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OPTIMUM PLASTIC DESIGN OF SPACE FRAMES

by

Mohsen M. K. Khalifa

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

Thesis Director's signature:

A handwritten signature in cursive script, reading "John E. Merwin", is written over a solid horizontal line.

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NOMENCLATURE

A	Cross-sectional area
B	Constraint error
D	Load factor
E	Modulus of elasticity
F	Flexibility matrix. Augmented function
F_x	Gradient of augmented function
G	Shear modulus
I	Moment of inertia
I_p	Polar moment of inertia
J	Torsional constant
K	Global stiffness matrix
M_{px}	Full plastic moment about the major axis
M_{py}	Full plastic moment about the minor axis
M_t	Torsional moment
M_x	Moment about x-axis
M_y	Moment about y-axis
N	Axial force
N_p	Plastic axial force
N_t	Torsional buckling force
P	External force vector
Q	Optimality error
R	Redundant vector

T_x	Reduced plastic moment about the major axis
T_y	Reduced plastic moment about the minor axis
V	Number of critical sections
W	Number of members
Y	Degree of indeterminacy
Z	Dummy variable vector
Z_x	Plastic section modulus with respect to the major axis
Z_y	Plastic section modulus with respect to the minor axis
d	Depth of cross-section
f	Objective function
f_x	Gradient of objective function
h	Frame height
k	Member stiffness matrix
l	General length
m	Number of constraints
n	Number of variables
p	Internal force vector
t	Thickness
x	Variable vector
z	Dummy variable
α	Ratio of applied moment to plastic moment
γ	Gradient step size
δ	Displacement vector

η	Ratio of applied axial force to plastic axial force
λ	Lagrange multiplier vector
φ	Constraint vector
φ_x	Constraint derivative matrix
φ_{xx}	Tensor of second derivatives
u	Lagrange multiplier vector
ρ	Restoration step size
σ_y	Yield stress
σ_{tb}	Stress at which torsional buckling occur
Δ	Displacement vector

CHAPTER I

INTRODUCTION

1.1 Background

The problem of optimum design is that of designing a structure, with a given geometry and required ultimate loads, such that its cost function is minimized. The total weight of the structure furnishes a good measure of its cost (14). Thus the problem can be viewed as that of minimizing the weight of the structure subject to constraints so as to lead to both a safe and a statically admissible internal force distribution in the structure.

Considerable work has been done in the problem of optimum design of plane frames (2). In most of the work only the plastic analysis part of the problem was considered. Few attempts, however, have been made to develop optimum design solutions which consider the frame instability effect. Additional constraints for each member in the optimization process to prevent local buckling has been suggested by Pope (21). An attempt has also been made to consider both frame instability effect and deflection limitations under service loads by Nakamura and Little (20).

Analysis neglecting axial strains was shown to be inaccurate (9,26) and gives low lateral displacements and joint rotations. Reinschmidt, Cornell and Brotchie (22) suggested the use of a radical initial design to give the solution algorithm more freedom to maneuver, but Nakamura showed that a radical initial design may converge to a meaningless structure.

For space frames, the problem cannot be considered as that of optimizing sets of independent plane frames and then using superposition because of the interaction between these sets. In the work done by McVinnie and Gaylord (15) the material distribution in the space frame and the slenderness ratios of the columns were assumed, and no optimization procedure was considered. Stability analysis, using moment distribution and plastic mechanism analysis assuming two possible failure mechanisms, were considered separately. The study was limited to vertical symmetrical loading and symmetrical frames.

1.2 Scope of Present Work

This study deals with the problem of optimizing space frames. Of primary concern is the formulation of the problem and the development of methods for its solution.

Also of importance is the study of the interaction of stability and plastic mechanism failures. The formation of plastic hinges reduces the frame stiffness and instability of the frame may occur before the formation of a plastic mechanism.

An elastic-plastic analysis of the optimized space frames will provide limits on the range in which these frames remain elastic. Also this study will lead to a relation between the actual failure load and the plastic mechanism load for these frames.

In Chapter 2 the problem of optimizing a general space frame is formulated and the method for its solution is presented. The basic variables considered in the optimization procedure are the member cross-sectional areas and the set of redundants needed to convert the statically indeterminate space frame into a determinate one. The sequential gradient-restoration algorithm (18) employed in optimizing these space frames is also presented.

In Chapter 3 a detailed analysis of the optimality problem is presented. The internal forces at critical sections are expressed in terms of the redundants and the applied set of ultimate loads. The objective function is expressed in terms of the members cross-sectional areas.

The constraints in the optimization problem consists of plastic capacity, torsion and torsional buckling constraints. The problem is also linearized in this chapter, and the symmetric method (11) is employed in its solution.

In Chapter 4 the stability problem of space frames is studied. Failure of these frames, due to their instability, may occur after the formation of only a few plastic hinges and before a plastic mechanism has been reached. An elastic-plastic method of analysis is presented in this chapter to study this problem. This analysis also provides the order in which the plastic hinges will form in the space frame up to its failure.

In Chapter 5 the methods developed in the previous chapters are applied to simple space frames. The results are presented and discussed.

Chapter 6 gives the summary and the main conclusions of this work.

CHAPTER II

NONLINEAR OPTIMIZATION AND FRAME DESIGN

2.1 Description and Approach to General Problem

The problem of optimum plastic design is that of designing a structure, with a given geometry, so as to support given ultimate loads and simultaneously minimize its cost function. Economy factors depend on a large number of variables, such as the relative cost of labor and steel, cost of manufacturing and aesthetic considerations. For a given overall dimension of a structure the cost of manufacturing the connections does not usually vary appreciably from one design to another, and the total weight of the structure furnishes a good measure of its cost.

Thus, the problem can now be viewed as that of minimizing the weight of the structure, subject to constraints that will lead to a both safe and statically admissible internal force distribution in the structure.

Before formulating the problem, the following familiar assumptions for plastic design are stated.

1. The combination of internal forces at any point cannot exceed the plastic capacity of the section.

2. Because of the ductility of the structural metal, narrow zones of high curvature can develop at critical sections. This effect may be considered as concentrated at the critical sections and will be called a plastic hinge.

3. The structural metal is sufficiently ductile to sustain the large local strains in the narrow zones of high curvature without breaking.

4. The structure does not become unstable before the formation of a plastic mechanism.

5. The deformations of the structure are acceptable and they do not appreciably influence the equilibrium conditions of the structure.

6. Each joint of the structure is capable of transmitting the maximum plastic forces and moments of the connected members.

7. The loads applied to the structure are assumed to increase in proportion.

Structural steel possesses the ductility properties stipulated under 1, 2 and 3 above. For assumptions 4 and 5, a separate elastic-plastic analysis is performed in Chapter 4 to account for stability and deformation effects.

For a given set of loads acting on a structure, the potentially critical sections where a plastic hinge might form have a finite number and plastic capacity constraints need be formulated only at those critical sections. For concentrated loads these critical sections are limited to the ends of each member and to points where the concentrated loads are applied.

For a general space structure having W members, V critical sections and which is Y -times statically indeterminate, the problem can be formulated as explained below.

If the i^{th} member of the structure has a cross-sectional area A_i and length l_i then the objective function to be minimized is

$$f = \sum_{i=1}^W A_i l_i \quad (2.1)$$

As seen, this function is linear in the W unknown areas.

This Y -times statically indeterminate space structure can be converted to a convenient statically determinate structure by introducing a set of Y unknown redundants R . The six independent internal forces p (2 moments, 1 torque, 2 shears, 1 axial force) present at each of the V critical sections can be expressed through equilibrium equations as linear functions of these

Y unknown redundants and the given applied set of ultimate loads P.

$$p = w(R, P) \quad (2.2)$$

If the geometric shape of the cross-section is to be predetermined, and not to be optimized, then only one unknown is needed to represent the section property in the optimization procedure. This can be conveniently chosen to be the cross-sectional area A. All other section properties needed in the optimization procedure can be expressed as functions of the area A. Thus the plastic capacity function S of any section can be expressed in the form

$$S(p^*) = \psi(A, \sigma_y) \quad (2.3)$$

where,

p^* = any of the infinite combinations of
internal forces causing plastification
at the section

σ_y = yield stress of the material.

Equation 2.3 determines, in general, a nonlinear interaction surface in seven dimensional space, giving the relation between the area A and the six internal forces p^* .

The constraints preventing the combination of internal forces p at any of the critical sections from exceeding its plastic capacity can thus be expressed in the form

$$S(p) \leq \psi(A, \sigma_y) \quad (2.4)$$

Substituting by equation 2.2, the left hand side of inequality 2.4 can be expressed in terms of a function, Γ , of the redundants R and the applied set of ultimate loads P . Thus inequality 2.4 becomes

$$\Gamma(R, P) \leq \psi(A, \sigma_y) \quad (2.5)$$

There are V of these nonlinear inequality constraint, one for each of the V critical sections. The unknown variables in these constraints are the W cross-sectional areas A and the Y redundants R .

Specifically, the problem now is that of minimizing the objective function given by equation 2.1, subject to the V constraints of the form given by inequality 2.5 .

However, determining the interaction surface given by equation 2.3 is a complicated task. The fact that, up to high levels, some of the internal forces, namely the two shear forces, have a negligible effect on the plastic capacity of many structural sections can simplify the problem. If one further uncouples the torsional moment

from the interaction equation 2.3 and considers it separately, then the three internal forces left are the axial force and the two flexural moments. This greatly simplifies the interaction equation 2.3 and it can be rewritten in the form

$$S_r(p_r^*) = \psi_r(A, \sigma_y) \quad (2.6)$$

where,

$$p_r^* = \text{any set of axial force and flexural bending moments causing plastification at the section.}$$

Inequality 2.4 can be rewritten in the form

$$S_r(p_r) \leq \psi_r(A, \sigma_y) \quad (2.7)$$

where

$$p_r = \text{the set of axial force and flexural bending moments at any critical section.}$$

Similarly the inequality constraint 2.5 can be rewritten in the form

$$\Gamma_r(R, P) \leq \psi_r(A, \sigma_y) \quad (2.8)$$

Additional constraints are now needed to limit the amount of torsional moment in each member. The torsional moment M_t can be expressed, through equilibrium

equations, as a linear function of the unknown set of redundants and the given applied set of ultimate loads.

$$M_t = \theta(R, P) \quad (2.9)$$

If for the i^{th} member the torsional moment is M_{ti} and the allowed torsional moment that causes yielding of the section is M_{ti}^* , then the torsional constraints are

$$M_{ti} \leq M_{ti}^* \quad i = 1, 2, \dots, W \quad (2.10)$$

The shape of the cross-section considered in this work is that of a wide flange I section. For this shape it was reasonable to set the allowed torsional moment M_t^* to zero.

The problem of minimizing the objective function given by equation 2.1, subject to the V constraints of the form given by inequality 2.5, can now be replaced, for convenience, by that of minimizing the same objective function subject to the V constraints of the form given by inequality 2.8 and to the W constraints given by inequality 2.10.

These constraints will lead as previously mentioned to optimum values of the areas and the redundants that furnishes both a safe and a statically admissible internal force distribution in the space structure. Any additional

constraints that are felt needed to account for other effects, such as stability and deflection limitations, can also be added. As mentioned before, in this study the shape of the cross-section considered was a wide flange I section. It was found that additional constraints to prevent torsional buckling of these thin-walled sections are needed. These constraints can be written in the form

$$N_i \leq N_{Ti} \quad i = 1, 2, \dots, W \quad (2.11)$$

where N_i and N_{Ti} are the axial force and the torsional buckling force in the i^{th} member respectively. The axial force being a linear function of the unknown set of redundants and the known set of applied ultimate loads. The torsional buckling force is a function of the member cross-section and the material properties.

It should be again emphasized that due to assumptions 4 and 5 previously mentioned, no guarantee exists that the optimum space structure obtained will not fail due to instability before a plastic mechanism is reached. To account for this and for the deflections of the structure an elastic-plastic analysis of the optimized structure is performed in Chapter 4.

In summary, the problem now is that of minimizing the objective function given by equation 2.1, subject to

the constraints given by inequalities 2.8, 2.10 and 2.11. For any space structure, this nonlinear optimization problem involves a large number of unknowns. These being the cross-sectional areas of the members and the set of redundants needed to transfer the indeterminate space structure to a determinate one. In addition to these basic unknowns, a dummy unknown for each inequality constraint is needed to transfer it to an equality constraint.

A nonlinear constrained optimization problem, with such a number of variables, should be tackled carefully. Three different optimization methods were attempted. These were the sequential gradient-restoration method (18), the augmented penalty function method (17) and the modified quasilinearization method (19). The first two are first order methods and the third is a second order method in which a tensor for the second partial derivatives of the constraints with respect to the variables had to be computed. The one that seemed most suitable for the problem, with its large number of constraints, was the sequential gradient-restoration algorithm explained in Section 2.2.

To reduce the computation time, a good initial choice of the variables was needed to initiate the algorithm. This was done by linearizing the problem

(Section 3.7). Then the symmetric method (11) was used to obtain the optimum solution for this linear problem. This method is a version of the primal-dual approach to linear programming where the optimal solution is usually obtained in a small number of iterations. The number of iterations required is reduced by choosing the pivotal elements so as to satisfy the feasibility set as soon as possible and then move to the optimal solution as rapidly as possible. The optimal solution may be reached very efficiently by jumping over redundant constraints. A main advantage of this method is that the identity matrix is not included and thus a considerable saving in storage is achieved. A flow chart of this method is given in Appendix A. The solution obtained from the symmetric method was then used as the starting point for the non-linear optimization procedure used, which is the sequential gradient-restoration algorithm explained in the following section.

2.2 Sequential Gradient-Restoration Algorithm

The sequential gradient-restoration algorithm is due to Miele (16,18) and is presented here for the sake of completeness.

The problem considered is that of minimizing the objective function

$$f = f(x) \quad (2.12)$$

subject to the constraint

$$\varphi(x) = 0 \quad (2.13)$$

where,

f = scalar

x = variable vector

φ = constraint vector.

Letting

n = number of variables

m = number of constraints, $m < n$,

then the variable and constraint vectors are of dimensions $n \times 1$ and $m \times 1$ respectively.

2.2.1 Gradient Phase

It is assumed that the gradient phase is started with a nominal point x satisfying the constraint equation 2.13. The following augmented function can be introduced

$$F(x, \lambda) = f(x) + \lambda^T \varphi(x) \quad (2.14)$$

where

λ = Lagrange multiplier vector of dimension $m \times 1$.

Since at the nominal point x equation 2.13 is satisfied, the augmented function $F(x, \lambda)$ and the objective function $f(x)$ are numerically equal.

Expansion of the augmented function to first order gives

$$F(\tilde{x}, \lambda) = F(x, \lambda) + \delta F(x, \lambda) \quad (2.15)$$

where

$$\tilde{x} = \text{varied point.}$$

One needs that

$$F(\tilde{x}, \lambda) < F(x, \lambda) . \quad (2.16)$$

Thus the first variation $\delta F(x, \lambda)$ should be negative.

But

$$\delta F(x, \lambda) = F_x^T(x, \lambda) \Delta x \quad (2.17)$$

where

$$F_x(x, \lambda) = \begin{array}{l} \text{gradient of the augmented} \\ \text{function } F \text{ with respect to the} \\ \text{vector } x \end{array}$$

$$\Delta x = \begin{array}{l} \text{the displacement vector leading} \\ \text{from the nominal point } x \text{ to the} \\ \text{varied point } \tilde{x} . \end{array}$$

$$\tilde{x} = x + \Delta x \quad (2.18)$$

If one chooses the displacement vector Δx to have the same direction as the gradient of the augmented function, or

$$\Delta x = - \gamma F_x(x, \lambda) \quad (2.19)$$

where

$$\gamma = \text{gradient step size, } \gamma > 0$$

then substituting in equation 2.17, the first variation becomes

$$\delta F(x, \lambda) = - \gamma F_x^T(x, \lambda) F_x(x, \lambda) \quad (2.20)$$

Equation 2.20 shows that, for $\gamma > 0$, the first variation of the augmented function is negative. Therefore, for sufficiently small step size γ , the augmented function decreases during the gradient phase. This provides the needed descent property of the gradient phase given by inequality 2.16.

It is to be noted that the displacement vector Δx given by equation 2.19 is not yet known, since both the Lagrange multiplier vector λ and the step size γ are yet to be determined.

Determination of the Lagrange multiplier vector

The Lagrange multiplier vector λ is to be determined so as to satisfy the constraint equation 2.13 to first order. Expansion of the constraint to first order gives

$$\varphi(\tilde{x}) = \varphi(x) + \delta \varphi(x) \quad (2.21)$$

Thus

$$\delta\varphi(x) = 0 \quad (2.22)$$

But,

$$\delta\varphi(x) = \varphi_x^T(x) \Delta x \quad (2.23)$$

where

$\varphi_x(x)$ = constraint derivative matrix of dimension $n \times m$ giving the gradient of the vectorial function φ with respect to the vector x .

Substituting by equation 2.19 in equation 2.23, then

$$\delta\varphi(x) = - \gamma \varphi_x^T(x) F_x(x, \lambda) .$$

From equation 2.14

$$F_x(x, \lambda) = f_x(x) + \varphi_x(x) \lambda$$

where

$f_x(x)$ = gradient of the scalar function f with respect to the vector x .

Thus

$$\delta\varphi(x) = - \gamma \varphi_x^T(x) [f_x(x) + \varphi_x(x) \lambda]$$

Therefore for $\delta\varphi = 0$ one gets

$$\varphi_x^T(x) [f_x(x) + \varphi_x(x) \lambda] = 0$$

from which

$$\lambda = - [\varphi_x^T(x) \varphi_x(x)]^{-1} [\varphi_x^T(x) f_x(x)] \quad (2.24)$$

The first variation of $F(x, \lambda)$ can be written as

$$\delta F(x, \lambda) = \delta f(x) + \lambda^T \delta \varphi(x) \quad (2.25)$$

Because of equation 2.22, equation 2.25 reduces to

$$\delta F(x, \lambda) = \delta f(x) \quad (2.26)$$

Thus the objective function $f(x)$ and the augmented function $F(x, \lambda)$ behave identically to first order. Since the descent property for the augmented function has been demonstrated, it also holds for the objective function $f(x)$.

Attempting to avoid a restoration phase by determining the Lagrange multiplier vector so as to continue satisfying the constraint equation 2.13 exactly at the varied point as well will lead to the following system of equations

$$\varphi(\tilde{x}) = 0$$

from equations 2.18 and 2.19, thus

$$\varphi \left[x - \gamma F_x(x, \lambda) \right] = 0 \quad (2.27)$$

In general the constraints are nonlinear and the above is a system of m nonlinear equations in the m Lagrange multipliers.

Determination of the step size

From equations 2.18 and 2.19

$$\tilde{x} = x - \gamma F_x(x, \lambda) \quad (2.28)$$

Since λ is now known from equation 2.24, then equation 2.28 defines a one-parameter family of points $\tilde{x}$, for which the augmented function F takes the form

$$F[\tilde{x}, \lambda] = F[x - \gamma F_x(x, \lambda), \lambda] = F(\gamma)$$

The best value of the step size γ , along this direction, is that γ that gives the lowest value of the augmented function F ; thus it should satisfy

$$F_\gamma(\gamma) = 0 \quad (2.29)$$

But,

$$F_\gamma(\gamma) = -F_x^T(\tilde{x}, \lambda) F_x(x, \lambda)$$

Thus one gets the necessary condition

$$F_x^T(\tilde{x}, \lambda) F_x(x, \lambda) = 0 \quad (2.30)$$

Equation 2.30 shows that the best value of the step size γ is that that causes the orthogonality of the augmented function at the nominal point x and the varied point $\tilde{x}$. However, the constraint equation 2.13 was satisfied only to first order. Consequently this value

of the step size might result in a too large constraint error, and the large restoration displacements then needed will offset the large decrease that was gained in the augmented function. This is usually the case when the nominal point x is far from the constrained minimum. This necessitates imposing a limit on the value that the step size could take.

The constraint error B is

$$B(x) = \varphi^T(x) \varphi(x) \quad (2.31)$$

At the varied point $\tilde{x}$

$$B[\tilde{x}] = B[x - \gamma F_x(x, \lambda)] = B(\gamma)$$

Thus a suitable limit on the value of the step size γ is

$$B(\gamma) \leq B^* \quad (2.32)$$

where,

$$B^* = \text{prechosen maximum allowed constraint error.}$$

A one-dimensional search employing cubic interpolation was used to obtain a value for the step size γ that satisfies the inequality

$$F(\gamma) < F(0) \quad (2.33)$$

This is guaranteed to occur because of the descent property of the gradient phase previously demonstrated.

2.2.2 Restoration Phase

If the constraints are linear, they will continue to be satisfied at the varied point $\tilde{\mathbf{x}}$, and

$$\varphi(\tilde{\mathbf{x}}) = 0$$

Thus the error in the constraint will vanish at the varied point eliminating the need for a restoration phase. If, on the other hand, the constraint equation is nonlinear, then the constraint will not be satisfied at the varied point $\tilde{\mathbf{x}}$, and

$$\varphi(\tilde{\mathbf{x}}) \neq 0$$

Thus a restoration phase is needed before starting the next gradient phase. Specifically one needs to apply a displacement vector $\Delta\tilde{\mathbf{x}}$ so that a new point $\hat{\mathbf{x}}$ can be reached at which the constraint equation is satisfied.

$$\hat{\mathbf{x}} = \tilde{\mathbf{x}} + \Delta\tilde{\mathbf{x}} \quad (2.34)$$

$$\text{and} \quad \varphi(\hat{\mathbf{x}}) = 0 \quad (2.35)$$

Expansion to first order gives

$$\varphi(\hat{\mathbf{x}}) = \varphi(\tilde{\mathbf{x}}) + \delta\varphi(\tilde{\mathbf{x}}) \quad (2.36)$$

Thus,

$$\varphi(\tilde{\mathbf{x}}) + \delta\varphi(\tilde{\mathbf{x}}) = 0$$

or,

$$\varphi(\tilde{\mathbf{x}}) + \varphi_{\mathbf{x}}^T(\tilde{\mathbf{x}}) \Delta\tilde{\mathbf{x}} = 0 \quad (2.37)$$

To prevent the variation $\Delta\tilde{\mathbf{x}}$ from becoming too large in case of highly nonlinear constraints, equation 2.37 is imbedded into the one-parameter family of equations

$$\rho\varphi(\tilde{\mathbf{x}}) + \varphi_{\mathbf{x}}^T(\tilde{\mathbf{x}}) \Delta\tilde{\mathbf{x}} = 0 \quad (2.38)$$

where

$$\rho = \text{restoration step size} \quad 0 \leq \rho \leq 1$$

This is a system of m equations in n unknown displacements $\Delta\tilde{\mathbf{x}}$. This freedom in restoring the constraint is to be controlled in a convenient way by causing the least disturbance to the decreased value of the augmented function at point $\tilde{\mathbf{x}}$. This is done by requiring the least-square change of the position vector. Thus the function to be minimized is

$$J = \frac{1}{2} \Delta\tilde{\mathbf{x}}^T \Delta\tilde{\mathbf{x}} \quad (2.39)$$

This minimization is subject to the linearized constraint 2.38. The augmented function is thus

$$F^* = \frac{1}{2} \Delta\tilde{\mathbf{x}}^T \Delta\tilde{\mathbf{x}} + \rho\mu^T [\rho\varphi(\tilde{\mathbf{x}}) + \varphi_{\mathbf{x}}^T(\tilde{\mathbf{x}}) \Delta\tilde{\mathbf{x}}] \quad (2.40)$$

where,

$$\mu = \text{Lagrange multiplier vector of dimension } m \times 1.$$

The optimal value for the displacement vector $\Delta \tilde{x}$ should satisfy the relation

$$F_{\Delta \tilde{x}}^* = 0 \quad (2.41)$$

Thus

$$\Delta \tilde{x} + \rho \varphi_x(\tilde{x}) \mu = 0$$

or,

$$\Delta \tilde{x} = - \rho \varphi_x(\tilde{x}) \mu \quad (2.42)$$

Substitution in the constraint equation 2.38 gives

$$\rho \varphi(\tilde{x}) - \rho \varphi_x^T(\tilde{x}) \varphi_x(\tilde{x}) \mu = 0$$

or,

$$\mu = \left[\varphi_x^T(\tilde{x}) \varphi_x(\tilde{x}) \right]^{-1} \varphi(\tilde{x}) \quad (2.43)$$

Equation 2.43 gives the Lagrange multiplier vector μ .

The descent property of the restoration phase can be demonstrated as follows.

From equation 2.31 the constraint error at point $\tilde{x}$ is

$$B(\tilde{x}) = \varphi^T(\tilde{x}) \varphi(\tilde{x})$$

The first variation of $B(\tilde{x})$ is thus

$$\delta B(\tilde{x}) = 2 \varphi^T(\tilde{x}) \delta \varphi(\tilde{x})$$

or,

$$\delta B(\tilde{x}) = 2 \varphi^T(\tilde{x}) \varphi_x^T(\tilde{x}) \Delta \tilde{x} \quad (2.44)$$

Substitution by equation 2.38 in equation 2.44 gives

$$\delta B(\tilde{x}) = -2 \rho \varphi^T(\tilde{x}) \varphi(\tilde{x})$$

or,

$$\delta B(\tilde{x}) = -2 \rho B(\tilde{x}) \quad (2.45)$$

Since $B(\tilde{x}) > 0$, equation 2.45 shows that for $\rho > 0$ the first variation of the constraint error is negative. Therefore for sufficiently small step size ρ , the constraint error decreases during the restoration phase. This provides the needed descent property.

Determination of the step size

The restoration step size ρ can be conveniently determined by starting with $\rho = 1$ and if the constraint error B did not decrease, then by successively bisecting ρ it is guaranteed, by the descent property, that a decrease in the constraint error can be achieved.

The restoration phase is to be performed interactively until the constraint is restored to the desired degree of accuracy, that is, until

$$B(x) \leq \epsilon_1 \quad (2.46)$$

where,

$$\epsilon_1 = \text{prechosen small number.}$$

The following order of magnitude analysis is needed to demonstrate the descent property of the whole algorithm.

From equation 2.19 it is seen that the gradient displacement has the order

$$\Delta x = O(\gamma) \quad (2.47)$$

Expansion of the constraint function to second order gives

$$\varphi(\tilde{x}) = \varphi(x) + \varphi_x^T(x) \Delta x + \frac{1}{2} \Delta x^T \varphi_{xx}(x) \Delta x \quad (2.48)$$

where

$$\varphi_{xx} = \text{tensor of second partial derivatives of the vectorial function } \varphi$$

In equation 2.48, the first term vanishes since the nominal point x was chosen satisfying the constraint equation 2.13. The second term also vanishes since the Lagrange multiplier vector λ for the gradient phase was chosen so as to produce the displacement vector Δx satisfying the constraint to first order. Thus from equations 2.47 and 2.48 it can be seen that

$$\varphi(\tilde{x}) = O(\gamma^2) \quad (2.49)$$

Thus the restoration displacement $\Delta \tilde{x}$ has the order

$$\Delta \tilde{x} = O(\gamma^2) \quad (2.50)$$

It is seen from equations 2.47 and 2.50 that the restoration displacement $\Delta\tilde{x}$ can be neglected with respect to the gradient displacement Δx . This consequently guarantees the descent property of the whole algorithm since the restoration phase will preserve the descent property of the gradient phase, which was previously proved.

After restoration the following inequality should be checked.

$$f(\hat{x}) < f(x) \quad (2.51)$$

If inequality 2.51 is satisfied, the next gradient phase can be started. If it is not satisfied, the step size γ of the gradient phase should be reduced, by bisecting process for example, and after restoration, if inequality 2.51 is not yet satisfied, then a further reduction of the step size γ is needed. It is guaranteed from the descent property of the whole algorithm that satisfaction of inequality 2.51 will occur.

Stopping condition

The error in the optimality condition Q is

$$Q = F_x^T(x, \lambda) F_x(x, \lambda) \quad (2.52)$$

The algorithm is stopped when this error is less than some small number ϵ_2

$$Q(x, \lambda) \leq \epsilon_2 \quad (2.53)$$

2.2.3 Summary of the Algorithm

A brief summary of the algorithm is presented diagrammatically in the flow chart of Figure 2.1. However, several additional remarks are mentioned below.

For the algorithm to start, a nominal point x satisfying the constraint equation 2.13 is needed. If this is not the case, the algorithm should simply start with the restoration phase first and, after restoration, a gradient phase can be started. Another remark is that after each gradient phase, the constraint error B is calculated and if it is found to satisfy the inequality 2.46 then the restoration phase is actually bypassed.

2.3 Summary of Approach

The approach employed in analyzing the problem is briefly summarized below.

The space frame considered is first converted, as presented in Section 3.1, to a statically determinate frame by introducing a set of unknown redundants. The

FIGURE 2.1

internal forces at the critical sections are then expressed in terms of these unknown redundants and the known set of applied ultimate loads.

The frame members are chosen to have wide flange I-sections. For these sections, a plastic capacity interaction surface limiting the amount of axial force and flexural bending moments is developed in Section 3.2.

In Section 3.3 the constraints for the optimization procedure are presented. First, plastic capacity constraints are developed. These constraints limit the combination of the axial force and the two flexural moments about both principal axes not to exceed, at each critical section, the plastic capacity of that section. Secondly, torsional buckling constraints are developed. These constraints limit the amount of axial force in each member not to exceed the torsional buckling load of that member. Finally, torsion constraints are developed. These constraints limit the amount of torsional moment in each member.

The unknown variables vector is then formulated in Section 3.4. This vector consists of the member areas, the redundants and the dummy variables arising from the inequality constraints.

In Section 3.5 the partial derivatives of all the constraints with respect to the variables are calculated. The constraint derivative matrix, needed in the sequential gradient restoration algorithm, is then formulated.

The weight of the frame is to be minimized and the objective function representing that is given in Section 3.6.

To obtain the optimum solution for the nonlinear problem the problem is first linearized in Section 3.7. The optimum solution for this linear problem is then used as the starting point for the sequential gradient restoration algorithm explained before in Section 2.2.

Finally, to account for stability and deflection effects on the failure load, a step-by-step elastic-plastic analysis is performed in Chapter 4 for the optimized frame. This analysis is continued until failure of the frame is achieved.

CHAPTER III

FORMULATION OF OPTIMALITY PROBLEM

3.1 Internal Force Vectors

The space structure considered is the simple frame shown in Figure 3.1. This frame is 24 times statically indeterminate. A convenient main system is that shown in Figure 3.2, where the six internal forces present at each of the four cuts are considered as the redundants.

The space frame is considered to be loaded by any set of concentrated loads acting at mid-span of each beam and at the four joints of the frame. This limits the critical sections in the frame to the twenty positions shown in Figure 3.3.

3.1.1 Bending Moment Vectors

If the x and y axes, and the positive sense for the moments M_x and M_y about them are as shown in Figure 3.4, it can be easily seen that the elements of the bending moment vectors M_x and M_y at the twenty points are:

$$M_x(1) = R_4 + M_{xl}(1)$$

$$M_x(2) = R_3 \frac{l_k}{2} + R_4 + M_{xl}(2)$$

FIGURE 3.1

FIGURE 3.2

FIGURE 3.3

FIGURE 3.4

$$M_{\mathbf{x}}(3) = R_3 \ell_k + R_4 + M_{\mathbf{x}\ell}(3)$$

$$M_{\mathbf{x}}(4) = R_{10} + M_{\mathbf{x}\ell}(4)$$

$$M_{\mathbf{x}}(5) = R_9 \frac{\ell_h}{2} + R_{10} + M_{\mathbf{x}\ell}(5)$$

$$M_{\mathbf{x}}(6) = R_9 \ell_h + R_{10} + M_{\mathbf{x}\ell}(6)$$

$$M_{\mathbf{x}}(7) = R_{16} + M_{\mathbf{x}\ell}(7)$$

$$M_{\mathbf{x}}(8) = R_{15} \frac{\ell_k}{2} + R_{16} + M_{\mathbf{x}\ell}(8)$$

$$M_{\mathbf{x}}(9) = R_{15} \ell_k + R_{16} + M_{\mathbf{x}\ell}(9)$$

$$M_{\mathbf{x}}(10) = R_{22} + M_{\mathbf{x}\ell}(10)$$

$$M_{\mathbf{x}}(11) = R_{21} \frac{\ell_h}{2} + R_{22} + M_{\mathbf{x}\ell}(11)$$

$$M_{\mathbf{x}}(12) = R_{21} \ell_h + R_{22} + M_{\mathbf{x}\ell}(12)$$

$$M_{\mathbf{x}}(13) = R_3 \ell_k + R_4 - R_{23} + M_{\mathbf{x}\ell}(13)$$

$$M_{\mathbf{x}}(14) = -R_2^h + R_3 \ell_k + R_4 - R_{19}^h - R_{23} + M_{\mathbf{x}\ell}(14)$$

$$M_{\mathbf{x}}(15) = R_4 + R_{11} + M_{\mathbf{x}\ell}(15)$$

$$M_{\mathbf{x}}(16) = -R_2^h + R_4 + R_7^h + R_{11} + M_{\mathbf{x}\ell}(16)$$

$$M_{\mathbf{x}}(17) = -R_{11} + R_{15} \ell_k + R_{16} + M_{\mathbf{x}\ell}(17)$$

$$M_{\mathbf{x}}(18) = -R_7^h - R_{11} - R_{14}^h + R_{15} \ell_k + R_{16} + M_{\mathbf{x}\ell}(18)$$

$$M_{\mathbf{x}}(19) = R_{16} + R_{23} + M_{\mathbf{x}\ell}(19)$$

$$M_{\mathbf{x}}(20) = -R_{14}^h + R_{16} + R_{19}^h + R_{23} + M_{\mathbf{x}\ell}(20)$$

(3.1)

and,

$$M_{\mathbf{Y}}(1) = -R_6 + M_{\mathbf{Y}\ell}(1)$$

$$M_{\mathbf{Y}}(2) = R_1 \frac{\ell_k}{2} - R_6 + M_{\mathbf{Y}\ell}(2)$$

$$M_{\mathbf{Y}}(3) = R_1 \ell_k - R_6 + M_{\mathbf{Y}\ell}(3)$$

$$M_{\mathbf{Y}}(4) = -R_{12} + M_{\mathbf{Y}\ell}(4)$$

$$M_{\mathbf{Y}}(5) = R_7 \frac{\ell_h}{2} - R_{12} + M_{\mathbf{Y}\ell}(5)$$

$$M_{\mathbf{Y}}(6) = R_7 \ell_h - R_{12} + M_{\mathbf{Y}\ell}(6)$$

$$M_{\mathbf{Y}}(7) = -R_{18} + M_{\mathbf{Y}\ell}(7)$$

$$M_{\mathbf{Y}}(8) = R_{13} \frac{\ell_k}{2} - R_{18} + M_{\mathbf{Y}\ell}(8)$$

$$M_{\mathbf{Y}}(9) = R_{13} \ell_k - R_{18} + M_{\mathbf{Y}\ell}(9)$$

$$M_{\mathbf{Y}}(10) = -R_{24} + M_{\mathbf{Y}\ell}(10)$$

$$M_{\mathbf{Y}}(11) = R_{19} \frac{\ell_h}{2} - R_{24} + M_{\mathbf{Y}\ell}(11)$$

$$M_{\mathbf{Y}}(12) = R_{19} \ell_h - R_{24} + M_{\mathbf{Y}\ell}(12)$$

$$M_{Y(13)} = R_5 + R_{22} + M_{Y\ell(13)}$$

$$M_{Y(14)} = R_1^h + R_5 - R_{20}^h + R_{22} + M_{Y\ell(14)}$$

$$M_{Y(15)} = -R_5 + R_9 \ell_h + R_{10} + M_{Y\ell(15)}$$

$$M_{Y(16)} = -R_1^h - R_5 - R_8^h + R_9 \ell_h + R_{10} + M_{Y\ell(16)}$$

$$M_{Y(17)} = R_{10} + R_{17} + M_{Y\ell(17)}$$

$$M_{Y(18)} = -R_8^h + R_{10} + R_{13}^h + R_{17} + M_{Y\ell(18)}$$

$$M_{Y(19)} = -R_{17} + R_{21} \ell_h + R_{22} + M_{Y\ell(19)}$$

$$M_{Y(20)} = -R_{13}^h - R_{17} - R_{20}^h + R_{21} \ell_h + R_{22} + M_{Y\ell(20)}$$

(3.2)

where $M_{x\ell}$ and $M_{Y\ell}$ are the bending moment vectors in the main system about the x and y axes due to the applied set of loads.

3.1.2 Torsional Moment Vector

If counter-clockwise torsional moment is considered as positive, the elements of the torsional moment vector M_t in the eight members are:

$$M_t(1) = R_5$$

$$M_t(2) = M_Y(10) - M_Y(3) = -R_1 \ell_k + R_6 - R_{24} + M_{Y\ell}(10) - M_{Y\ell}(3)$$

$$M_t(3) = R_{11}$$

$$M_t(4) = M_Y(1) - M_Y(6) = -R_6 - R_7 \ell_h + R_{12} + M_{Y\ell}(1) - M_{Y\ell}(6)$$

$$M_t(5) = R_{17}$$

$$M_t(6) = M_Y(4) - M_Y(9) = -R_{12} - R_{13} \ell_k + R_{18} + M_{Y\ell}(4) - M_{Y\ell}(9)$$

$$M_t(7) = R_{23}$$

$$M_t(8) = M_Y(7) - M_Y(12) = -R_{18} - R_{19} \ell_h + R_{24} + M_{Y\ell}(7) - M_{Y\ell}(12)$$

(3.3)

3.1.3 Axial Force Vector

If compression force is considered as positive, the elements of the axial force vector N in the eight members are:

$$N_1 = -R_2 + N\ell_{(1)}$$

$$N_2 = -R_3 + R_{21} + N\ell_{(2)}$$

$$N_3 = -R_8 + N\ell_{(3)}$$

$$N_4 = R_3 - R_9 + N\ell_{(4)}$$

$$N_5 = -R_{14} + N\ell_{(5)}$$

$$N_6 = R_9 - R_{15} + N\ell_{(6)}$$

$$N_7 = -R_{20} + N\ell_{(7)}$$

$$N_8 = R_{15} - R_{21} + N\ell_{(8)} \quad (3.4)$$

where $N\ell$ is the axial force vector in the main system due to the applied set of loads.

3.2 Section Plastic Capacity

Every section in a member is, in general, subject to six internal forces, namely a thrust, two shearing forces, two bending moments and a torsional moment. For highly loaded short spans, the two shearing forces might be large enough to cause plastification of the web or the flanges, and thus reduce the plastic bending capacity of the I-section. However, for practical cases, the two shearing forces have a negligible effect on the plastic bending capacity of the section (14), and their effect is neglected in this study.

The effect of torsional moment on the plastic bending capacity of an I section is not small (5). Except after the complete deterioration of the flexural stiffness at a joint, due to plastic hinge formation, the flexural stiffness is very large compared to the torsional stiffness and the amount of torsional moment carried by a member is very small. However, additional constraints were imposed in the optimization procedure to limit the amount of torsional moments in each member (see section 3.3.3). Thus the three internal forces left are the axial force and the two flexural moments. The interaction between these forces can be studied in a non-dimensional manner by introducing the following terms.

$$\begin{aligned}
\alpha_x &= |M_x|/M_{px} \\
\alpha_y &= |M_y|/M_{py} \\
\eta &= |N|/N_p
\end{aligned}
\tag{3.5}$$

where,

$$\begin{aligned}
M_{px} &= \text{full plastic moment about the major axis} \\
M_{py} &= \text{full plastic moment about the minor axis} \\
N_p &= \text{full plastic axial force.}
\end{aligned}$$

Massonnet (14) showed that a plot of the interaction curves between η , α_x and η , α_y for the available wide flange sections were found to be contained in a very thin strip and were best approximated (Figure 3.5 and 3.6) by the equations

$$\begin{aligned}
\alpha_x &= 1 & \text{for } 0 \leq \eta \leq 0.15 \\
\alpha_x &= 1.18(1-\eta) & \text{for } 0.15 \leq \eta \leq 1
\end{aligned}
\tag{3.6}$$

and

$$\begin{aligned}
\alpha_y &= 1 & \text{for } 0 \leq \eta \leq 0.4 \\
\alpha_y &= 1.19(1-\eta^2) & \text{for } 0.4 \leq \eta \leq 1
\end{aligned}
\tag{3.7}$$

For the interaction between M_x and M_y the following relation is due to Horne (6). In this analysis it is assumed that the web takes the axial thrust and that the

FIGURE 3.5

FIGURE 3.6

contribution of the web to the major axis bending moment may be neglected.

The section considered is an I section in which the flanges are of width b and thickness t , the centroids of the flanges being distance d apart (Figure 3.7). The full plastic moment about the major axis $x-x$, in absence of bending about the minor axis, is

$$M_{px} = btd\sigma_y$$

Similarly, for bending about the minor axis,

$$M_{py} = \frac{1}{2} b^2 t \sigma_y$$

Under combined bending, since the axial load is assumed taken by the web, the neutral axis must intersect the flanges on opposite sides of the web at the same distance $\frac{1}{2} rb$ from the web, as shown by A-A and B-B. The moments about the major and minor axes are now $\alpha_x M_{px}$ and $\alpha_y M_{py}$, where

$$\alpha_x M_{px} = rbtd\sigma_y = rM_{px}$$

and

$$\alpha_y M_{py} = (1 - r^2)b^2 t \sigma_y / 2 = (1 - r^2)M_{py}$$

Therefore

$$\alpha_x = r$$

and

$$\alpha_y = 1 - r^2$$

FIGURE 3.7

FIGURE 3.8

Thus

$$\alpha_x^2 + \alpha_y = 1 \quad (3.8)$$

For the combined interaction of M_x , M_y and N , it will be assumed that the relation between the reduced M_{px} and M_{py} , due to the axial force N will have the form

$$u \cdot \alpha_x^2 + v \cdot \alpha_y = 1 \quad (a)$$

where u and v are suitable scaling factors to be determined so as to be compatible with equations 3.6 and 3.7 in case of α_y or α_x is zero respectively.

(a) For $0 \leq \eta \leq 0.15$

There is no reduction in the plastic moments about either the major or the minor axes, thus

$$u = v = 1$$

and relation (a) yields equation 3.8.

(b) For $0.15 \leq \eta \leq 0.4$

There is no reduction in the plastic moment about the minor axis, thus

$$v = 1$$

For $\alpha_y = 0$, equation 3.6 gives

$$\alpha_x = 1.18(1 - \eta)$$

Substituting in equation (a) one gets

$$u[1.18(1 - \eta)]^2 + 0 = 1$$

or

$$u = \frac{1}{[1.18(1 - \eta)]^2} = \frac{1}{\beta^2}$$

where

$$\beta = 1.18(1 - \eta) \quad (3.9)$$

Substitution in equation (a) gives

$$\alpha_x^2 + \beta^2 \alpha_y^2 = \beta^2 \quad (3.10)$$

(c) For $0.4 \leq \eta \leq 1$

There is reduction in the plastic moment about both the major and minor axes, and a similar analysis yields the values

$$u = \frac{1}{\beta^2}$$

and

$$v = \frac{1}{1.19(1 - \eta^2)} = \frac{1}{\tau}$$

where

$$\tau = 1.19(1 - \eta^2) \quad (3.11)$$

Substitution in equation (a) gives

$$\tau \alpha_x^2 + \beta^2 \alpha_y^2 = \beta^2 \tau \quad (3.12)$$

The interaction between $|M_x|$, $|M_y|$ and $|N|$ is thus represented in a non-dimensional form by the interaction surface of Figure 3.9 determined by equations 3.8, 3.10 and 3.12.

3.3 Constraint Vector

3.3.1 Plastic Capacity Constraints

The point in the interaction diagram determined by the combination of $|M_x|$, $|M_y|$ and $|N|$ acting on a section should lie inside or on the interaction surface (Figure 3.9) determined by equations 3.8, 3.10 and 3.12.

Due to the discontinuity of the derivatives at $\eta = 0.15$ and $\eta = 0.4$, an assumption is needed to prevent the anticipated instability of solution that may occur because of the fact that the applicable constraints depend on the optimum solution, which in turn depend on the assumed constraints. Since the value of the axial force in the beams is usually small, it was reasonable to assume that the interaction surface for the beam sections is represented by equation 3.8 (part a in Figure 3.9). For the columns, however, it was reasonable to assume that the value of η will exceed 0.15 . Since torsional buckling of members having economical sections was found to occur, for normal values of yield stress, at values of η less than 0.4

FIGURE 3.9

(see Section 3.3.2), the interaction surface for the column sections was represented by equation 3.10 (part b in Figure 3.9).

Thus the plastic capacity constraints for the twelve critical sections in the beams are

$$\alpha_{xj}^2 + \alpha_{yj} \leq 1 \quad j = 1, 2, \dots, 12$$

These inequality constraints can be converted to equality constraints by introducing a set of dummy variables Z as follows

$$\alpha_{xj}^2 + \alpha_{yj} - 1 \leq 0$$

or

$$\alpha_{xj}^2 + \alpha_{yj} - 1 + Z_j^2 = 0 \quad j = 1, 2, \dots, 12 \quad (3.13)$$

For the columns, the plastic capacity constraints for its eight critical sections are

$$\alpha_{xj}^2 + \beta_j^2 \alpha_{yj} \leq \beta_j^2 \quad j = 13, 14, \dots, 20$$

Similarly these inequality constraints can be converted to the equality constraints

$$\alpha_{xj}^2 + \beta_j^2 (\alpha_{yj} - 1) + Z_j^2 = 0 \quad j = 13, 14, \dots, 20 \quad (3.14)$$

3.3.2 Torsional Buckling Constraints

A study for the standard steel beam sections made by Bleich (3) has indicated that since the "equivalent torsional radius of gyration" is in most cases greater than the flexural radius of gyration about the minor axis, torsional buckling is not likely to occur except for very short columns.

Since, however, a minimum weight design is sought, a study on the economical sections shows that torsional buckling might be a problem. For thin-walled open sections having two perpendicular axes of symmetry so that the center of twist and the centroid coincide, and for relatively long members so that free warping can take place, it can be shown (4) that

$$\sigma_{tb} = \frac{GJ}{I_p} \quad (3.15)$$

where,

σ_{tb} = normal stress at which torsional buckling occur

G = shear modulus

J = torsional constant

I_p = polar moment of inertia

An average value for the ratio J/I_p for economical wide flange sections [provided by the Manual of Steel Construction (13)] was found to be 0.00159. If further the shear modulus G for steel is taken equal to 11150 ksi, then

$$\sigma_{tb} = 11150 \times 0.00159 = 17.7 \text{ ksi}$$

This consequently limits the value of η to

$$\eta \leq \frac{A\sigma_{tb}}{A\sigma_y} = \frac{17.7}{\sigma_y}$$

Thus the torsional buckling constraints for the eight members are

$$\eta_k \leq \frac{17.7}{\sigma_y}$$

or

$$\eta_k - \frac{17.7}{\sigma_y} + z_j^2 = 0 \quad \begin{array}{l} k = 1, 2, \dots, 8 \\ j = k + 20 \end{array} \quad (3.16)$$

3.3.3 Torsion Constraints

Since the torsional stiffness of the wide flange sections is very small compared to its flexural stiffness, additional constraints are needed to limit the amount of torsional moment in each member. The absence of such constraints in the optimization procedure has led to meaningless

bending moment distributions. For example, for the simple symmetric case of loading shown in Figure 3.10, the set of redundants obtained without including any torsion constraints in the optimization procedure were as shown in Figure 3.11, where bending of the columns were avoided by transmitting the corner moments as torsional moment in beams 3 and 7 (Figure 3.12). A symmetric bending moment diagram was not even possible since it would have caused spinning of the two beams 3 and 7.

It is thus assumed that the members can carry no torsional moment, and the torsional moment constraints for the eight members are

$$M_{tk} = 0 \quad k = 1, 2, \dots, 8 \quad (3.17)$$

where M_{tk} are given by equation 3.3 .

This brings up the total number of constraints for the problem to

$$m = 12 + 8 + 8 + 8 = 36$$

3.4 Variable Vector

The variable representing the section property was chosen to be the cross-section area. This implies that the shape of the cross-section is predetermined and is not included in the optimization procedure. This shape was

FIGURE 3.10

FIGURE 3.11

FIGURE 3.12

chosen to be the wide flange I section. A study of the relations between the area and the other needed section properties for wide flange I sections was studied. Since a minimum weight design is sought, this study was limited to economical sections as obtained from the plastic design selection table of the Manual of Steel Construction (13). In this table the shapes were listed in groups by descending order of plastic section modulus with respect to the major axis, Z_x . The economical sections are the lightest in weight in each group. Since the shape of the cross-section is fixed, and not optimized, the ratio of Z_y/Z_x is about constant, as can be seen from Figure 3.13. Thus, for this fixed shape, the chosen economical sections are also economical for bending about the minor axis. If the frame columns are subject to large forces causing moments about the cross-sections minor axes, other shapes with higher Z_y/Z_x ratio can be used.

Previous studies (8,23) have shown that economical sections possess a relation similar to those of geometrically similar shapes, and that the plastic section moduli Z_x and Z_y can be expressed as

$$Z_x = C_x A^k \quad (b)$$

$$Z_y = C_y A^k \quad (c)$$

where k was found to be very close to its value of $\frac{3}{2}$ for geometrically similar shapes, and thus relations (b) and (c) can be rewritten as

$$Z_x = C_x A^{3/2} \quad (3.18)$$

$$Z_y = C_y A^{3/2} \quad (3.19)$$

A least square fit for the economical sections provided by the Manual of Steel Construction (Figure 3.13) gives the values

$$C_x = 1.667 \quad (3.20)$$

$$C_y = 0.286 \quad (3.21)$$

Accordingly, the full plastic moment about the major and minor axes M_{px} and M_{py} and the full plastic axial force N_p can be written as follows

$$\begin{aligned} M_{px} &= \sigma_y Z_x = \sigma_y C_x A^{3/2} \\ M_{py} &= \sigma_y Z_y = \sigma_y C_y A^{3/2} \\ N_p &= \sigma_y A \end{aligned} \quad (3.22)$$

FIGURE 3.13

The other variables, besides the section areas A are the redundants R and the dummy variables Z in equations 3.13, 3.14 and 3.16 arising from the inequality constraints. Thus the variables vector x is

$$x^T = [A^T, R^T, Z^T] \quad \begin{array}{l} A \text{ is } 8 \times 1 \\ R \text{ is } 24 \times 1 \\ Z \text{ is } 28 \times 1 \end{array} \quad (3.23)$$

and this brings up the total number of variables for the problem to

$$n = 8 + 24 + 28 = 60$$

3.5 Constraint Derivative Matrix

The two dimensional array ϕ_x of the partial derivatives of the vectorial function ϕ with respect to the variables vector x composed of A , R and Z vectors is arranged as an $n \times m$ array where the $(i,j)^{th}$ element is the partial derivative of the j^{th} constraint with respect to the i^{th} variable. The ϕ_x array has the form shown in Figure 3.14. For convenience, the following partial derivatives needed in forming this array can be calculated first. Equations 3.5, 3.9 and 3.22 are being repeatedly used in their calculation.

$$\frac{\partial}{\partial A_i} \alpha_j = \frac{\partial}{\partial A_i} \frac{|M_j|}{M_{pj}} = \frac{\partial}{\partial A_i} \frac{|M_j|}{\sigma_Y C A_i^{3/2}} = -1.5 \frac{1}{A_i} \frac{|M_j|}{\sigma_Y C A_i^{3/2}} = -\frac{1.5}{A_i} \alpha_j \quad (d)$$

$$\frac{\partial}{\partial A_i} \alpha_j^2 = 2 \alpha_j \frac{\partial \alpha_j}{\partial A_i} = -\frac{3}{A_i} \alpha_j^2 \quad (e)$$

$$\frac{\partial}{\partial A_i} \eta_j = \frac{\partial}{\partial A_i} \frac{|N_j|}{N_{pj}} = \frac{\partial}{\partial A_i} \frac{|N_j|}{\sigma_Y A_i} = -\frac{1}{A_i} \frac{|N_j|}{\sigma_Y A_i} = -\frac{1}{A_i} \eta_j \quad (f)$$

$$\frac{\partial}{\partial A_i} \beta_j = \frac{\partial}{\partial A_i} [1.18(1-\eta_j)] = -1.18 \frac{\partial \eta_j}{\partial A_i} = \frac{1.18}{A_i} \eta_j \quad (g)$$

$$\frac{\partial}{\partial A_i} \beta_j^2 = 2 \beta_j \frac{\partial \beta_j}{\partial A_i} = \frac{2.36}{A_i} \beta_j \eta_j \quad (h)$$

$$\frac{\partial}{\partial R_i} \alpha_j = \frac{\partial}{\partial R_i} \frac{|M_j|}{M_{pj}} = \frac{1}{M_{pj}} \frac{\partial}{\partial R_i} |M_j| = \frac{\text{sgn}(M_j)}{M_{pj}} \frac{\partial}{\partial R_i} M_j \quad (i)$$

$$\begin{aligned}
\frac{\partial}{\partial R_i} \alpha_j^2 &= 2 \alpha_j \frac{\partial}{\partial R_i} \alpha_j = 2 \alpha_j \frac{\text{sgn}(M_j)}{M_{pj}} \frac{\partial}{\partial R_i} M_j \\
&= 2 \frac{|M_j|}{M_{pj}} \frac{\text{sgn}(M_j)}{M_{pj}} \frac{\partial}{\partial R_i} M_j = \frac{2 M_j}{M_{pj}^2} \frac{\partial}{\partial R_i} M_j
\end{aligned} \tag{j}$$

$$\frac{\partial}{\partial R_i} \eta_j = \frac{\partial}{\partial R_i} \frac{|N_j|}{N_{pj}} = \frac{1}{N_{pj}} \frac{\partial}{\partial R_i} |N_j| = \frac{\text{sgn}(N_j)}{N_{pj}} \frac{\partial}{\partial R_i} N_j \tag{k}$$

$$\begin{aligned}
\frac{\partial}{\partial R_i} \beta_j &= \frac{\partial}{\partial R_i} [1.18(1-\eta_j)] = -1.18 \frac{\partial \eta_j}{\partial R_i} \\
&= -1.18 \frac{\text{sgn}(N_j)}{N_{pj}} \frac{\partial}{\partial R_i} N_j
\end{aligned} \tag{l}$$

$$\frac{\partial}{\partial R_i} \beta_j^2 = 2 \beta_j \frac{\partial \beta_j}{\partial R_i} = -2.36 \beta_j \frac{\text{sgn}(N_j)}{N_{pj}} \frac{\partial}{\partial R_i} N_j \tag{m}$$

The above relations will be repeatedly used in Sections 3.5.1 and 3.5.2 in developing the ϕ_x matrix.

3.5.1 Plastic Capacity Constraints

(A) For beams

From equation 3.13,

$$\varphi_j = \alpha_{xj}^2 + \alpha_{yj} - 1 + z_j^2$$

(i) Partial derivatives with respect to areas

$$\begin{aligned} \varphi_{x(i,j)} &= \frac{\partial}{\partial A_i} (\alpha_{xj}^2 + \alpha_{yj} - 1 + z_j^2) \\ &= \frac{\partial}{\partial A_i} (\alpha_{xj}^2 + \alpha_{yj}) \\ &= -\frac{3}{A_i} \alpha_{xj}^2 - \frac{1.5}{A_i} \alpha_{yj} \end{aligned}$$

Thus

$$\varphi_{x(i,j)} = -\frac{1.5}{A_i} (2 \alpha_{xj}^2 + \alpha_{yj}) \quad (3.24)$$

Equation 3.24 gives the elements of blocks (I) in Figure 3.14.

FIGURE 3.14

(ii) Partial derivatives with respect to
redundants

$$\begin{aligned}
 \varphi_{x(i+8,j)} &= \frac{\partial}{\partial R_i} (\alpha_{xj}^2 + \alpha_{yj} - 1 + z_j^2) \\
 &= \frac{\partial}{\partial R_i} (\alpha_{xj}^2 + \alpha_{yj}) \\
 &= \frac{2M_{xj}}{M_{pxj}^2} \frac{\partial}{\partial R_i} M_{xj} + \frac{\text{sgn}(M_{yj})}{M_{pyj}} \frac{\partial}{\partial R_i} M_{yj}
 \end{aligned}$$

Thus

$$\varphi_{x(i+8,j)} = m_{bxj} \frac{\partial}{\partial R_i} M_{xj} + m_{byj} \frac{\partial}{\partial R_i} M_{yj} \quad (3.25)$$

where,

$$m_{bxj} = \frac{2M_{xj}}{M_{pxj}^2} \quad (3.26)$$

$$m_{byj} = \frac{\text{sgn}(M_{yj})}{M_{pyj}} \quad (3.27)$$

The different elements are thus obtained by direct substitution of equations 3.1 and 3.2 in equations 3.25, 3.26 and 3.27. These elements formed the four blocks (II) in Figure 3.14.

(iii) Partial derivatives with respect to dummy variables

$$\begin{aligned}\varphi_{x(i+32,j)} &= \frac{\partial}{\partial z_i} (\alpha_{xj}^2 + \alpha_{yj} - 1 + z_j^2) \\ &= \frac{\partial}{\partial z_i} z_j^2\end{aligned}$$

Thus

$$\begin{aligned}\varphi_{x(i+32,j)} &= 2 z_i & i = j & \quad i = 1, 2, \dots, 12 \\ &= 0 & i \neq j & \quad (3.28)\end{aligned}$$

(B) For columns

From equation 3.14

$$\varphi_j = \alpha_{xj}^2 + \beta_j^2 (\alpha_{yj} - 1) + z_j^2$$

(i) Partial derivatives with respect to areas

$$\begin{aligned} \varphi_{x(i,j)} &= \frac{\partial}{\partial A_i} [\alpha_{xj}^2 + \beta_j^2 (\alpha_{yj} - 1) + z_j^2] \\ &= \frac{\partial}{\partial A_i} [\alpha_{xj}^2 + \beta_j^2 (\alpha_{yj} - 1)] \\ &= \frac{\partial}{\partial A_i} \alpha_{xj}^2 + \beta_j^2 \frac{\partial}{\partial A_i} (\alpha_{yj} - 1) + (\alpha_{yj} - 1) \frac{\partial}{\partial A_i} \beta_j^2 \\ &= -\frac{3}{A_i} \alpha_{xj}^2 - \frac{1.5}{A_i} \alpha_{yj} \beta_j^2 + \frac{2.36}{A_i} \beta_j \eta_j (\alpha_{yj} - 1) \\ &= -\frac{3}{A_i} \alpha_{xj}^2 + \frac{\beta_j \alpha_{yj}}{A_i} [2.36 \eta_j - 1.5 \beta_j] - \frac{2.36}{A_i} \eta_j \beta_j \\ &= -\frac{3}{A_i} \alpha_{xj}^2 + \frac{\beta_j \alpha_{yj}}{A_i} \{2.36 \eta_j - 1.5 [1.18 (1 - \eta_j)]\} \\ &\quad - \frac{2.36}{A_i} \eta_j \beta_j \end{aligned}$$

Thus

$$\varphi_{x(i,j)} = -\frac{3}{A_i} \alpha_{xj}^2 + \frac{\beta_j \alpha_{yj}}{A_i} (4.13 \eta_j - 1.77) - \frac{2.36}{A_i} \eta_j \beta_j \quad (3.29)$$

Equation 3.29 gives the elements of blocks (III) in Figure 3.14.

(ii) Partial derivatives with respect to
redundants

$$\begin{aligned} \varphi_{x(i+8,j)} &= \frac{\partial}{\partial R_i} [\alpha_{xj}^2 + \beta_j^2 (\alpha_{yj} - 1) + z_j^2] \\ &= \frac{\partial}{\partial R_i} [\alpha_{xj}^2 + \beta_j^2 (\alpha_{yj} - 1)] \\ &= \frac{\partial}{\partial R_i} \alpha_{xj}^2 + \beta_j^2 \frac{\partial}{\partial R_i} (\alpha_{yj} - 1) + (\alpha_{yj} - 1) \frac{\partial}{\partial R_i} \beta_j^2 \\ &= \frac{2M_{xj}}{M_{pxj}^2} \frac{\partial}{\partial R_i} M_{xj} + \frac{\text{sgn}(M_{yj})}{M_{pyj}} \beta_j^2 \frac{\partial}{\partial R_i} M_{yj} \\ &\quad + 2.36 \beta_j \frac{\text{sgn}(N_j)}{N_{pj}} (1 - \alpha_{yj}) \frac{\partial}{\partial R_i} N_j \end{aligned}$$

Thus

$$\varphi_{x(i+8,j)} = m_{cxj} \frac{\partial}{\partial R_i} M_{xj} + m_{cyj} \frac{\partial}{\partial R_i} M_{yj} + n_j \frac{\partial}{\partial R_i} N_j \quad (3.30)$$

where,

$$m_{cxj} = \frac{2M_{xi}}{M_{pxj}^2} \quad (3.31)$$

$$m_{cyj} = \beta_j^2 \frac{\text{sgn}(M_{yj})}{M_{pyj}} \quad (3.32)$$

$$n_j = 2.36 \beta_j \frac{\text{sgn}(N_j)}{N_{pj}} (1 - \alpha_{yj}) \quad (3.33)$$

The different elements are thus obtained by direct substitution of equations 3.1, 3.2 and 3.4 in equations 3.30, 3.31, 3.32 and 3.33. These elements formed the eight blocks (IV) and (V) in Figure 3.14.

(iii) Partial derivatives with respect to dummy variables

$$\begin{aligned} \varphi_{x(i+32,j)} &= \frac{\partial}{\partial Z_i} [\alpha_{xj}^2 + \beta_j^2 (\alpha_{yj} - 1) + Z_j^2] \\ &= \frac{\partial}{\partial Z_i} Z_j^2 \end{aligned}$$

Thus

$$\begin{aligned}\varphi_{x(i+32,j)} &= 2 z_i & i = j & \quad i = 13,14,\dots,20 \\ &= 0 & i \neq j & \quad\quad\quad (3.34)\end{aligned}$$

3.5.2 Torsional Buckling Constraints

From equation 3.16

$$\varphi_j = \eta_k - \frac{17.7}{\sigma_y} + z_j^2$$

(i) Partial derivatives with respect to areas

$$\begin{aligned}\varphi_{x(i,j)} &= \frac{\partial}{\partial A_i} \left(\eta_k - \frac{17.7}{\sigma_y} + z_j^2 \right) \\ &= \frac{\partial}{\partial A_i} \eta_k\end{aligned}$$

$$\varphi_{x(i,j)} = - \frac{1}{A_i} \eta_k \quad (3.35)$$

- (ii) Partial derivatives with respect to
redundants

$$\begin{aligned}
 \varphi_{x(i+8,j)} &= \frac{\partial}{\partial R_i} \left(\eta_k - \frac{17.7}{\sigma_y} + z_j^2 \right) \\
 &= \frac{\partial}{\partial R_i} \eta_k \\
 \varphi_{x(i+8,j)} &= \frac{\text{sgn}(N_k)}{N_{pk}} \frac{\partial}{\partial R_i} N_k \quad (3.36)
 \end{aligned}$$

The different elements are obtained by direct
substitution of equation 3.4 in equation 3.36.

- (iii) Partial derivative with respect to dummy
variables

$$\begin{aligned}
 \varphi_{x(i+32,j)} &= \frac{\partial}{\partial z_i} \left(\eta_k - \frac{17.7}{\sigma_y} + z_j^2 \right) \\
 &= \frac{\partial}{\partial z_i} z_j^2 \\
 \varphi_{x(i+32,j)} &= 2 z_i \quad i = j \quad i = 21, 22, \dots, 28 \\
 &= 0 \quad i \neq j \quad (3.37)
 \end{aligned}$$

3.5.3 Torsion Constraints

From equation 3.17

$$\varphi_j = M_{tk}$$

These torsion constraints are not functions of the areas or the dummy variables. Thus the partial derivatives of these constraints with respect to both the areas and the dummy variables vanishes. For the redundants

$$\varphi_{x(i+8,j)} = \frac{\partial}{\partial R_i} M_{tk} \quad (3.38)$$

The different elements are obtained by direct substitution of equation 3.3 in equation 3.38.

3.6 Objective Function

The function to be minimized was given by equation 2.1. For the problem considered, it can be written as

$$f = l_k (A_1 + A_5) + l_h (A_3 + A_7) + h (A_2 + A_4 + A_6 + A_8) \quad (3.39)$$

As seen, this function is linear in the eight unknown areas.

3.7 Linearized Problem

The nonlinear optimization method used is the sequential gradient-restoration algorithm. For this first order method, and with such a large number of unknowns, convergence was observed to be slow. To avoid long computation times, a good initial choice of the variables was needed to initiate the algorithm. This was done by linearizing the problem. The symmetric method was then used to obtain the optimum solution for this linear problem. This solution was used as the starting point for the sequential gradient-restoration algorithm. Although two optimization methods were thus employed to reach the nonlinear optimum solution, it still proved to be faster than only using the sequential gradient-restoration algorithm and guessing a starting point.

To linearize the problem, the constraints and the objective function previously developed have to be linearized with respect to the unknown variables. The variables for the linearized problem are taken to be the eight plastic moments M_{px} and the twenty-four redundants R .

The linearization of a constraint was done so as to best approximate the nonlinear constraint. Such linearization will give a solution closer to the nonlinear

optimum solution than if the constraint was linearized so that its safe region lies within that of the nonlinear constraint. With such approximation, the solution obtained might violate the original nonlinear constraints. This unsafety can always be corrected by the restoration phase of the sequential gradient-restoration algorithm. The linearization of the problem is explained below.

3.7.1 Plastic Capacity Constraints

The plastic capacity constraints for the beams were given in Section 3.3.1 by

$$\alpha_x^2 + \alpha_y \leq 1$$

This inequality was approximated by the linear constraint

$$\alpha_x + \alpha_y \leq 1.1 \quad (3.40)$$

Using definition 3.5, this can be rewritten as

$$\frac{|M_x|}{M_{px}} + \frac{|M_y|}{M_{py}} \leq 1.1$$

or

$$|M_x| + \frac{M_{px}}{M_{py}} |M_y| \leq 1.1 M_{px}$$

From equations 3.18, 3.19, 3.20 and 3.21, it can be seen that

$$\frac{M_{px}}{M_{py}} = \frac{1.667}{0.286} = 5.829 \quad (3.41)$$

Therefore,

$$- |M_x| - 5.829 |M_y| + 1.1 M_{px} \geq 0 \quad (3.42)$$

The absolute values of M_x and M_y appearing in the above inequality have to be eliminated to be able to apply the symmetric method. If a variable is positive, its absolute value is equal to itself. If, on the other hand, a variable is negative, its absolute value is equal to minus itself. Thus this inequality constraint can be replaced by the four following constraints giving the possible combination of signs for M_x and M_y .

$$\begin{aligned} - M_x - 5.829 M_y + 1.1 M_{px} &\geq 0 & (a) \\ M_x + 5.829 M_y + 1.1 M_{px} &\geq 0 & (b) \\ - M_x + 5.829 M_y + 1.1 M_{px} &\geq 0 & (c) \\ M_x - 5.829 M_y + 1.1 M_{px} &\geq 0 & (d) \end{aligned} \quad (3.43)$$

These four constraints have to be applied at each of the twelve critical sections in the beams. Thus the number of plastic capacity constraints needed for the beams is forty-eight.

Toakley's method (25) for reducing the number of constraints in the tableau was considered. However, since two absolute values are present in each constraint, two equations need to be satisfied implicitly for each critical section rather than only one. In addition, in Toakley's method several complete cycles of the reduced tableau are needed before all the effective constraints become present in the tableau and a solution can be obtained. Thus this approach was not used. For the columns, the plastic capacity constraints were given in Section 3.3.1 by

$$\alpha_x^2 + \beta^2 \alpha_y \leq \beta^2$$

For the range of η between zero and 0.4, this inequality can be approximated by

$$\alpha_x + 0.91 \alpha_y + 0.55 \eta \leq 1.07 \quad (3.44)$$

Using definition 3.5, this can be rewritten as

$$\frac{|M_x|}{M_{px}} + 0.91 \frac{|M_y|}{M_{py}} + 0.55 \eta \leq 1.07$$

or,

$$|M_x| + 0.91 \frac{M_{px}}{M_{py}} |M_y| + 0.55 M_{px} \eta \leq 1.07 M_{px}$$

From equation 3.41, this can be rewritten as

$$- |M_x| - 5.304 |M_y| + (1.07 - 0.55\eta) M_{px} \geq 0 \quad (3.45)$$

Following the previous argument, this inequality can be replaced by

$$\begin{aligned}
 - M_x - 5.304 M_y + (1.07 - 0.55\eta)M_{px} &\geq 0 & (a) \\
 M_x + 5.304 M_y + (1.07 - 0.55\eta)M_{px} &\geq 0 & (b) \\
 - M_x + 5.304 M_y + (1.07 - 0.55\eta)M_{px} &\geq 0 & (c) \\
 M_x - 5.304 M_y + (1.07 - 0.55\eta)M_{px} &\geq 0 & (d)
 \end{aligned} \tag{3.46}$$

These four constraints have to be applied at each of the eight critical sections in the columns. Thus the number of plastic capacity constraints needed for the columns is thirty-two.

3.7.2 Torsional Buckling Constraints

The torsional buckling constraints were given in Section 3.3.2 by

$$\eta \leq \frac{17.7}{\sigma_y}$$

Using definition 3.5, this can be rewritten as

$$- \frac{|N|}{N_p} + \frac{17.7}{\sigma_y} \geq 0$$

or,

$$- |N| + \frac{17.7}{\sigma_y} N_p \geq 0$$

or,

$$- |N| + \frac{17.7}{\sigma_y} \sigma_y A \geq 0$$

or,

$$- |N| + \frac{17.7}{\sigma_y} \sigma_y \frac{A^{3/2}}{A^{1/2}} \geq 0$$

From equations 3.20 and 3.22, this can be rewritten as

$$- |N| + \frac{17.7}{\sigma_y} \frac{M_{px}}{1.667 A^{1/2}} \geq 0$$

An average value for $A^{1/2}$ for economical wide flange sections (provided by the Manual of Steel Construction) was found to be 5.745. Thus

$$- |N| + s M_{px} \geq 0 \quad (3.47)$$

where,

$$s = 17.7/(\sigma_y \times 9.577)$$

Since only one absolute value appears in the above inequality, it can be replaced by the two inequalities

$$\begin{aligned} -N + s M_{px} &\geq 0 & (a) \\ N + s M_{px} &\geq 0 & (b) \end{aligned} \quad (3.48)$$

These two constraints have to be applied to each of the eight members of the frame. Thus the number of torsional buckling constraints needed is sixteen.

3.7.3 Torsion Constraints

The torsion constraints were given in Section 3.3.3 by

$$M_t = 0$$

In the symmetric method only inequality constraints can be treated. The above equality constraint can be converted to the two inequality constraints

$$M_t \geq 0$$

and

$$M_t \leq 0$$

These constraints can be rewritten as

$$M_t \geq 0 \quad (a)$$

$$- M_t \geq 0 \quad (b)$$

(3.49)

These two constraints have to be applied to each of the eight members of the frame. Thus the number of torsion constraints needed is sixteen.

Thus it is seen from the above discussion that the total number of constraints for the problem is

$$m = 48 + 32 + 16 + 16 = 112$$

3.7.4 Objective Function

The objective function previously given in Section 3.6 is linear in the areas. Since the unknowns to be optimized for the linear problem were taken to be the plastic moments instead of the areas, the objective function had to be expressed linearly in terms of the plastic moments M_{px} .

From equation 2.1

$$f = \sum_{i=1}^W A_i \ell_i$$

or,

$$f = \sum_{i=1}^W \frac{A_i^{3/2}}{A_i^{1/2}} \ell_i$$

Using equation 3.18, this can be rewritten as

$$f = \frac{1}{C_x \sigma_y} \sum_{i=1}^W \frac{M_{pxi}}{A_i^{1/2}} \ell_i$$

As previously mentioned, an average value of $A^{1/2}$ for economical wide flange sections was found to be 5.745. Thus

$$f = \frac{1}{5.745 C_x \sigma_y} \sum_{i=1}^W M_{pxi} \ell_i$$

or

$$f = \frac{1}{C} \left[\ell_k (M_{px1} + M_{px5}) + \ell_h (M_{px3} + M_{px7}) \right. \\ \left. + h (M_{px2} + M_{px4} + M_{px6} + M_{px8}) \right] \quad (3.50)$$

where

$$C = 5.745 C_x \sigma_y$$

3.7.5 Initial Tableau

In the symmetric method only positive variables are allowed. Any variable R that might be either positive or negative can be expressed as the difference between two positive variables R_a and R_b

$$R = R_a - R_b$$

As mentioned before, the variables considered are the twenty-four redundants and the eight plastic moments. The redundants might have either positive or negative values. Toakley's method (25) by which only one positive variable can represent each redundant in the tableau can be used but was not applied. Thus each redundant was expressed as the difference between two positive variables in the manner explained before. Accordingly, forty-eight variables were needed to represent the twenty-four

redundants. For the plastic moments, only positive values can take place. Accordingly, only one variable is needed for each plastic moment, and eight variables are thus needed to represent the eight plastic moments. Thus the total number of variables for the problem is

$$n = 48 + 8 = 56$$

For the general linear optimization problem of minimizing the function

$$f = - \sum_{j=1}^n c_j x_j \quad (3.51)$$

subject to the constraints

$$b_i + \sum_{j=1}^n a_{ij} x_j \geq 0 \quad i = 1, 2, \dots, m \quad (3.52)$$

The initial tableau for the symmetric method has the form shown in Figure 3.15. A flow chart of the symmetric method algorithm as adapted to this formulation is given in Appendix A.

For the problem considered, the tableau is 113 x 57. This tableau is formed by substituting equations 3.1, 3.2, 3.3 and 3.4 in the inequality constraints 3.43, 3.46, 3.48 and 3.49. The values of η in inequality 3.46 were assumed.

To avoid wasteful indexing and to save computer time, the tableau was stored in a one-dimensional array and pivoting in the tableau was performed on that one-dimensional array.

FIGURE 3.15

CHAPTER IV

ELASTIC-PLASTIC ANALYSIS

In the previous study an optimally designed space frame was obtained through a nonlinear optimization procedure. The constraints employed were plastic capacity, torsional and torsional buckling constraints. No consideration has been taken to the frame deflections or to its stability since any set of constraints added in the optimization procedure to account for them will have to be functions of the deteriorating stiffness of the frame.

Results on plane frames (1) have shown that geometrical nonlinearity is important even for short frames, and the frame stiffness is thus dependent on the resulting deflections. In addition to this interdependence relation, the formation of a plastic hinge causes a sudden drop in the frame stiffness. This discontinuity in stiffness might have caused instability in the optimization procedure.

Thus it was necessary to consider the interaction of stability and plastic mechanism failures by following the behavior of the optimized structure under increasing load intensity up to failure which may be due to the formation of a mechanism or due to the instability of the deteriorating structure at some stage before the plastic mechanism failure.

Since stability was not considered in the optimization procedure, the failure load obtained is essentially an upper limit on the actual failure load.

To avoid oscillation problems (12) in the iterative procedure needed to predict the load factor for a plastic hinge, especially when more than one hinge is about to form at the same time, the level of axial force in all members was assumed constant through the loading process. This level being in each member equal to the optimal value of axial force obtained from the optimization procedure. This high level of axial forces will essentially lead to a lower limit on the actual failure load.

The difference between the upper limit obtained by nonlinear optimization and the lower limit obtained by this procedure was small and justifies the approach used. Further, no oscillation problems occurred.

A flow chart for the procedure used is shown in Figure 4.1 and is explained in the following sections.

FIGURE 4.1

4.1 Interaction Diagram

For simplicity, the interaction surface in Figure 3.9 given by equations 3.8, 3.10 and 3.12 can be linearized and represented by the equation

$$\alpha_x + C_1 \alpha_y + C_2 \eta = C_3 \quad (4.1)$$

where the coefficients C_1 , C_2 and C_3 are determined so that the linearization will best represent the nonlinear interaction surface.

For a known level of axial force in each member, η is consequently known, and equation 4.1 can be rewritten as

$$\alpha_x + C_1 \alpha_y = C_3 - C_2 \eta$$

For $\alpha_y = 0$, α_x is given by

$$\alpha_x = C_3 - C_2 \eta$$

Thus the value of the reduced plastic moment about the major axis T_x is given by

$$T_x = (C_3 - C_2 \eta) M_{px} \quad (4.2)$$

Similarly, the value of the reduced plastic moments about the minor axis T_y is given by

$$T_y = \left[(C_3 - C_2 \eta) / C_1 \right] M_{py} \quad (4.3)$$

Thus for each critical point the interaction diagram shown in Figure 4.2 can be obtained by substituting the value of η corresponding to the optimal axial force at this point in equations 4.2 and 4.3.

4.2 Stiffness Terms

It is convenient to state the following stiffness terms for an axially loaded member of length l and an axial force N (Figure 4.3). For derivation of these standard terms, see for example (3), (10) and (24). These terms were continuously used in the development of the different stiffness matrices.

The lateral stiffness k_f is

$$k_f = \frac{12EI}{l^3} \varphi_1 \quad (4.4)$$

where,

E = modulus of elasticity

I = moment of inertia

The accompanying end moment k_{fm} is

$$k_{fm} = \frac{6EI}{l^2} \varphi_2 \quad (4.5)$$

FIGURE 4.2

FIGURE 4.3

The rotational stiffness k_m is

$$k_m = \frac{4EI}{l} \varphi_3 \quad (4.6)$$

The far end moment k_{mm} is

$$k_{mm} = \frac{2EI}{l} \varphi_4 \quad (4.7)$$

where,

$$\varphi_1 = \frac{S(1+C)}{6} \pm \frac{Nl^2}{12EI} \quad (4.8)$$

The plus sign being for axial tension, and the minus sign for axial compression.

$$\varphi_2 = \frac{S(1+C)}{6} \quad (4.9)$$

$$\varphi_3 = \frac{S}{4} \quad (4.10)$$

$$\varphi_4 = \frac{SC}{2} \quad (4.11)$$

and

$$S = \frac{12\psi}{4\psi^2 - \xi^2} \quad (4.12)$$

$$C = \frac{\xi}{2\psi} \quad (4.13)$$

For axial compression

$$\xi = \frac{6(\varphi \csc \varphi - 1)}{\varphi^2} \quad (4.14)$$

$$\psi = \frac{3(1 - \varphi \cot \varphi)}{\varphi^2} \quad (4.15)$$

For axial tension

$$\xi = \frac{6(1 - \varphi \operatorname{csch} \varphi)}{\varphi^2} \quad (4.16)$$

$$\psi = \frac{3(\varphi \coth \varphi - 1)}{\varphi^2} \quad (4.17)$$

In the above

$$\varphi = \ell \sqrt{\frac{N}{EI}} \quad (4.18)$$

The torsional stiffness k_t is

$$k_t = \frac{GJ}{\ell} \quad (4.19)$$

The axial stiffness k_a is

$$k_a = \frac{EA}{\ell} \quad (4.20)$$

4.3 Stiffness Matrices

In the formulation of the stiffness matrices the global axes are taken as shown in Figure 4.4. The x and y axes for each member are taken as given before in Figure 3.4. The individual member stiffness matrices and the global stiffness matrix of the space frame can be obtained as indicated in the following sections.

It should be mentioned that the stiffness matrices formulated here are those before the development of any plastic hinge in the frame. After the development of a plastic hinge, the corresponding stiffness matrices will be modified accordingly.

4.3.1 Member Stiffness Matrices

The set of internal forces at the ends of any member can be calculated from the equation

$$p = k\delta \quad (4.21)$$

where,

p = internal force vector

δ = member displacement vector

k = member stiffness matrix

Considering six degrees of freedom at each end of a member, the member stiffness matrices for beams $B_1, \dots, B_8$

FIGURE 4.4

FIGURE 4.5

FIGURE 4.6

(Figure 4.5) are thus 12×12 . For the columns $C_1, \dots, C_4$, however, the twelve internal forces at top and bottom of each column can always be represented in terms of the six displacement components at the top of each column, and a 12×6 stiffness matrices for the columns are thus formed.

Thus, for the eight beams and the four columns, eight 12×12 and four 12×6 stiffness matrices were formulated by simply placing the different stiffness terms developed in Section 4.2 in their appropriate locations. For programming convenience, these matrices were arranged in two tensors, $8 \times 12 \times 12$ for the beams, and $4 \times 12 \times 6$ for the columns.

4.3.2 Global Stiffness Matrix

The space frame considered has 48 degrees of freedom, six at each of the eight possible loading points (Figure 4.6). The displacement vector Δ can be calculated from the equation

$$P = K\Delta \quad (4.22)$$

where

$$\begin{aligned} P &= \text{external force vector} \\ \Delta &= \text{global displacement vector} \\ K &= \text{global stiffness matrix} \end{aligned}$$

The global stiffness matrix was obtained by combining the member stiffness matrices.

4.4 Stiffness Modifications

For an axially loaded member with an end hinge and acted upon by an axial force N (Figure 4.7), the stiffness terms were obtained by the following derivation.

For the beam with no hinges shown in Figure 4.8 acted upon by the same axial force N the different stiffness terms for this beam were given in Section 4.2. Since the axial force is the same in both beams, superposition can be used to derive the stiffness terms for the hinged beam.

For a unit lateral displacement, the end forces for the non-hinged beam are as shown in Figure 4.9a. To eliminate the moment at end 2 of the member, case (b) with its end forces shown in Figure 4.9b is superposed so as to cause the moment at end 2 to vanish. Superposition of the two cases yields case (c) in Figure 4.9. The net lateral force kh_f is

$$\begin{aligned} kh_f &= \frac{12EI}{l^3} \varphi_1 - \frac{6EI}{l^3} \varphi_2 (1 + C) \\ &= \frac{3EI}{l^3} [4\varphi_1 - 2\varphi_2 (1 + C)] \end{aligned}$$

FIGURE 4.7

FIGURE 4.8

FIGURE 4.9

From equations 4.9 and 4.10

$$1 + C = \frac{3}{2} \frac{\varphi_2}{\varphi_3}$$

Therefore

$$kh_f = \frac{3EI}{l^3} \left[4\varphi_1 - 3 \frac{\varphi_2^2}{\varphi_3} \right]$$

and thus the lateral stiffness for the hinged beam kh_f is

$$kh_f = \frac{3EI}{l^3} \kappa \quad (4.23)$$

where

$$\kappa = \frac{[4\varphi_1\varphi_3 - 3\varphi_2^2]}{\varphi_3} \quad (4.24)$$

The net moment at the fixed end is kh_{fm} is

$$kh_{fm} = \frac{6EI}{l^2} \varphi_2 - \frac{6EI}{l^2} \varphi_2 C$$

or,

$$kh_{fm} = \frac{6EI}{l^2} (1 - C) \varphi_2 \quad (4.25)$$

In a similar manner, the rotational stiffness for the hinged member kh_m is

$$kh_m = \frac{6EI}{l} (1 - C) \varphi_2 \quad (4.26)$$

which corresponds to a lateral force of $\frac{6EI}{l^2} (1 - C) \varphi_2$,

and this checks the symmetry of the stiffness matrix.

Any modified stiffness matrix, due to the formation of a hinge, can be viewed as the difference between the current stiffness matrix and a modification matrix

$$k_h = k_c - k_{mod} \quad (4.27)$$

where

k_h = modified stiffness matrix due to the formation of a hinge

k_c = current stiffness matrix

k_{mod} = modification matrix

The elements of a modification matrix thus being equal to the difference between the stiffness terms of a non-hinged and a hinged beam.

Thus for the twelve possible hinge locations in the beams, and the eight possible hinge locations in the columns (Figure 3.3), twelve 12 x 12 and eight 12 x 6 modification matrices were formulated by simply placing the modification elements in their appropriate locations. For programming convenience, these matrices were arranged in two tensors 12 x 12 x 12 for hinges in the beams, and 8 x 12 x 6 for hinges in the columns.

The method described by Jennings and Majid (7), in which the stiffness matrix is systematically increased and then inverted for each plastic hinge, is both time

consuming and requires a large storage, and thus was not considered.

4.5 Flexibility Matrix

The global stiffness matrix was inverted to get the flexibility matrix F of the structure, thus

$$\Delta = K^{-1}P = FP$$

It is to be noticed that a complete inverse of the stiffness matrix was obtained rather than a solution of the system of simultaneous equations 4.22 since further modifications due to the formation of plastic hinges in the structure will be directly done on the flexibility matrix rather than on the stiffness matrix.

4.6 Load Factor for Plastic Hinge

Under no load on the structure, the bending moments M_x and M_y at any point are zeros, and are represented by point "O" in Figure 4.10.

FIGURE 4.10

Since the level of axial load is assumed constant in all members through the loading process, equations 4.21 and 4.22 are linear, and M_x and M_y will increase linearly with the load factor D . This linear relation, represented by line 0-a, will continue up to the value of D that causes a plastic hinge at some point in the structure. At this stage, the stiffness of the structure changes due to this hinge, and the relation between D , M_x and M_y will assume a different linear relation represented by line a-b. If the structure did not collapse either due to the

formation of a plastic mechanism or combined stability-plastic collapse, this procedure will continue until the combined value of M_x and M_y reaches the interaction line $m-n$. At this moment, a plastic hinge has been reached at this point. A modification of the corresponding member stiffness matrix and the global flexibility matrix is then needed to account for the deteriorating stiffness of the structure, and to allow no more moments at this point.

For clarity, a more detailed flow chart is shown in Figure 4.11 and is explained as follows.

Under a zero load on the structure, the load factor D and the moments at each point about the major and minor axes are set equal to zero.

At any current structural state of the frame, with a corresponding flexibility matrix F , the displacement vector Δ of that structure when loaded by the set of ultimate loads P (which corresponds to a load factor $D = 1$) can be obtained from

$$\Delta = FP$$

where

$$F = \text{current flexibility matrix of the structure}$$

The corresponding set of internal forces p can thus be determined from

$$p = k\delta$$

FIGURE 4.11

where,

k = current members stiffness matrices.

Thus the bending moments about the major and minor axes ΔM_x and ΔM_y at each critical point due to the external load P can be readily extracted from p .

It should be mentioned here that the above steps merely furnish the current linear relation between ΔM_x and ΔM_y at each critical point. The increments of moments that are actually added in each step are $(\Delta D \cdot \Delta M_x)$ and $(\Delta D \cdot \Delta M_y)$ where ΔD is the increment of load factor needed to reach the next plastic hinge. This increment of load factor is determined as follows.

If the current moments at any point M_x and M_y , and the linear relation for the new structural state determined by ΔM_x and ΔM_y are as shown in Figure 4.12, the load factor increment ΔD needed to reach a plastic hinge at this point can be determined as follows.

FIGURE 4.12

From Figure 4.12, it can be seen that

$$\Delta D = \frac{M_{xu} - M_x}{\Delta M_x} \quad (a)$$

also,

$$\Delta D = \frac{M_{yu} - M_y}{\Delta M_y} \quad (b)$$

$$M_{yu} = T_y - \frac{T_y}{T_x} M_{xu} \quad (c)$$

Substituting by (c) in (b)

$$\Delta D = \frac{T_Y - \frac{T_Y}{T_X} M_{xu} - M_Y}{\Delta M_Y}$$

Thus

$$M_{xu} = T_X - \frac{T_X}{T_Y} M_Y - \frac{T_X}{T_Y} \Delta D \Delta M_Y$$

Substituting in (a), thus

$$\Delta D = \frac{T_X - \frac{T_X}{T_Y} M_Y - \frac{T_X}{T_Y} \Delta D \Delta M_Y - M_X}{\Delta M_X}$$

$$\Delta D \left(\Delta M_X + \frac{T_X}{T_Y} \Delta M_Y \right) = T_X - \frac{T_X}{T_Y} M_Y - M_X$$

Therefore

$$\Delta D = \left[T_X - \frac{T_X}{T_Y} M_Y - M_X \right] / \left[\Delta M_X + \frac{T_X}{T_Y} \Delta M_Y \right] \quad (4.28)$$

This equation is applied to all critical points where a plastic hinge is still to form, and the smallest value of the load factor increment ΔD will then determine the location for the next plastic hinge. The load factor D and the internal forces are then increased to this new level.

4.7 Modifications due to Plastic Hinge Formation

The position for the next plastic hinge has been determined in Section 4.6 corresponding to the smallest value of the load factor increment ΔD . The modification in the appropriate member stiffness matrix is then done by subtracting the modification matrix corresponding to this plastic hinge location from the current member stiffness matrix to give the modified new member stiffness matrix.

The global stiffness matrix is to be also modified by subtracting the columns of the modification matrix from the correct columns of the current stiffness matrix.

Since these modifications are limited to a few columns each time, it was possible to avoid repeatedly inverting the modified global stiffness matrix by doing the corresponding modifications directly on the flexibility matrix using the procedure described in Appendix B and yielding the equation

$$f_{kl}^* = f_{kl} - \frac{\sum_{m=1}^N f_{km} a_m f_{il}}{1 + \sum_{n=1}^N f_{in} a_n} \quad (4.29)$$

where,

- f^* = element of the new flexibility matrix
- f = elements of the old flexibility matrix
- a = vector of changes to the i^{th} column
of the stiffness matrix.

The vector of changes to the i^{th} column is composed of zeros and minus the elements of the corresponding column in the modification matrix k_{mod} of that hinge location.

For each plastic hinge developed, equation 4.29 was applied for all corresponding vectors of changes, one by one in succession.

Failure Condition

Failure of the structure occurs when the stiffness matrix becomes singular. This is the case when the denominator in equation 4.29 becomes zero.

$$1 + \sum_{n=1}^N f_{in} a_n = 0 \quad (4.30)$$

The elements of the modified flexibility matrix thus becoming infinitely large leading consequently to infinite displacements.

All these above steps were repeated until failure of the structure was achieved.

CHAPTER V

EXAMPLES AND RESULTS

5.1 General

The application of the methods developed in the previous chapters will be illustrated by reference to seven simple space frames. For comparison reasons, all these frames are taken to be fixed at column bases and to have the same dimensions (Figure 5.1). The members cross-sections are considered to have a wide flange shape and to be oriented as shown in Figure 5.1. The yield stress σ_y , the modulus of elasticity E and the shear modulus G for steel are taken to be

$$\begin{aligned}\sigma_y &= 50 \text{ ksi} \\ E &= 29000 \text{ ksi} \\ G &= 11150 \text{ ksi}\end{aligned}\tag{5.1}$$

In the following examples, the loads shown are the required ultimate loads on the frame in kips. It is assumed that any local failure in the members cross-sections will be satisfactorily prevented by stiffeners or any other suitable means as provided by specifications.

FIGURE 5.1

For each example, the linear and nonlinear constrained optimization methods were employed in succession. The minimum weight obtained and the optimum values for the members cross-sectional areas are presented. Also presented are the bending moment distribution, the points at which plastic hinges formed and the failure mechanism of the frame.

In case of vertical loads, the frame might fail due to instability at some stage before the formation of a plastic mechanism. For these cases, the optimally obtained cross-sectional areas were increased to the next available economical wide flange sections, as provided by the Manual of Steel Construction. For this new frame a step by step elastic-plastic analysis was performed up to failure. The order in which the plastic hinges formed and the load factors at their formation are presented. The load factor at any stage being defined as the ratio of the load at that stage to the required ultimate load on the frame.

Since the cross-sectional areas were increased, the frame was slightly strengthened and its plastic collapse load will thus slightly exceed the required ultimate loads. To find the load factor for plastic mechanism failure of this frame, an elastic-plastic analysis was performed

setting the axial forces in all members to zero. This was done in order to avoid frame instability and thus to permit enough plastic hinges to form to cause plastic mechanism failure of the frame. This load factor is also given for comparison.

The examples follow in Section 5.2. Comments and discussion of the results and their implications are presented in Section 5.3.

5.2 Examples

5.2.1 Example 1

In this example the set of ultimate loads acting on the frame are as shown in Figure 5.2.

Optimizing this case resulted the following minimum weight for the frame

$$W = 38860.8 \times \gamma$$

where

$$\gamma = \text{weight of steel per cubic inch.}$$

The optimum values obtained for the members cross-sectional areas are given in Table 5.1.

The set of redundants obtained led to the bending moment distribution shown in Figure 5.3. The moments given are in kip-inch.

FIGURE 5.2

Member	Area (in ²)	Member	Area (in ²)
1	56.0	5	56.0
2	48.7	6	48.7
3	--	7	--
4	48.7	8	48.7

TABLE 5.1

FIGURE 5.3

FIGURE 5.4

The points at which plastic hinges formed and the failure mechanism are shown in Figure 5.4. Torsional buckling occurred in the four columns.

This case of loading involves vertical loads and stability might be a problem. Thus an elastic-plastic analysis is needed. For this analysis the closest safe economical wide flange sections (as provided by the Manual of Steel Construction) were chosen for each of the optimum areas of Table 5.1. This led to a slightly strengthened frame. The chosen wide flange sections are given in Table 5.2.

Analyzing this frame up to failure, the order in which the plastic hinges formed and the load factors at their formation are given in Figure 5.5 and Table 5.3. It can be seen that the failure of this frame was due to its lateral buckling about the columns' minor axes after the two plastic hinges had formed.

The lower limit on the actual failure load is thus 1.0380. It is seen that the load factor exceeds unity because of the slight strengthening of the frame. To find the plastic collapse load for this slightly strengthened frame, which furnishes an upper limit on the failure load, an elastic-plastic analysis was performed setting the axial forces in all members to zero in order

Member	Section	Member	Section
1	W 36 x 194	5	W 36 x 194
2	W 36 x 170	6	W 36 x 170
3	W 12 x 22	7	W 12 x 22
4	W 36 x 170	8	W 36 x 170

TABLE 5.2

FIGURE 5.5

Hinge	Load Factor
1	1.0380

TABLE 5.3

to reach the plastic mechanism failure. If the frame was not strengthened, this load factor would have been unity. However, because of the slight strengthening of the frame, the load factor at plastic mechanism failure was found to be 1.2673. Thus the ratio of the limits obtained on the actual failure load is

$$r = \frac{1.0380}{1.2673} = 0.819$$

5.2.2 Example 2

In this example the set of ultimate loads acting on the frame are as shown in Figure 5.6.

Optimizing this case resulted the following minimum weight for the frame

$$W = 25250.5 \times \gamma$$

The optimum values obtained for the members cross-sectional areas are given in Table 5.4.

The set of redundants obtained led to the bending moment distribution shown in Figure 5.7. The moments given are in kip-inch.

The points at which plastic hinges formed and the failure mechanism are shown in Figure 5.8.

FIGURE 5.6

Member	Area (in ²)	Member	Area (in ²)
1	34.4	5	34.4
2	33.5	6	33.5
3	--	7	--
4	33.5	8	33.5

TABLE 5.4

FIGURE 5.7

FIGURE 5.8

5.2.3 Example 3

In this example the set of ultimate loads acting on the frame are as shown in Figure 5.9.

Optimizing this case resulted the following minimum weight for the frame

$$W = 38128.8 \times \gamma$$

The optimum values obtained for the members cross-sectional areas are given in Table 5.5.

The set of redundants obtained led to the bending moment distribution shown in Figure 5.10. The moments given are in kip-inch.

The points at which plastic hinges formed and the failure mechanism are shown in Figure 5.11.

This case of loading involves vertical loads and stability might be a problem. Thus an elastic-plastic analysis is needed. For this analysis, the closest safe economical wide flange sections were chosen for each of the optimum areas of Table 5.5. This led to a slightly strengthened frame. The chosen wide flange sections are given in Table 5.6.

Analyzing this frame up to failure, the order in which the plastic hinges formed and the load factors at their formation are given in Figure 5.12 and Table 5.7.

FIGURE 5.9

Member	Area (in ²)	Member	Area (in ²)
1	56.1	5	56.1
2	59.5	6	33.9
3	--	7	--
4	33.9	8	59.5

TABLE 5.5

FIGURE 5.10

FIGURE 5.11

Member	Section	Member	Section
1	W 36 x 194	5	W 36 x 194
2	W 33 x 220	6	W 30 x 116
3	W 12 x 22	7	W 12 x 22
4	W 30 x 116	8	W 33 x 220

TABLE 5.6

FIGURE 5.12

Hinge	Load Factor
1	1.0112

TABLE 5.7

As mentioned before, the load factor exceeds unity because of the slight strengthening of the frame. To find the plastic collapse load for this slightly strengthened frame, an elastic-plastic analysis was performed setting the axial forces in all members to zero in order to reach the plastic mechanism failure. Because of the slight strengthening of the frame, the load factor at plastic mechanism failure exceeded unity and reached a value of 1.0325. Thus the ratio of the limits obtained on the actual failure load is

$$r = \frac{1.0112}{1.0325} = 0.979$$

5.2.4 Example 4

In this example the set of ultimate loads acting on the frame are as shown in Figure 5.13.

Optimizing this case resulted the following minimum weight for the frame

$$W = 50400.0 \times \gamma$$

The optimum values obtained for the members cross-sectional areas are given in Table 5.8.

The set of redundants obtained led to the bending moment distribution shown in Figure 5.14. The moments given are in kip-inch.

FIGURE 5.13

Member	Area (in ²)	Member	Area (in ²)
1	56.0	5	56.0
2	55.0	6	55.0
3	38.0	7	38.0
4	55.0	8	55.0

TABLE 5.8

FIGURE 5.14

FIGURE 5.15

The points at which plastic hinges formed are shown in Figure 5.15. Torsional buckling occurred in the four columns.

Since this case of loading also involves vertical loads, an elastic-plastic analysis of the frame was needed to study the frame stability. For this analysis the closest safe economical wide flange sections were chosen for each of the optimum areas of Table 5.8. These sections are given in Table 5.9.

Analyzing this frame up to failure, the order in which the plastic hinges formed and the load factors at their formation are given in Figure 5.16 and Table 5.10.

Again the load factors exceed unity because of the slight strengthening of the frame. As before, to find the plastic collapse load for this slightly strengthened frame, an elastic-plastic analysis was performed setting the axial forces in all members to zero. The load factor obtained at plastic mechanism failure was 1.1864. Thus the ratio of the limits obtained on the actual failure load is

$$r = \frac{1.0815}{1.1864} = 0.912$$

Member	Section	Member	Section
1	W 36 x 194	5	W 36 x 194
2	W 36 x 194	6	W 36 x 194
3	W 33 x 130	7	W 33 x 130
4	W 36 x 194	8	W 36 x 194

TABLE 5.9

FIGURE 5.16

Hinge	Load Factor
1	1.0581
2	1.0815

TABLE 5.10

5.2.5 Example 5

In this example the set of ultimate loads acting on the frame are as shown in Figure 5.17.

Optimizing this case resulted the following minimum weight for the frame

$$W = 32669.1 \times \gamma$$

The optimum values obtained for the members cross-sectional areas are given in Table 5.11.

The set of redundants obtained led to the bending moment distribution shown in Figure 5.18. The moments given are in kip-inch.

The points at which plastic hinges formed and the failure mechanism are shown in Figure 5.19.

5.2.6 Example 6

In this example the set of ultimate loads acting on the frame are as shown in Figure 5.20.

Optimizing this case resulted the following minimum weight for the frame

$$W = 56234.4 \times \gamma$$

The optimum values obtained for the members cross-sectional areas are given in Table 5.12.

FIGURE 5.17

Member	Area (in ²)	Member	Area (in ²)
1	11.1	5	11.1
2	50.1	6	76.6
3	23.2	7	15.0
4	75.4	8	48.7

TABLE 5.11

FIGURE 5.18

FIGURE 5.19

The set of redundants obtained led to the bending moment distribution shown in Figure 5.21. The moments given are in kip-inch.

The points at which plastic hinges formed and the failure mechanism are shown in Figure 5.22.

As previously explained, an elastic-plastic analysis is needed to study the stability of this frame. For this analysis the chosen economical wide flange sections for each of the optimum areas of Table 5.12 are given in Table 5.13.

Analyzing this frame up to failure, the order in which the plastic hinges formed and the load factors at their formation are given in Figure 5.23 and Table 5.14.

The plastic collapse load for this slightly strengthened frame was obtained by performing an elastic-plastic analysis in the manner explained before. The load factor obtained at plastic mechanism failure was 1.1172. Thus the ratio of the limits obtained on the actual failure load is

$$r = \frac{1.0649}{1.1172} = 0.953$$

FIGURE 5.21

FIGURE 5.22

Member	Section	Member	Section
1	W 33 x 220	5	W 33 x 220
2	W 36 x 245	6	W 36 x 300
3	W 30 x 116	7	W 33 x 130
4	W 33 x 130	8	W 36 x 300

TABLE 5.13

FIGURE 5.23

Hinge	Load Factor
1	0.7911
2	0.9660
3	0.9995
4	1.0135
5	1.0208
6	1.0649

TABLE 5.14

5.2.7 Example 7

In this example the set of ultimate loads acting on the frame are as shown in Figure 5.24.

Optimizing this case resulted the following minimum weight for the frame

$$W = 26340.0 \times \gamma$$

The optimum values obtained for the members cross-sectional areas are given in Table 5.15.

The set of redundants obtained led to the bending moment distribution shown in Figure 5.25. The moments given are in kip-inch.

The points at which plastic hinges formed and the failure mechanism are shown in Figure 5.26. Torsional buckling occurred in column a-b.

For the elastic-plastic analysis of the frame, the chosen economical wide flange sections for each of the optimum areas of Table 5.15 are given in Table 5.16.

Analyzing this frame up to failure, the order in which the plastic hinges formed and the load factors at their formation are given in Figure 5.27 and Table 5.17.

The plastic collapse load factor for this slightly strengthened frame was found to be 1.1102. Thus the ratio of the limits obtained on the actual failure load is

$$r = \frac{1.0628}{1.1102} = 0.957$$

FIGURE 5.24

Member	Area (in ²)	Member	Area (in ²)
1	59.5	5	--
2	36.1	6	24.8
3	33.5	7	--
4	60.0	8	--

TABLE 5.15

FIGURE 5.25

FIGURE 5.26

Member	Section	Member	Section
1	W 33 x 220	5	W 12 x 22
2	W 33 x 130	6	W 24 x 84
3	W 30 x 116	7	W 12 x 22
4	W 33 x 220	8	W 12 x 22

TABLE 5.16

FIGURE 5.27

Hinge	Load Factor
1	1.0018
2	1.0617
3	1.0618
4	1.0628

TABLE 5.17

5.3 Comments and Discussion of Results

5.3.1 Optimization Analysis

In the previous examples the frames, subject to the required sets of ultimate loads, were optimized. The function to be minimized was the weight of the frame. It is clearly seen from examples 4, 5, 6 and 7 that the problem of optimizing a space frame cannot be considered as that of optimizing independent plane frames and then using superposition. The double bending of the frame columns causes interaction between these cases. It can be easily seen that the bending moment distribution in these examples would have differed from what is presented here if each of these space frames was optimized as a set of independent plane frames.

The examples also clearly show the importance of considering torsional buckling of the columns, particularly when only large vertical loads act on the frame. For examples 1, 4 and 7 torsional buckling of some or all of the frame columns occurred.

A study of examples 2 and 5 shows that bending of the columns about their minor axes greatly affects the internal force distribution in the frame. Adding two lateral forces of less than one-third of the forces in example 2 greatly changed the bending moment distribution

from that of Figure 5.7 to that of Figure 5.18. For expected high lateral forces causing moments about the columns' minor axes, other cross-sectional shapes with higher Z_y/Z_x ratio will result in a more economical frames.

The effect of neglecting torsion constraints in the optimization procedure have led to meaningless internal force distribution in the frame, as was previously discussed in Section 3.3.3. This was overcome by imposing torsion constraints to limit the amount of torsional moment in each member. However, another approach that will reduce the computation time will be to consider only five internal forces at each of the beam cuts and neglect the internal forces causing torsion in the beams. The torsion constraints will thus be only needed for the columns. For the simple space frame considered, this approach will reduce the number of variables and constraints, each by four.

5.3.2 Elastic-Plastic Analysis

For the examples where vertical loads are present, and stability might cause an early failure to the frame, an elastic-plastic analysis was performed. Since the frames were first optimized and a good distribution of

the material was achieved, the frames remained elastic up to at least 70% of the plastic mechanism load. Plastic hinges then started forming in succession at very close load factors and the frame finally failed. It is to be noted that since each optimal area, obtained from the continuous solution, was increased to the next available economical wide flange section, the frame was slightly strengthened. The plastic collapse load for the new frame only increased slightly over the required ultimate loads. Discrete optimization techniques thus seems to be of little advantage. For larger space frames, however, the difference is expected to increase and discrete optimization might result in a considerable saving. It should be noted that discretization of the frame changed the pattern in which the plastic hinges formed and the failure mechanism, but slightly affected the failure load of the frame.

A remark to be mentioned concerning example 6 is that the frame did not collapse after the formation of the fifth plastic hinge. The frame was still able to resist a slightly higher load until the sixth plastic hinge formed. The results showed that since the moments at points 1, 4 and 5 were kept constant and equal to the plastic moments of these sections, the frame was able to

adjust itself and absorb the required slight increase in moments by slightly increasing the torsional moments in beams a-b and c-d.

The conditions at service loads were also checked and both the deflections and stresses were found to be satisfactory.

The ratio of the limits obtained on actual failure load varied from 0.819 to 0.979. Thus actual failure of such short space frames, due to the interaction of stability and plastic mechanism failures, is expected not to occur until approximately 90% of the plastic collapse load has been reached. It should be mentioned, however, that this ratio is expected to decrease, and stability becomes more important if higher space frames are considered.

5.3.3 Sequential Gradient-Restoration Algorithm

As previously mentioned, the sequential gradient-restoration algorithm was employed in optimizing these space frames. The number of variables and constraints for any of these frames were 60 and 36 respectively.

Concerning the gradient phase, the limit imposed on the one-dimensional search for a gradient step size was

$$B^* \leq 30 \quad (5.2)$$

Thus the error in the constraint at the end of the gradient phase did not exceed 30. A constraint error B^* greater than 30 was found to generally increase the computation time. This is because a larger allowed constraint error led to a large gradient step size. After restoration the value of the objective function was found to have increased and successive bisections of the gradient step size was thus needed.

Concerning the restoration phase, the restoration algorithm was employed iteratively until the error in the constraint satisfied the inequality

$$B \leq 10^{-1} \quad (5.3)$$

Finally, the whole sequential gradient-restoration algorithm was terminated when the error in the optimal condition Q satisfied the inequality

$$Q \leq 10^{-1} \quad (5.4)$$

5.3.4 Computer Time

Computations were performed on the Burroughs B5500 computer, which has 32,000 48-bit words of core plus magnetic disk and tape storage. The times for addition and multiplication are about 20 and 50 microseconds, respectively. The computation time needed varied from

one example to the other depending on the case of loading. For the optimization analysis of the space frame, on the average, approximately 15 minutes were needed for each case. For the elastic-plastic analysis, on the average, approximately 3 minutes were needed for each case.

CHAPTER VI

SUMMARY AND CONCLUSIONS

In this work the problem of optimizing a space frame has been formulated and a general method for its solution has been presented. The function to be minimized was the weight of the frame. This function was subject to constraints so as to lead to a both safe and statically admissible internal force distribution in the space frame.

Because space frames are highly statically indeterminate and generally have a large number of members and possible loading points, the optimization problem of such structures involves a large number of variables and constraints. For such large problems, both linear and nonlinear constrained optimization methods of analysis were presented.

In the optimization analysis presented herein, the variable representing the cross-section property was chosen to be its area. Relations between the area and the plastic section moduli, based on a least square fit, for economical wide flange sections has been obtained. Also a nonlinear nondimensional interaction surface has been developed determining the combinations of internal

forces that causes plastification of a wide flange section.

The stability problem of space frames has also been studied. The formation of plastic hinges reduces the stiffness of the frame, and its instability may occur after only several plastic hinges had formed and before the formation of a plastic mechanism. A step by step elastic-plastic method of analysis has been presented that results the order in which the plastic hinges will form and that provides a lower limit on the actual failure load.

The different methods developed have been applied to simple space frames. The main results and conclusions of this study are summarized below.

1. The results obtained by linear optimization of the space frames only differed slightly from those obtained by nonlinear optimization. It is suggested that a linear optimization analysis of the space frame be performed followed by applying the sequential gradient-restoration algorithm only for a few iterations.

2. Neglecting torsion constraints in the optimization analysis resulted in meaningless internal force distribution in the space frame. Torsion constraints thus need to be considered in such optimization problems.

3. Torsional buckling of the frame columns might occur if in the optimization analysis these columns were considered to have economical wide flange cross-sections. This was particularly the case when only large vertical loads acted on the frames.

4. The ratio of the plastic section moduli with respect to the major and minor axes (Z_y/Z_x) affects the optimum internal force distribution in the space frame and its weight. For expected large forces causing moments about the cross-sections minor axes, a cross-sectional shape with high Z_y/Z_x ratio is recommended.

5. For the optimized space frames, the deflections and stresses at service loads were found to be satisfactory.

6. For the optimized space frames, the frames remained elastic until at least 70% of the plastic mechanism load has been reached. Plastic hinges then started forming in succession with small increases in load, and the frames finally failed due to the interaction of stability and plastic mechanism failures.

7. Failure of single-story space frames, due to their instability after the formation of only several plastic hinges, is unlikely to occur at a load less than 90% of the plastic mechanism load.

8. This study has clearly demonstrated that the problem of optimizing a space frame cannot be considered as that of optimizing a set of independent plane frames and then using superposition. Double bending of the frame columns and the overall stability of these frames prevent such analysis.

9. Discrete optimization of small space frames seems to be of little advantage in reducing the weight of the frame. For large space frames, however, discrete optimization might result in a considerable saving.

10. Discretization of the space frames changed the pattern in which the plastic hinges formed and the failure mechanisms, but slightly affected the failure loads of these frames.

11. In employing the modified quasilinearization algorithm, the constraint error and the optimality error were compared after each iteration. If the optimality error was found greater than the constraint error, it was found advantageous to recalculate the Lagrange multiplier vector so as to minimize the optimality error with respect to it. This resulted in a considerable reduction in the optimality error and consequently in the overall error.

12. The sequential gradient-restoration algorithm was found to be convenient for these optimization problems with their large number of variables and constraints.

13. In employing the sequential gradient-restoration algorithm, it was found that imposing a large limit on the one-dimensional search for a gradient step size increased the computation time. A large limit led to a large gradient step size. After restoration the value of the objective function was found to have increased, and successive bisections of the gradient step size was thus needed.

Further work remains to be done for larger space frames. To permit the treatment of such large problems, these frames can be considered as composed of simple space frame units as those studied in this work. A unit by unit optimum design procedure can then be followed. The formulations and techniques developed herein should be very useful in these further studies.

APPENDIX A

Symmetric Method

APPENDIX B

Flexibility Modifications

APPENDIX B

Flexibility Modifications

Consider a stiffness matrix $[K]$ of size $N \times N$ and its inverse $[F]$. If the i^{th} column of K is to be modified by adding a vector $\{a\}$ to it, the corresponding new inverse $[F]^*$ can be determined without making a formal inversion of the modified stiffness matrix $[K]^*$ as follows.

The modified stiffness matrix $[K]^*$ can be expressed as

$$[K]^* = [K] + [A] \quad (\text{B.1})$$

If $\{e_i\}$ is the unit vector of spread i , then the matrix $[A]$ can be expressed as

$$[A] = \{a\} \{e_i\}^T \quad (\text{B.2})$$

Performing the following matrix operations

$$\begin{aligned} [A] [F] [A] &= \{a\} \{e_i\}^T [F] \{a\} \{e_i\}^T \\ &= \{a\} \left(\{e_i\}^T [F] \{a\} \right) \{e_i\}^T \\ &= \left(\{e_i\}^T [F] \{a\} \right) [A] \end{aligned} \quad (\text{B.3})$$

and,

$$[K]^* [F] [A] [F] = [K+A] [F] [A] [F] = [A] [F] + [A] [F] [A] [F] \quad (\text{B.4})$$

Substituting by the value of the product $[A][F][A]$ from equation B.3 into equation B.4, one gets

$$\begin{aligned} [K^*][F][A][F] &= \left(1 + \{e_i\}^T [F] \{a\}\right) [A][F] \\ &= \left(1 + \{e_i\}^T [F] \{a\}\right) ([A][F] + [K][F] - [I]) \end{aligned} \quad (B.5)$$

Using equation B.1, equation B.5 can be rewritten as

$$[K^*][F][A][F] = \left(1 + \{e_i\}^T [F] \{a\}\right) ([K^*][F] - [I]) \quad (B.6)$$

Premultiplying both sides of equation B.6 by $[F^*]$, one gets

$$[F][A][F] = \left(1 + \{e_i\}^T [F] \{a\}\right) ([F] - [F^*])$$

Solving for $[F^*]$, one gets

$$[F^*] = [F] - \left(\frac{1}{1 + \{e_i\}^T [F] \{a\}} \right) [F] \{a\} \{e_i\}^T [F] \quad (B.7)$$

Equation B.7 gives the new flexibility matrix $[F^*]$, in terms of the original flexibility matrix $[F]$ and the vector of changes $\{a\}$ to the i^{th} column of the stiffness matrix $[K]$.

Equation B.7 can be written in the expanded form

$$f_{kl}^* = f_{kl} - \frac{\sum_{m=1}^N f_{km} \times a_m \times f_{il}}{\left(1 + \sum_{n=1}^N f_{in} \times a_n\right)} \quad (\text{B.8})$$

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