \sum_{j=0}^{q_k-1} \phi(x + j/q_k)$, and let $A_k = \{x: \phi^{(q_k)}(x) = \psi_k(x)\}$. Now let k be even; the proof is similar for k odd. Let $i_j \theta$ be defined as in Corollary 2.5, and set

$$D'_k = \bigcup_{j=0}^{q_k-1} [-i_j \theta, -j/q_k) \quad D''_k = D'_k + t = \{x + t: x \in D'_k\}$$

$$A'_k = D'_k \setminus (D'_k \cap D''_k) \quad A''_k = D''_k \setminus (D'_k \cap D''_k)$$

By Corollary 2.5ii), the measures of D'_k and D''_k are $\frac{1}{2}(q_k-1)|q_k \theta|$. Therefore we certainly have that the measures of A'_k and A''_k are each less than $\frac{1}{2}q_k|q_k \theta|$. Putting $q = q_k$ in Lemma 7.1 and selecting the B and B' which so arise, we have $A'_k \subseteq B$, $A''_k \subseteq B'$, with the consequence that $\mu(B \setminus A'_k)$ and $\mu(B' \setminus A''_k)$ are each greater than $||q_k t|| - \frac{1}{2}q_k|q_k \theta|$.

Now select $\{q_{k_j}\}$ for infinitely many j with

$\lim_{j \rightarrow \infty} (||q_{k_j} t|| - \frac{1}{2}q_{k_j}|q_{k_j} \theta|) = \varepsilon$, and set $\varepsilon_0 = \varepsilon/2$. Also set $C_k =$

$B_k \setminus A'_k$, $C'_k = B'_k \setminus A''_k$. Then for infinitely many j , $\mu(C_{k_j}) > \varepsilon_0$ and

$\mu(C'_{k_j}) > \varepsilon_0$. Trivially $\phi^{(q_k)}(x)$ is constant on C_k , because $\phi^{(q_k)}|_{C_k} = \psi_k$

restricted to B. Similarly, $\phi^{(q_k)}$ is constant on C'_k , and by Lemma 7.1 the difference of these two constants has absolute value 1.

The following result was proved by Koksma. We present a proof as well (see [5, p.143]).

Lemma 7.3. Let $x_1 \leq x_2 \leq \dots \leq x_N$ be N points in $(0,1)$ and define $x_0 = 0$, $x_{N+1} = 1$. Let f be of bounded variation on $[0,1]$. Then

$$\left| \frac{1}{N} \sum_{n=1}^N f(x_n) - \int_X f(t) dt \right| \leq (\text{Var } f) \cdot D_N^*,$$

where D_N^* , the discrepancy of $\{x_i\}$, $1 \leq i \leq N$, is equal to $\max_i (\max |x_i - i/N|, |x_{i+1} - i/N|)$.

Proof. The first step is to prove

$$(7.4) \quad \frac{1}{N} \sum_{n=1}^N f(x_n) - \int_X f(t) dt = \sum_{n=0}^N \int_{x_n}^{x_{n+1}} (t - n/N) df(t).$$

Now the right side of (7.4) equals $\int_X t df(t) - \frac{1}{N} \sum_{n=0}^N (f(x_{n+1}) - f(x_n)) \cdot n$
 $= t \cdot f(t) \Big|_0^1 - \int_X f(t) dt + \frac{1}{N} \sum_{n=0}^{N-1} f(x_{n+1}) - f(1)$, which is the left side of (7.4).

Now for n fixed between 0 and N we have $|t - n/N| \leq \max(|x_n - n/N|, |x_{n+1} - n/N|) \leq D_N^*$ for $x_n \leq t \leq x_{n+1}$, so that the desired result follows immediately.

For our purposes we fix any $x \in X$ and let $\{x_i\}$, $1 \leq i \leq N$, be $\{x + j\theta\}$, $1 \leq j \leq q_k$. Also put $f = \phi$. We obtain

Corollary 7.5. For all x in X and $k > 0$ we have $|\phi^{(q_k)}(x)| \leq 2$.

§8. Proof of Theorem 4. Notations are as in the previous section. Also, we are assuming Theorem 3, so that, in particular, the results of Lemma 7.2 are true. Let $A \cap Y$ be strictly invariant ($TA = A$) and of positive measure. We will show that $A = Y$ up to a null set. In what follows, equations involving sets may only be true modulo a null set; the context will indicate this.

We begin with the proof of part i). By Lemma 7.2. we pick a subsequence $Q' = \{q_1' < q_2' < \dots\}$ of the sequence of denominators for θ such that

$$(8.1) \quad \mu\{x: \phi^{(q_k')}(x) = \alpha_k'\} \geq \varepsilon_0$$

where $|\alpha_k'| \geq \frac{1}{2}$ and $\alpha_k' \rightarrow \alpha'$, $|\alpha'| \geq \frac{1}{2}$, and $\varepsilon_0 > 0$ is as in the statement of the lemma. Now define the following sets for $x \in X$ and $\alpha \in [-2, 2]$.

$$D_{Q'}^\alpha = \{x: \liminf_{k \rightarrow \infty} |\phi^{(q_k')}(x) - \alpha| = 0\}$$

$$B_{Q'}(x) = \{\text{limit points of } \{\phi^{(q_k')}(x)\}, k \geq 1\}.$$

Clearly $\alpha \in B_{Q'}(x)$ if and only if $x \in D_{Q'}^\alpha$. Also, since

$$\liminf_{k \rightarrow \infty} |\phi^{(q_k')}(x) - \phi^{(q_k')}(x + \theta)| = \liminf_{k \rightarrow \infty} |\phi(x) - \phi(x + q_k'\theta)| = 0 \text{ for}$$

$x \neq 0, t$, it is true that $B_{Q'}(x) = B_{Q'}(x + \theta)$ for $x \neq 0, t$. By (8.1),

$$\mu\{y: \phi^{(q_k')}(y) = \alpha_k' \text{ for infinitely many } k\} \geq \varepsilon_0. \text{ As a result, } \mu(D_{Q'}^{\alpha'})$$

$\geq \varepsilon_0$. Therefore $\mu(D_{Q'}^{\alpha'}) = 1$, since $x \mapsto B_{Q'}(x)$ is easily seen to be a measurable function. We have proved

Lemma 8.2. There exists $\alpha' \in [-2, 2]$, $|\alpha'| \geq \frac{1}{2}$, with $\mu(D_{Q'}^{\alpha'}) = 1$.

Let the α' named in this lemma be fixed now for the rest of the proof of part i).

For $\beta \in E$ define $B_\beta = \{x: (x, \beta) \in A\}$. We will show that $C = \{(x, \beta) \in Y: (x, \beta) \in A \text{ and } (x, \beta + \alpha') \notin A\}$ has zero measure. Indeed, suppose C has positive measure. C is T -invariant, so $C \in \mathcal{Z}$ in the language of EV 2 (in Section 7). But $\rho_{\alpha'} C \cap C = \emptyset$, so that $\rho_{\alpha'} C \neq C$. Therefore α' is not an essential value of ϕ , by EV 2. But this contradicts.

Lemma 8.3. If $\mu(D_{Q'}^{\alpha'}) = 1$, then $\alpha' \in E(\phi)$.

Proof. Let Z be a set of positive measure in X , and fix $\varepsilon > 0$. Choose a subsequence $Q'' \subseteq Q'$ such that $\chi_Z(x + q_k''\theta) \rightarrow \chi_Z(x)$ a.e. (x) . Note that the q_k'' and α_k'' , along with α' , satisfy (8.1) just as did the full sequence.

Now for some $k_1 > 0$ there is a set $Z_1 \subseteq Z$ of positive measure with $Z_1 + q_k''\theta \subseteq Z \pmod{0}$ for all $k \geq k_1$. That is, for all $k \geq k_1$, we have

$$Z_1 \subseteq Z \cap R_{\theta}^{-q_k''} Z.$$

Define, for ε fixed as above, $Z_{1,j} = \{x \in Z_1: |\phi^{(q_j'')}(x) - \alpha'| < \varepsilon\}$.

$Z_1 = \bigcup_{j=1}^{\infty} Z_{1,j}$; indeed, almost every x in Z_1 belongs to infinitely many

$Z_{1,j}$, since $\mu(D_{Q''}^{\alpha'}) = 1$. Therefore infinitely many $Z_{1,j}$ have positive

measure; pick one of them, say $Z_{1,h}$, with $h \geq k_1$. Immediately,

$$Z_{1,h} \subseteq (Z \cap R_{\theta}^{-q_h''} Z \cap \{x: |\phi^{(q_h'')}(x) - \alpha'| < \varepsilon\}), \text{ so that } \alpha' \in E(\phi), \text{ as}$$

claimed.

Therefore C has measure 0 in Y , or $B_\beta \subseteq B_{\beta+\alpha'}$, almost all β . Using

the fact that $-\alpha'$ is also an essential value, we get $B_\beta \subseteq B_{\beta-\alpha'}$, for almost all β , and hence $B_\beta = B_{\beta+\alpha'}$, for almost all β . Thus, defining $\Phi(x) = \{\beta: (x, \beta) \in A\}$, we may regard $\Phi(x)$ as a subset of $E/\alpha'\mathbb{Z}$, defined only up to a null set.

Now if U and V are measurable sets in $E/\alpha'\mathbb{Z}$, define $d(U, V) = \lambda(U \Delta V)$, where λ denotes Lebesgue measure on $E/\alpha'\mathbb{Z}$. Note that $\liminf_{k \rightarrow \infty} d(\Phi(x + q_k \theta), \Phi(x)) = 0$ a.e. (x).

By Lemma 7.2 there is a fixed constant $\varepsilon_0 > 0$ and for infinitely many k sets C_k, C'_k of measure greater than ε_0 in X with $\phi^{(q_k)}(x) \equiv \lambda_k$ on C_k and $\phi^{(q_k)}(x) \equiv \lambda_k \pm 1$ on C'_k , where the $\pm$ need not be determined.

Since we have $\Phi(x + q_k \theta) = \Phi(x) + \phi^{(q_k)}(x)$ for all k and x , we have the following relations for any preassigned $\varepsilon > 0$: first, for each large enough k given by Lemma 7.2, there is a subset C_k of C_k of positive measure for which $d(\Phi(x) + \lambda_k, \Phi(x)) < \varepsilon$. For these k , $d(\Phi(x) + \lambda_k, \Phi(x)) < \varepsilon$ a.e. (x) by the ergodicity of R_θ . Similarly, using sets C'_k and C'_k , we have $d(\Phi(x) + \lambda_k \pm 1, \Phi(x)) < \varepsilon$ a.e. (x) for large k . Therefore $d(\Phi(x) + \lambda_k, \Phi(x) + \lambda_k \pm 1) < 2\varepsilon$ a.e. (x), so that $\Phi(x) = \Phi(x) + 1$ a.e. (x). This proves that 1 also is an essential value of ϕ , by EV 2. Therefore by EV 1, $\mathbb{Z} \subseteq E(\phi)$.

Let $F = E/\mathbb{Z}$. Define $T_1: X \times F \rightarrow X \times F$ by $T_1(x, y) = (x + \theta, (y + \phi(x)) \bmod 1) = (x + \theta, y - t)$, and define the cocycle ϕ_1 from X to F by $\phi_1(x) = \phi(x) + \mathbb{Z}$. As remarked before, T is ergodic on Y if and only if T_1 is ergodic on $X \times F$. But T_1 is ergodic trivially, since $1, \theta$, and t are rationally independent. Theorem 4i) is proved.

The following example shows the necessity of the assumption that $1, \theta$, and t are rationally independent. Let $\theta = [\overline{3, 2}] = [3, 2, 3, 2, \dots]$, and set $t = \sum_{i=1}^{\infty} q_{2i-1} \theta$. Then $t \notin \mathbb{Z}\theta = \Gamma^0(\theta)$; this is because θ has bounded partial quotients. But $2t = \sum_{i=1}^{\infty} 2q_{2i-1} \theta = 1 - \theta$.

Now we prove part ii); the proof is essentially that of [1, Theorem 4]. Let $t = p/q$ in lowest terms, and again suppose $A \subset Y$ is T -invariant and of positive measure. We show $A = Y$ up to a null set. To this end, define

$$B_j = \{x: (x, j/q) \in A\}, \quad C_{j,k} = B_j \cap B_{j+k}^c.$$

Let $\{q_{k_i}\}$ be a subsequence of the denominators of θ with $(q_{k_i}, q) = 1$.

This is possible since q_i and q_{i+1} are relatively prime for all i .

Also assume the q_{k_i} have been picked so that $\chi_C(x + q_{k_i} \theta) \rightarrow \chi_C(x)$

a.e.(x), where $C = C_{j, k_i}$. Now define

$$D_h = \{x: \phi^{(q_{k_i})}(x) = h/q \text{ for infinitely many } i\}.$$

It is easily shown that each D_h is R_θ -invariant; being measurable, it is of measure 0 or 1.

Now $\phi^{(q_{k_i})}(x) = r - q_{k_i}(p/q)$ for some r between 0 and q_{k_i} , so that $(q\phi^{(q_{k_i})}(x), q) = 1$ for every k and x . As a result, D_h is empty unless $(h, q) = 1$. Therefore by Corollary 7.5 we have

$$X = \bigcup_{\substack{j=-2q \\ (j,q)=1}}^{2q} D_j,$$

so that some D_j has measure 1. Henceforth let j be fixed with this property. Then a proof along the lines of Lemma 8.2 shows that

$B_k = B_{k+j}$ for all k ; that is, $C_{k,j}$ has measure 0 for all k . Now if $j = \pm 1$, we are done, for then $A = Y$. So suppose otherwise; we now have $|j| \geq 2$ and $(j,q) = 1$.

Define $\Phi(x) = \{k/q: (x, k/q) \in A\}$. As in the proof of part i) we shall consider $\Phi(x) \subseteq \{i/q\}_{i=0}^{j-1}$ for all x . But now the fact that $\Phi(x+\theta) = \Phi(x) + \phi(x) \pmod{j/q}$ shows that $\text{card}(\Phi(x)) = \text{card}(\Phi(x+\theta))$. Since $g(x) = \text{card}(\Phi(x))$ is measurable, it is a.e. constant, say $g(x) \equiv m$ a.e. (x) , $m \neq 0$. More is true. Consider the values of $\Phi(x)$ as lying in the set of j^{th} complex roots of 1. Almost every $\Phi(x)$ is m of these j points. But again, by ergodicity, there can be only one configuration for these m points (up to rotation).

Now there are only j possible configurations, all rotations of one another, for $\Phi(x)$. Cyclically number these configurations as $0, 1, 2, \dots, j-1$, and let $G(x)$ be the number assigned to the configuration for $\Phi(x)$. Also define $\psi(x) = q \cdot \phi(x) \pmod{j}$. Then $G(x)$ is a measurable function from X to $\{0, 1, \dots, j-1\}$ and satisfies $G(x+\theta) = G(x) + \psi(x) \pmod{j}$. But this contradicts the following result.

Theorem [1, Theorem 1]. Let F be the cyclic group of order j , and let $\psi(x)$ be defined as above. Then there is no measurable F -valued solution to $f(x+\theta) = f(x) + \psi(x)$.

Because of this contradiction we are forced to assume $A = Y$ up

to a null set, finishing ii) and Theorem 4.

It remains to demonstrate Theorem 3.

§9. Continued Fractions II. We record here some calculations to be used in the following two sections. As in [3, Section 1] we introduce the notation

$$a'_i = [a_i; a_{i+1}, a_{i+2}, \dots] = a_i + 1/a'_{i+1} \quad (i \geq 1),$$

$$q'_i = a'_i q'_{i-1} + q'_{i-2}, \quad r_i = q'_{i-1}/q'_i.$$

We then have

Proposition 9.1. For all $j \geq 0$, i) $a'_{j+1} q'_j = q'_{j+1}$ ii) $||q_j \theta|| = (q'_{j+1})^{-1}$.

Proof. Both statements follow by induction on j . First we have $q'_1 = a'_1 q'_0 + q'_{-1} = a'_1 = 1/\theta$, which proves the initial setp for both i) and ii). Now assume i) holds for all $j < k$; we seek to verify $q'_{k+1} = a'_{k+1} q'_k$. But $q'_{k+1} = a'_{k+1} (q'_k + (q'_{k-1}/a'_{k+1}))$, so that we are reduced to showing

$$q'_k - q'_k = q'_{k+1}/a'_{k+1}.$$

By the respective definitions of q'_k and q'_k we have $q'_k - q'_k = (a'_k - a'_k) q'_{k-1} = q'_{k-1}/a'_{k+1}$, which was required. Finally, assume ii) holds for $j > k$. Then $||q_k \theta|| = ||q'_{k-2} \theta|| - a'_k ||q'_{k-1} \theta|| = 1/q'_{k-1} - a'_k/q'_k = |1 - (a'_k/a'_k)|/q'_{k-1} = (1/a'_{k+1})(1/q'_k) = 1/q'_{k+1}$, and the proposition is proved.

Corollary 9.2. For all $j \geq 0$, i) $q_j ||q_j \theta|| = (a'_{j+1} + r_j)^{-1}$, ii) $q_j ||q_{j+1} \theta|| = (a'_{j+2})^{-1} (a'_{j+1} + r_j)^{-1}$.

Lemma 9.3. For all $j \geq -1$, $q_{j+1} ||q_j \theta|| + q_j ||q_{j+1} \theta|| = 1$.

Proof. If $j = -1$, we have $q_0 = ||q_{-1} \theta|| = 1$ and $q_{-1} = 0$, which give the result. For $j \geq 0$ we may write $q_{j+1} ||q_j \theta|| + q_j ||q_{j+1} \theta|| =$

$$q_{j+1}/q'_{j+1} + q_j/(a'_{j+2}q'_{j+1}) = (a'_{j+2}q_{j+1} + q_j)/(a'_{j+2}q'_{j+1}) = q'_{j+2}/q'_{j+2} = 1.$$

Lemma 9.4. Let the canonical representation for $t \in X \setminus \mathbb{Z}\theta$ be

$$t = \sum_{i=0}^{\infty} b_i q_i \theta. \text{ Then for large } j,$$

$$\text{i) } 0 < \sum_{i=0}^j b_i q_i < q_{j+1}$$

$$\text{ii) } \sum_{i=j}^{\infty} b_i q_i \theta \text{ lies in the shorter arc of } X \text{ joining } -q_{j-1} \theta \text{ and}$$

$$-q_j \theta. \text{ Also, } \sum_{i=j}^{\infty} b_i q_i \theta \text{ lies in the same element of the partition}$$

$\{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$ of X as does $q_k \theta$, where b_k is the first nonzero term in $\{b_j, b_{j+1}, \dots\}$.

Proof. Part i) is immediate from the construction of the canonical representation, especially Lemma 3.2. As for ii), the shorter arc in X joining $q_j \theta$ and $q_{j+1} \theta$ contains 0 for large j , by 2.1iii). Thus we have that $\sum_{i=j}^{\infty} b_i q_i \theta$ lies in the shorter arc joining $\sum_{i=0}^{\infty} (\max b_{j+2i}) q_{j+2i} \theta$ and $\sum_{i=0}^{\infty} (\max b_{j+2i+1}) q_{j+2i+1} \theta$. But these last two quantities are $-q_{j-1} \theta$ and $-q_j \theta$, respectively, by 2.2ii). Further, suppose $b_k > 0$, $k \geq j$, and k minimally so. Then $b_{k+1} < a_{k+2}$, by

canonicity. Therefore $\sum_{i=j}^{\infty} b_i q_i \theta$ lies in the shorter arc joining

$$b_k q_k \theta + \sum_{i=1}^{\infty} (\max b_{k+2i}) q_{k+2i} \theta \text{ and } b_k q_k \theta + (a_{k+2} - 1) q_{k+1} \theta +$$

$\sum_{i=1}^{\infty} (\max b_{k+2i+1}) q_{k+2i+1}^{\theta}$. The endpoints of this arc are $b_k q_k^{\theta} - q_{k+1}^{\theta}$ and $b_k q_k^{\theta} - q_{k+1}^{\theta} - q_k^{\theta}$, and both of these quantities lie in the same element of $\{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$ as does q_k^{θ} . The lemma is verified.

Lemma 9.5. Let θ have bounded partial quotients. Then there exist constants δ, η depending only on θ such that for all $i \geq 1$,

i) $a_i' a_{i+1}' > 2 + \delta$, ii) $r_i > \delta$, iii) $q_i / q_i' < 1 - \eta$.

Proof. Choose $\delta = (1 + M)^{-1}$, $\eta = (2 + M)^{-2}$, where $M = \sup_j \{a_j\}$. Then $a_i' a_{i+1}' = 1 + a_i a_{i+1}' \geq 1 + a_{i+1}' \geq 2 + 1/a_{i+2} > 2 + \delta$. Next $r_i = q_{i-1} / q_i = q_{i-1} / (a_i q_{i-1} + q_{i-2}) = 1 / (a_i + r_{i-1}) > \delta$. The proof of iii) is just as straightforward, and it is omitted.

Corollary 9.6. If θ has bounded partial quotients, then for some $c > 0$ depending only on θ it is true for all $i \geq 0$ that

$$||q_{i+2}^{\theta}|| / ||q_i^{\theta}|| < \frac{1}{2} - c.$$

Proof. By Proposition 9.1, $||q_{i+2}^{\theta}|| / ||q_i^{\theta}|| = (a_{i+2}' a_{i+3}')^{-1}$.

Now apply Lemma 9.5i).

§10. Proof of Theorem 3. We assume θ has bounded partial quotients. In this case, $\Gamma^0(\theta) = \mathbb{Z}\theta$. Choose $t \in X \setminus \mathbb{Z}\theta$, and write it canonically as $t = \sum_{i=0}^{\infty} b_i q_i \theta$. Define the coefficients of t to be the sequence $\{b_i\}$. We seek an infinite subsequence $S = \{q_{i_j}\}$ of the denominators of θ with

$$(10.1) \quad \lim_{j \rightarrow \infty} (||q_{i_j} t|| - b_{i_j} q_{i_j} ||q_{i_j} \theta||) = c > 0.$$

The major idea is to evaluate $||q_{i_j} t||$ using the canonical representation for t ; the proof then simply examines a collectively exhaustive set of cases for the coefficients of t .

$$\text{Notation: } I_i = \sum_{j=0}^{i-1} b_j q_j, \quad II_i = b_i q_i q_i \theta, \quad III_i = \sum_{j=i+1}^{\infty} b_j q_j \theta.$$

Each particular case being considered for the coefficients of t will be represented symbolically as

$$\begin{array}{c} i_j \\ \hline \dots A B C \dots \\ \dots X Y Z \dots \end{array},$$

where the table indicates that $\{q_{i_j}\}$ has been found with $\{b_{i_j}\} \equiv B$ and

$\{a_{i_j+1}\} \equiv Y$, $\{b_{i_j-1}\} \equiv A$ and $\{a_{i_j}\} \equiv X$, $\{b_{i_j+1}\} \equiv C$ and $\{a_{i_j+2}\} \equiv Z$,

etc. If there are restrictions on X , Y , A , B , etc., they will be stated in parentheses. Each table, of course, will be of finite length.

$$\begin{array}{c} \text{Case A.} \quad \begin{array}{c} i_j \\ \hline L \\ M \end{array} \quad (M \geq 4, L \notin \{0, 1, M-1, M\}) \end{array}$$

In this case $S = \{q_{i_j}\}$ satisfies condition A (that is, (10.1)).

For we have

$$||q_{i_j} t|| = ||(I_{i_j})q_{i_j} \theta + II_{i_j} + q_{i_j} (III_{i_j})||.$$

Now $||II_{i_j}|| = ||Lq_{i_j}|| |q_{i_j} \theta|$ by Proposition 2.6, and the sense of

II_{i_j} is that of $q_{i_j} \theta$ (see Proposition 2.1iii) for the definition of

sense). By Proposition 2.6 and Lemma 9.4i), $||I_{i_j} q_{i_j} \theta|| = I_{i_j} ||q_{i_j} \theta|| \in$

$(0, q_{i_j} ||q_{i_j} \theta||)$, and the sense is that of $q_{i_j} \theta$. Finally, by Lemma

9.4ii), $q_{i_j} (III_{i_j})$ has at most the magnitude $q_{i_j} ||q_{i_j+1} \theta||$ in the sense

of $q_{i_j} \theta$, and it has at most the magnitude $q_{i_j} ||q_{i_j} \theta||$ in the sense

opposed to $q_{i_j} \theta$. We obtain from these estimates

$$q_{i_j} t \in ((L-1)q_{i_j} ||q_{i_j} \theta||, (L+1)q_{i_j} ||q_{i_j} \theta|| + q_{i_j} ||q_{i_j+1} \theta||).$$

Therefore, in order to verify (10.1), it is enough to find a constant $c > 0$ such that for all large j we have the following relations:

$$(10.2a) \quad (L+1)q_{i_j} ||q_{i_j} \theta|| + q_{i_j} ||q_{i_j+1} \theta|| < 1 - \frac{1}{2}q_{i_j} ||q_{i_j} \theta|| - c$$

$$(10.2b) \quad (L-1)q_{i_j} ||q_{i_j} \theta|| > \frac{1}{2}q_{i_j} ||q_{i_j} \theta|| + c.$$

Using Corollary 9.2, (10.2a) reduces to

$$(L+1)/(M'+r_{i_j}) + 1/(a'_{i_j+2})(M'+r_{i_j}) < 1 - 1/(2(M'+r_{i_j})) - c,$$

where M' is an abuse of notation meaning $M' = a'_{i_j+1} = M + 1/a'_{i_j+2}$.

Therefore we seek

$$L + 3/2 + 1/a'_{i_j+2} < M' + r_{i_j} - c$$

$$L + 3/2 < M + r_{i_j} - c.$$

Here, since $L \leq M - 2$ by assumption and $r_{i_j} > 0$ for all j , it is clear

that $c = \frac{1}{2}$ will suffice. As for (10.2b), we require

$$(L - 3/2)/(M' + r_{i_j}) > c$$

$$L - 3/2 > c.$$

which is also true by assumption for any $c < \frac{1}{2}$. Case A is verified.

Henceforth we may assume $b_i \in \{0, 1, a_{i+1}-1, a_{i+1}\}$ for all large i , since otherwise we could invoke Case A.

<p>Case B1.</p> $\begin{array}{c} i_j \\ \hline 0 \ 1 \ 0 \\ K \ L \ M \ (L \geq 2) \end{array}$	<p>Case B2.</p> $\begin{array}{c} i_j \\ \hline 0 \ L-1 \ 0 \\ K \ L \ M \ (L \geq 2) \end{array}$
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For either of these cases $S = \{q_{i_j}\}$ is a valid choice to satisfy

condition A. Since the proofs involve the same inequalities and the same estimates for B1 and B2, we give the details only for B1. In this case we have

$$||q_{i_j} t|| = ||(I_{i_j})q_{i_j} \theta + q_{i_j} (q_{i_j} \theta) + q_{i_j} (III_{i_j})||.$$

By Lemma 9.4i) and the assumption $b_{i_j-1} = 0$, $0 < I_{i_j} < q_{i_j-1}$, and we

also have that the contribution of $q_{i_j} (III_{i_j})$ is an addition of at

most $q_{i_j} ||q_{i_j+1} \theta||$ or a subtraction of $q_{i_j} ||q_{i_j+2} \theta||$. Therefore we

wish to verify both of the following inequalities:

$$(10.3a) \quad (q_{i_j} + q_{i_j-1}) ||q_{i_j} \theta|| + q_{i_j} ||q_{i_j+1} \theta|| < 1 - \frac{1}{2} q_{i_j} ||q_{i_j} \theta|| - c$$

$$(10.3b) \quad q_{i_j} ||q_{i_j} \theta|| - q_{i_j} ||q_{i_j+2} \theta|| > \frac{1}{2} q_{i_j} ||q_{i_j} \theta|| + c.$$

First, (10.3a) reduces to

$$(3/2 + r_{i_j}) / (L' + r_{i_j}) + (1/M') (1 / (L' + r_{i_j})) < 1 - c$$

$$3/2 + r_{i_j} + 1/M' < L' + r_{i_j} - c = L + 1/M' + r_{i_j} - c,$$

where this last equation follows from the abuse of notation introduced after (10.2b). But now we have

$$3/2 < L - c,$$

which is true since $L \geq 2$. Next, (10.3b) reduces to

$$\frac{1}{2} (1/L' + r_{i_j}) - (1/(M'a'_{i_j+3})) (1/(L' + r_{i_j})) < c$$

$$\frac{1}{2} - 1/(M'a'_{i_j+3}) < c,$$

which is true by Lemma 9.5i). This verifies B1; B2 is proved similarly.

Cases C1-C6:	C1. $\frac{i_j}{\begin{smallmatrix} 1 & 1 \\ L & M \end{smallmatrix}} \quad (L, M \geq 2)$	C2. $\frac{i_j}{\begin{smallmatrix} 1 & M-1 \\ L & M \end{smallmatrix}} \quad (L, M \geq 2)$	C3. $\frac{i_j}{\begin{smallmatrix} L-1 & 1 \\ L & M \end{smallmatrix}} \quad (L, M \geq 2)$
	C4. $\frac{i_j}{\begin{smallmatrix} L-1 & M-1 \\ L & M \end{smallmatrix}} \quad (L, M \geq 2)$	C5. $\frac{i_j}{\begin{smallmatrix} L & 1 \\ L & M \end{smallmatrix}} \quad (M \geq 2)$	C6. $\frac{i_j}{\begin{smallmatrix} L & M-1 \\ L & M \end{smallmatrix}} \quad (M \geq 2)$

In these cases a slightly different proof is used. We shall prove that for each of these cases, either $S = \{q_{i_j}\}$ or $\{q_{i_j+1}\}$ satisfies

condition A, without determining which. The proof will be presented

only for C1, the others involving similar inequalities and estimates.

In order that $S = \{q_{i_j}\}$, it is sufficient that the following are both true:

$$(10.4a) \quad (I_{i_j} + q_{i_j}) ||q_{i_j} \theta|| - q_{i_j} ||q_{i_j+1} \theta|| + q_{i_j} ||III_{i_j+1}|| < 1 - \frac{1}{2} q_{i_j} ||q_{i_j} \theta|| - c$$

$$(10.4b) \quad (I_{i_j} + q_{i_j}) ||q_{i_j} \theta|| - q_{i_j} ||q_{i_j+1} \theta|| + q_{i_j} ||III_{i_j+1}|| > \frac{1}{2} q_{i_j} ||q_{i_j} \theta|| + c.$$

Here, $||III_{i_j+1}||$ is an abuse of notation meaning

$$\sum_{i=i_j+2}^{\infty} (-1)^{i-i_j} b_i ||q_i \theta||; \text{ clearly these two quantities are either}$$

equal or are the negative of each other. Also, in (10.4a,b) we are using the assumptions of C1 in that the coefficients of $q_{i_j} ||q_{i_j} \theta||$ and $q_{i_j} ||q_{i_j+1} \theta||$ are +1 and -1, respectively, corresponding to $b_{i_j} = b_{i_j+1} = 1$.

On the other hand, for $S = \{q_{i_j+1}\}$, it is sufficient to establish both of these:

$$(10.4c) \quad (I_{i_j} + q_{i_j} + q_{i_j+1}) ||q_{i_j+1} \theta|| - q_{i_j+1} ||III_{i_j+1} \theta|| < 1 - \frac{1}{2} q_{i_j+1} ||q_{i_j+1} \theta|| - c$$

$$(10.4d) \quad (I_{i_j} + q_{i_j} + q_{i_j+1}) ||q_{i_j+1} \theta|| - q_{i_j+1} ||III_{i_j+1} \theta|| > \frac{1}{2} q_{i_j+1} ||q_{i_j+1} \theta|| + c$$

Rename these last four inequalities a, b, c, and d for short. We show that one of each of the following pairs of inequalities must be true: a and c, a and d, b and c, b and d. When this is done, then it will follow that either $S = \{q_{i_j}\}$ or $\{q_{i_j+1}\}$ is valid.

As for the pair a and c, multiply a on both sides by q_{i_j+1} , and multiply c on both sides by q_{i_j} . Adding these inequalities together and simplifying via Lemma 9.3, we obtain

$$(10.5) \quad (I_{i_j} + q_{i_j}) < q_{i_j+1} + q_{i_j} - \frac{1}{2} q_{i_j+1} q_{i_j} (||q_{i_j} \theta|| + ||q_{i_j+1} \theta||) - c q_{i_j}.$$

If (10.5) is true, then either a or c is true. In (10.5) we estimate

$I_{i_j} + q_{i_j} < q_{i_j+1}$, so that we seek

$$(10.6) \quad q_{i_j+1} (||q_{i_j} \theta|| + ||q_{i_j+1} \theta||) < 2 - c.$$

But clearly $q_{i_j+1} ||q_{i_j} \theta|| = q_{i_j+1} / q'_{i_j+1}$ by Proposition 9.1ii), and

the fact that θ has bounded partial quotients implies that $q_{i_j+1} / q'_{i_j+1} < 1 - c$, some positive c , for all j . Also, $||q_{i_j+1} \theta|| < ||q_{i_j} \theta||$.

Together, these facts imply (10.6).

Now look at the pair b and d. Again multiply b on both sides by q_{i_j+1} , and multiply d on both sides by q_{i_j} to obtain (after collecting terms)

$$(10.7) \quad (I_{i_j} + q_{i_j}) > \frac{1}{2} q_{i_j+1} q_{i_j} (||q_{i_j} \theta|| + ||q_{i_j+1} \theta||) + c q_{i_j}.$$

Note that the constant c in (10.7) differs from the one in (10.5).

Now estimating $I_{i_j} > 0$, we arrive at (10.6), which has already been

verified. Therefore either b or d holds also.

For the pair a and d, form the linear combination $a(||q_{i_j+1} \theta||) + d(||q_{i_j} \theta||)$ (it should be clear what this means) to obtain, after

clearing terms,

$$(10.8) \quad ||III_{i_j+1}||^{+\frac{1}{2}(q_{i_j}+q_{i_j+1})} (||q_{i_j}\theta|| \cdot ||q_{i_j+1}\theta||) < (2-c) ||q_{i_j+1}\theta||.$$

Now estimate $\frac{1}{2}(q_{i_j} + q_{i_j+1}) ||q_{i_j}\theta|| < q_{i_j+1} ||q_{i_j}\theta|| < 1$; what results

is

$$(10.9) \quad ||III_{i_j+1}|| < (1 - c) ||q_{i_j+1}\theta||.$$

Since $b_{i_j+1} \equiv 1$, canonicity forces $\max ||III_{i_j+1}|| = ||q_{i_j+1}\theta|| -$

$||q_{i_j+2}\theta||$. This, coupled with $||q_{i_j+2}\theta|| / ||q_{i_j+1}\theta|| = 1/(a'_{i_j+3}) > c$

> 0 , all j , implies (10.9), which in turn proves that either a or d holds.

For the pair b and c, from the linear combination $b(||q_{i_j+1}\theta||) + c(||q_{i_j}\theta||)$ and clear terms. We then seek to show

$$(10.10) \quad ||q_{i_j}\theta|| + ||III_{i_j+1}|| > (1 - c) ||q_{i_j+1}\theta||.$$

Since $||III_{i_j+1}|| = \sum_{i=i_j+2}^{\infty} (-1)^{i-i_j} b_i ||q_i\theta||$, we estimate

$||III_{i_j+1}|| > -||q_{i_j+2}\theta||$, whereupon (10.10) becomes

$$(10.11) \quad ||q_{i_j}\theta|| - ||q_{i_j+2}\theta|| > (1 + c) ||q_{i_j+1}\theta||,$$

which is true, since the left hand side of (10.11) equals $M ||q_{i_j+1}\theta|| \geq$

$2 ||q_{i_j+1}\theta||$. This verifies Case C1, and the other cases are treated

the same way.

Now notice that if $\{b_{i_j}\} \equiv 1$ and $\{a_{i_j+1}\} \equiv M \geq 2$, then one of the following cases must apply: B1, C1, C2, C3, C5. Therefore if none of these is to apply to t , we may conclude $b_i \in \{0, a_{i+1}-1, a_{i+1}\}$ for all large i . Also, if $\{b_{i_j}\} \equiv a_{i_j+1}-1$, where $\{a_{i_j+1}\} \equiv M \geq 2$, then either B2, C2, C3, C4, or C6 must apply. Thus we now are assured that $b_i \in \{0, a_{i+1}\}$ for all large i , if none of the above cases applies. We continue by inspecting the lengths of the strings of consecutive zeroes in $\{b_i\}$.

§11. Proof of Theorem 3 (continued). We now have $b_i \in \{0, a_{i+1}\}$ for all $i > i_0$. Suppose that infinitely often the strings of zeroes in $\{b_i\}_{i>i_0}$ have even (but not necessarily the same) lengths. That is,

suppose we have

$$\begin{array}{l} \text{Case D.} \quad \begin{array}{c} i_j \\ \hline K \ 0 \ . \ . \ . \ 0 \quad M \\ K \ L_1 \ . \ . \ . \ L_{2k_j} \quad M \quad (k_j \geq 1, \text{ all } j) \end{array} \end{array}$$

Here the notation is ambiguous if $k_j \rightarrow \infty$. If this cannot be avoided, let us then only require that $\{a_{i_j+h}\}$ be constant for $0 \leq h \leq 3$, as well as $h = 2k_j + 2$. In this setting $S = \{q_{i_j+1}\}$ satisfies condition A if $L_1 \geq 2$, and $S = \{q_{i_j}\}$ if $L_1 = 1$.

Suppose $L_1 \geq 2$. In order that $S = \{q_{i_j+1}\}$, it is sufficient that

$$(11.1a) \quad I_{i_j+1} ||q_{i_j+1}^\theta|| + q_{i_j+1} (III_{i_j+1}) < 1 - \frac{1}{2} q_{i_j+1} ||q_{i_j+1}^\theta|| - c$$

$$(11.1b) \quad I_{i_j+1} ||q_{i_j+1}^\theta|| + q_{i_j+1} (III_{i_j+1}) > \frac{1}{2} q_{i_j+1} ||q_{i_j+1}^\theta|| + c.$$

For (11.1a), estimate $I_{i_j+1} < q_{i_j+1}$. Also, III_{i_j+1} contributes at most

$||q_{i_j+2}^\theta||$ in the sense of $q_{i_j+1}^\theta$, assuming $k_j \equiv 1$. Therefore we want

to demonstrate

$$q_{i_j+1} ||q_{i_j+1}^\theta|| + q_{i_j+1} ||q_{i_j+2}^\theta|| < 1 - \frac{1}{2} q_{i_j+1} ||q_{i_j+1}^\theta|| - c$$

$$3/2 + 1/L_2' < L_1' + r_{i_j+1} - c$$

$$3/2 < L_1 + r_{i_j+1} - c,$$

which is true since $L_1 \geq 2$. Then for (11.1b), we estimate the contribution of III_{i_j} as 0 and put $I_{i_j+1} > Kq_{i_j}$. What we then seek to show is this:

$$Kq_{i_j} ||q_{i_j+1}\theta|| > \frac{1}{2}q_{i_j+1} ||q_{i_j+1}\theta|| + c$$

$$2(q_{i_j+1} - q_{i_j-1})/q_{i_j+1} > 1 + c,$$

which is true because $q_{i_j+1} > (2 + c) q_{i_j-1}$, some $c > 0$.

Now if $L_1 = 1$, in order that $S = \{q_{i_j}\}$ it is sufficient that

$$(11.2a) \quad (I_{i_j} + Kq_{i_j}) ||q_{i_j}\theta|| + q_{i_j} (III_{i_j}) < 1 - \frac{1}{2}q_{i_j} ||q_{i_j}\theta|| - c$$

$$(11.2b) \quad (I_{i_j} + Kq_{i_j}) ||q_{i_j}\theta|| + q_{i_j} (III_{i_j}) > \frac{1}{2}q_{i_j} ||q_{i_j}\theta|| + c.$$

In (11.2a) we estimate $I_{i_j} < q_{i_j-1}$ and $III_{i_j} < 0$ (since any contribution to III_{i_j} must be in the sense opposed to $q_{i_j}\theta$) to obtain

$$(q_{i_j-1} + Kq_{i_j}) ||q_{i_j}\theta|| < 1 - \frac{1}{2}q_{i_j} ||q_{i_j}\theta|| - c$$

$$r_{i_j} + K + \frac{1}{2} < K' + r_{i_j} - c.$$

But this is true because $L_1 = 1$ forces $1/L_1' \in (\frac{1}{2}, 1)$. In (11.2b) estimate $I_{i_j} > 0$, and estimate III_{i_j} as a subtraction of no more than

$||q_{i_j+2}\theta||$ (as in the estimate of III_{i_j+1} for (11.1a) to obtain

$$Kq_{i_j} ||q_{i_j}\theta|| - q_{i_j} ||q_{i_j+2}\theta|| > \frac{1}{2}q_{i_j} ||q_{i_j}\theta|| + c.$$

This inequality is true if it is true for $K = 1$. For $K = 1$ we seek to

verify

$$\frac{1}{2}q_{i_j} ||q_{i_j} \theta|| - q_{i_j} ||q_{i_j+2} \theta|| > c,$$

which is true by Corollary 9.6. Case D is complete.

Henceforth we may assume the existence of a new index i_1 such that whenever a string of zeroes occurs in $\{b_i\}_{i>i_1}$, the length of that string is odd. (Estimates for III in Case D are not valid for odd strings of zeroes, because the sense of III is opposed to what it is in the even length case.) Note that Proposition 2.2i) and the assumption that $t \notin \mathbb{Z}\theta$ force the lengths of those odd strings to be three or more infinitely often. In the table notation we have

$$(11.3) \quad \begin{array}{ccccccc} & i_j & & & & & \\ & \hline K & 0 & . & . & . & 0 & M \\ K & L_1 & . & . & . & L_{2k_j+1} & M \quad (k_j \geq 1). \end{array}$$

This table will serve all the remaining cases (the E series). Again k_j is not necessarily constant, and if $k_j \rightarrow \infty$ is forced, it will suffice to assume $\{a_{i_j+h+1}\}$ and $\{a_{i_j+2k_j+h+1}\}$ are constant for $-3 \leq h \leq 3$. In this setting either $S = \{q_{i_j}\}$, $\{q_{i_j+1}\}$, or $\{q_{i_j+2}\}$ will satisfy condition A.

In order that $S = \{q_{i_j}\}$ it is sufficient that both of the following be satisfied:

$$(11.4a) \quad (I_{i_j} + Kq_{i_j}) ||q_{i_j} \theta|| + q_{i_j} ||III_{i_j}|| < 1 - \frac{1}{2}q_{i_j} ||q_{i_j} \theta|| - c$$

$$(11.4b) \quad (I_{i_j} + Kq_{i_j}) ||q_{i_j} \theta|| + q_{i_j} ||III_{i_j}|| > \frac{1}{2}q_{i_j} ||q_{i_j} \theta|| + c.$$

Here $||III_{i_j}||$ denotes $\sum_{i=i_j+4}^{\infty} (-1)^{i-i_j} b_i ||q_i \theta||$; it is an abuse of notation similar to that in (10.4). Since the zero-strings have odd lengths, III_{i_j} must carry the sense of $q_{i_j} \theta$. Therefore estimating $||III_{i_j}||$ as 0, (11.4b) is verified. We leave (11.4a) for the moment and consider sufficient conditions for $S = \{q_{i_j+1}\}$. They are the following:

$$(11.4c) \quad (I_{i_j} + Kq_{i_j}) ||q_{i_j+1} \theta|| - q_{i_j+1} ||III_{i_j}|| < 1 - \frac{1}{2} q_{i_j+1} ||q_{i_j+1} \theta|| - c$$

$$(11.4d) \quad (I_{i_j} + Kq_{i_j}) ||q_{i_j+1} \theta|| - q_{i_j+1} ||III_{i_j}|| > \frac{1}{2} q_{i_j+1} ||q_{i_j+1} \theta|| + c$$

Then, for $S = \{q_{i_j+2}\}$, it is sufficient that

$$(11.4e) \quad (I_{i_j} + Kq_{i_j}) ||q_{i_j+2} \theta|| + q_{i_j+2} ||III_{i_j}|| > 1 - \frac{1}{2} q_{i_j+2} ||q_{i_j+2} \theta|| - c$$

$$(11.4f) \quad (I_{i_j} + Kq_{i_j}) ||q_{i_j+2} \theta|| + q_{i_j+2} ||III_{i_j}|| < \frac{1}{2} q_{i_j+2} ||q_{i_j+2} \theta|| + c.$$

However, (11.4e) is always true. For we estimate $||III_{i_j}|| <$

$||q_{i_j+3} \theta||$ and $I_{i_j} < q_{i_j-1}$. Then (11.4e) becomes

$$q_{i_j+1} ||q_{i_j+2} \theta|| + q_{i_j+2} ||q_{i_j+3} \theta|| < 1 - \frac{1}{2} q_{i_j+2} ||q_{i_j+2} \theta|| - c$$

$$\frac{1}{2} + r_{i_j+2} + 1/L'_3 < L'_2 + r_{i_j} - c$$

$$\frac{1}{2} < L_2 - c,$$

which is true, as claimed.

The four remaining inequalities in (11.4) we now abbreviate, in order, as a, c, d, and f. We wish to verify either a or both of c and d or f.

First, either a or c must hold. To see this, form the linear combination $a(q_{i_j+1}) + c(q_{i_j})$ as in Case C1, and estimate $I_{i_j} < q_{i_j-1}$.

After clearing terms one obtains (10.6), which has been verified already. See C1 for the calculations.

As a result, we seek to verify either a or d or f. While this cannot be verified directly in general, it is easily established in a great many circumstances. Moreover, for the exceptional cases, either a or d must hold.

We look to the triple (a, d, f) first to see when one of these is true. Forming the linear combination $a + L_1 d + f$, it is sufficient that

$$(11.5) \quad q_{i_j} ||q_{i_j} \theta|| + L_1 q_{i_j+1} ||q_{i_j+1} \theta|| + q_{i_j+2} ||q_{i_j+2} \theta|| < 2 - c.$$

Omitting the calculations, (11.5) is verified whenever any of the following three conditions are met (Case E1):

$$(11.6a) \quad K = 1 \text{ and } L_1 \leq 2L_2'$$

$$(11.6b) \quad K \geq 2 \text{ and } L_2 \geq 2$$

$$(11.6c) \quad K \geq 2, L_2 = 1, \text{ and } L_1 \leq 2(L_2' + 1)/(2 - L_2').$$

Thus in the presence of one of the above three conditions, either $S = \{q_{i_j}\}$, $\{q_{i_j+1}\}$, or $\{q_{i_j+2}\}$ satisfies condition A. We now treat the outstanding cases, for some of which the following trick will prove useful.

Recall $t = m\theta + \sum_{i=i_1+1}^{\infty} b_i q_i \theta$, where i) $b_i \in \{0, a_{i+1}\}$ for all $i > i_1$,

and ii) the lengths of the strings of zeroes in $\{b_i\}$, $i > i_1$, are all

odd. In view of ii) the range $i > i_1$ for the sum $\sum_{i=i_1+1}^{\infty} b_i q_i \theta$ may be

replaced either by $\{i: i > i_1, i \text{ even}\}$ or $\{i: i > i_1, i \text{ odd}\}$. Call this index set I , and now let $s = \sum_I (a_{i+1} - b_i) q_i \theta$. It is true that

i) s is thus canonically represented, ii) $a_{i+1} - b_i \in \{0, a_{i+1}\}$ for all $i \in I$, and iii) $s + t \in \mathbb{Z}\theta$ (by Proposition 2.2i)). But then

$$||q_i t|| - ||q_i s|| \leq ||q_i (t+s)|| = ||n q_i \theta||, \text{ some } n \in \mathbb{Z}. \text{ Furthermore,}$$

$$\lim_{i \rightarrow \infty} ||n q_i \theta|| = 0. \text{ Therefore to verify condition A we may choose}$$

between t and s .

There are five exceptional cases left; we continue to refer to (11.3) for each of them.

Case E2. $K \geq 2$, $L_2 = 1$, $L_1 > 2(L_2' + 1)/(2 - L_2')$, and the zero-strings are infinitely often of length three.

Case E3. $K \geq 2$, $L_2 = 1$, $L_1 > 2(L_2' + 1)/(2 - L_2')$, and the zero-strings are infinitely often longer than 3.

Cases E4-E6. $K = 1$ and $L_1 > 2L_2'$.

Case E2. We show that either a or d (that is, either (11.4a) or (11.4d)) holds. We replace the condition $L_1 > 2(L_2' + 1)/(2 - L_2')$ by the evidently weaker condition $L_1 > 4$. Note that by referring to s instead of t , we are done in light of (11.6) for s unless $L_3 \geq 3$ (which is weaker than either $L_3 > 2(M' + 1)/(2 - M')$ for $M = 1$ or $L_3 > 2M'$). Therefore we assume $L_3 \geq 3$ as well. Now form the inequality $a(||q_{i_j+1} \theta||) + d(||q_{i_j} \theta||)$, which we seek to demonstrate.

Omitting the collecting of terms, and estimating $||III_{i_j}|| <$

$||q_{i_j+3}^\theta||$, the inequality becomes

$$(q_{i_j} + q_{i_j+1})||q_{i_j}^\theta|| \cdot ||q_{i_j+1}^\theta|| < (2 - c)(||q_{i_j+1}^\theta|| - ||q_{i_j+3}^\theta||)$$

$$(q_{i_j+1} + q_{i_j})||q_{i_j}^\theta|| < (2 - c)/L_2'$$

$$|(K+1)q_{i_j} + q_{i_j-1}|/(K'q_{i_j} + q_{i_j-1}) < (2 - c)/L_2'$$

$$(K + 1)L_2'q_{i_j} + L_2'q_{i_j-1} < (2 - c)K'q_{i_j} + (2 - c)q_{i_j-1}.$$

Since $L_2 = 1$, this will be true if

$$(K + 1)L_2' < (2 - c)K'.$$

Now estimate $L_2' < 4/3$, because $L_2 = 1$ and $L_3 \geq 3$. We then obtain

$$(4/3)(K + 1) < (2 - c)K',$$

and this is indeed true for all $K \geq 2$, no matter how large L_1 is.

Case E3. Again form the linear combination $a(||q_{i_j+1}^\theta||) +$

$d(||q_{i_j}^\theta||)$, and estimate $||III_{i_j}|| \leq ||q_{i_j+5}^\theta||$, which leads to

$$(q_{i_j+1} + q_{i_j})||q_{i_j}^\theta|| \cdot ||q_{i_j+1}^\theta|| < (2 - c)(||q_{i_j+1}^\theta|| - ||q_{i_j+5}^\theta||).$$

Now $||q_{i_j+5}^\theta||/||q_{i_j+1}^\theta|| = 1/(L_2'L_3'L_4'L_5')$, and $L_2' \dots L_5' > 5 + c$. (This

is seen by estimating $L_2 = L_3 = L_4 = L_5 = 1$ and proceeding as in

Lemma 9.5i).) With this, the desired inequality becomes

$$((K+1)q_{i_j} + q_{i_j-1})/(K'q_{i_j} + q_{i_j-1}) < (8 - c)/5$$

$$5(K + 1)q_{i_j} + 5q_{i_j-1} < (8-c)K'q_{i_j} + (8-c)q_{i_j-1}$$

$$5(K + 1) < (8-c)K',$$

which is true for all $K \geq 2$ and all L_1 .

Cases E4-E6 all carry the assumptions that i) $K = 1$ and ii) $L_1 > 2L_2'$. We replace ii) by $L_1 \geq 3$. In addition, the freedom to choose either s or t allows us to divide the remaining possibilities as follows:

Case E4. The zero-strings in $\{b_i\}$, $i > i_1$, are all of length 3.

Case E5. The zero-strings are infinitely often of length 5 or more, and $b_{i_j-2} = a_{i_j-1} \equiv J_2$, some integer J_2 .

Case E6. The zero-strings are infinitely often of length 5 or more, and $a_{i_j-1} \equiv J_2$, $b_{i_j-2} \equiv 0$.

In each of the following we shall establish (11.4d). Since we have established (11.4b) and one of (11.4a), (11.4c) already, it will follow that $S = \{q_{i_j}\}$ or $\{q_{i_j+1}\}$ satisfies condition A.

Case E4. Since $K = 1$ and $k_j \equiv 1$ (see (11.3) for the meaning of k_j), we have $M = K = 1$. Also we may assume $L_2 = 1$; other we are done upon switching to s instead of t . Similarly, $L_1 \geq 3$ and $L_3 \geq 3$. To verify (11.4d), it is enough to show

$$q_{i_j} ||q_{i_j+1}^\theta|| - q_{i_j+1} ||q_{i_j+3}^\theta|| > \frac{1}{2} q_{i_j+1} ||q_{i_j+1}^\theta|| + c.$$

(Here we estimated $I_{i_j} > 0$ and $||III_{i_j}|| < ||q_{i_j+3}^\theta||$.) This reduces to

$$\frac{1}{2} - q_{i_j-1}/q_{i_j+1} - 1/(L_2' L_3') > c.$$

This inequality is true because of two observations: i) $K = 1$ and $L_1 \geq 3$ force $q_{i_j-1}/q_{i_j+1} < \frac{1}{2} - c_1$, some $c_1 > 0$, and ii) $L_2 = 1$ and

$L_3 \geq 3$ force $L_2' L_3' > 4 + c_2$, some $c_2 > 0$.

Case E5. Our table now looks like the following:

$$\begin{array}{cccccccc} & & i_j & & & & & \\ \hline J_2 & 0 & 1 & 0 & . & . & . & 0 & M \\ J_2 & J_1 & 1 & L_1 & . & . & . & L_{2k_j+1} & M \quad (k_j \geq 2, \text{ all } j) \end{array}$$

In order to check (11.4d), it is sufficient that

$$\begin{aligned} (I_{i_j} + q_{i_j}) ||q_{i_j+1}^\theta|| - q_{i_j+1} ||q_{i_j+5}^\theta|| &> \frac{1}{2} q_{i_j+1} ||q_{i_j+1}^\theta|| + c \\ (q_{i_j+1} - q_{i_j-3}) ||q_{i_j+1}^\theta|| - q_{i_j+1} ||q_{i_j+5}^\theta|| &> \frac{1}{2} q_{i_j+1} ||q_{i_j+1}^\theta|| + c \\ 2(1 - (q_{i_j-3}/q_{i_j+1})) - 2/(L_2' L_3' L_4' L_5') &> 1 + c. \end{aligned}$$

From Case E3 we have $L_2' \dots L_5' > 5 + c$. It is just as straightforward from Proposition 2.1i) that $q_{i_j+1} > 5q_{i_j-3}$; just estimate $a_{i_j+1} = a_{i_j} = a_{i_j-1} = 1$. These facts immediately verify the desired inequality.

Case E6. We have

$$\begin{array}{cccccccc} & & i_j & & & & & \\ \hline 0 & 0 & 0 & 1 & 0 & . & . & . & 0 & M \\ J_3 & J_2 & J_1 & 1 & L_1 & . & . & . & L_{2k_j+1} & M. \end{array}$$

By switching to s instead of t , we are done in view of (11.6) for s and the preceding cases unless $J_2 = 1$ and $J_1 \geq 3$. Therefore, in order to demonstrate (11.4d), it is enough that

$$2(1 - (q_{i_j-1}/q_{i_j+1})) - 2/(L_2' \dots L_5') > 1 + c.$$

Be we may estimate $L_2' \dots L_5' > 5 + c_1$ and $q_{i_j-1}/q_{i_j+1} < \frac{1}{4} - c_2$ for some

positive constants c_1 and c_2 . Therefore the desired inequality is true, and the proof of Theorem 3 is thereby complete.

References

1. Conze, J. P. "Équirépartition et Ergodicité de Transformations Cylindriques," Preprint, Université de Rennes (1976).
2. Furstenberg, H. "Strict Ergodicity and Transformations of the Torus," American Journal of Mathematics, 88 (1961), 573-601.
3. Kesten, H. "On a Conjecture of Erdős and Szűsz Related to Uniform Distribution Mod 1," Acta Arithmetica, 12 (1966), 193-212.
4. Khinchin, A. Ya. Continued Fractions. Chicago: University of Chicago Press, 1964.
5. Kuipers, L., and H. Niederreiter. Uniform Distribution of Sequences. New York: John Wiley and Sons (Interscience), 1974.
6. Lesca, J. "Sur la Répartition Modulo 1 des Suites $(n\alpha)$," Seminaire Delange - Pisot - Poitou (théorie des nombres) 8e année (1966/67), no. 2.
7. Sataev, E. "On the Number of Invariant Measures for Flows on Orientable Surfaces," Math. USSR Izvestija, 9 (1975), no. 4, 813-830.
8. Schmidt, K. "Cohomology and Skew Products of Ergodic Transformations," Preprint, University of Warwick, 1974.
9. _____. "A Cylinder Flow Arising From Irregularity of Distribution," Preprint, University of Warwick, 1975.
10. Veech, W. A. "Strict Ergodicity in Zero-Dimensional Dynamical Systems and the Kronecker-Weyl Theorem Mod 2," Transactions of the American Mathematical Society, 140 (1969), 1-33.
11. _____. "Well Distributed Sequences of Integers," Transactions of the American Mathematical Society, 161 (1971), 63-70.
12. _____. "Topological Dynamics," Bulletin of the American Mathematical Society, 83 (1977), 775-830.