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SWIRLING FLOW OF A DISSOCIATED GAS

by

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NOMENCLATURE

a	speed of sound
A^*	throat area of nozzle
A_1, A_2, A_3, A_4	coefficients defined by equations (35)-(38)
C.R.	contraction ratio of nozzle
c_i	mass fraction of species i
C_p	heat capacity
c_1	constant in Sutherland formula
D_i	binary diffusion coefficient of species i
D_i^t	thermal diffusion coefficient of species i
D_o	dissociation energy
f	stream function defined for boundary layer flow
g	nondimensional enthalpy ratio = h_o/h_{oe}
h	static enthalpy
h_o	stagnation enthalpy
h_p	Planck's constant = 6.6256×10^{-27} erg-sec
k	thermal conductivity
k_B	Boltzmann's constant = 1.38054×10^{-16} erg/ $^{\circ}K$
l	$\rho\mu / \rho_e \mu_e$
M	Mach number
$\mathcal{M}$	molecular weight
m	mass of an atom
$\dot{m}$	mass flow rate
$\dot{m}_T$	total mass flow rate
N_{Le}	Lewis number
N_{pr}	Prandtl number
p	pressure

p_o	stagnation pressure
Q_{el}^a	electronic partition function for an atom
Q_{el}^{aa}	electronic partition function for a molecule
R	gas constant = R/m
$\mathcal{R}$	universal gas constant = 1545 ft-lb/mole °R
r	radial coordinate (in)
r_c	chamber radius (in)
r_v	core radius
$\bar{r}$	r/r_c
s	entropy
T	static temperature
T_o	stagnation temperature
t	time
$\vec{v}$	velocity vector
u	radial velocity component
v	tangential velocity component
w	axial velocity component
W_i	rate of production of species i
x	streamwise coordinate measured along nozzle wall
y	coordinate normal to nozzle wall
z	axial coordinate
α	dissociation fraction
β	pressure gradient parameter
δ^*	boundary layer displacement thickness
ϵ	turbulent eddy diffusivity
$\tilde{z}$	transformed normal coordinate
Γ'	circulation

Γ	vr
γ	ratio of specific heats for a perfect gas
δ_e	equilibrium isentropic exponent
δ_f	frozen isentropic exponent
χ	empirical constant for turbulent flow
μ	viscosity
Ω_μ	collision integral for viscosity
ω	angular velocity
ψ	stream function for inviscid flow
ρ	density
ρ_d	characteristic density for dissociation
σ	collision diameter of a molecule
Θ_d	characteristic temperature for dissociation = D_o/k_b
Θ_r	characteristic temperature for rotation
Θ_v	characteristic temperature for vibration
ξ	transformed streamwise coordinate
S	nondimensional mass fraction = c_1/c_{1e}

Subscripts

c	chamber
cl	centerline
e	edge conditions for boundary layer
i	i^{th} species
n	total number of species
o	stagnation conditions
t	throat
w	wall conditions for boundary layer

Superscripts

$\rightarrow$	vector quantity
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I INTRODUCTION

Swirling flow is the term that is normally applied to fluid flow in a streamtube in which a vortex is superimposed on an axial flow thru the streamtube. If the vortex axis is aligned with the axial flow direction, the resulting streamlines describe a helical path thru the tube. Centrifugal forces give rise to radial pressure gradients in the flow. This phenomenon becomes especially interesting in the case of a duct whose area is decreasing in the direction of flow. Here the conservation of momentum requires the tangential velocity of the fluid to increase as the area of the duct, and thus the effective radius of the vortex, decreases. In this way an initially slow swirling motion can be greatly accelerated by an area reduction with profound effects on the whole flowfield. This is precisely the case for an axisymmetric converging-diverging nozzle designed to accelerate a gas to supersonic speeds.

Physical examples of such situations are numerous. Small solid-fuel rockets are often spun about their longitudinal axis in order to obtain gyroscopic stabilization in flight. The spin rates can be as large as 20,000 rpm. The solid fuel spins with the motor casing and thus imparts a tangential velocity to the gas produced by the burning of the propellant. As the gas is accelerated thru the nozzle, the tangential velocity is greatly increased at the throat and serves to decrease the mass flow rate and thus the effective throat area. This causes the chamber pressure to rise, sometimes to dangerous levels. Lucy⁽¹⁾ has observed this experimentally as well

as the effects of spin on the actual burning process.

The decrease in effective throat area with increasing spin has been proposed as a means of controlling the chamber pressure and mass flow rate of both solid and liquid fueled rocket motors. A vortex valve could be used to impart a tangential velocity to the gases entering the nozzle. Such a control system has the advantage of being much lighter in weight and simpler than mechanical methods of adjusting the nozzle geometry. One example of such a system has been discussed recently by Walsh, et.al.⁽²⁾

Swirl may be introduced into a nozzle flow for other reasons. In certain proposed nuclear rocket designs, a vortex is used to stabilize the motor and keep the uranium fuel inside the chamber⁽³⁾. Another important application is in arc-jets where the working gas is often injected tangentially into the chamber. The swirling gas helps to stabilize the arc and promotes more uniform mixing if more than one species is involved. In these cases the swirl will again affect the mass flow rate and chamber pressure as described above. The presence of swirl may also affect the heat transfer due to changes in pressure gradient and velocity along the wall. It is therefore important that the fluid mechanics of swirling flow be well understood so that its effects can be analyzed and their magnitude predicted.

Previous analytical studies of the swirling flow problem are numerous and varied. All have one feature in common in that they treat only perfect gases with constant specific heat ratios. This simplification represents many real flow situations well and is a

definite computational advantage. At temperatures and pressures typical of many rocket combustion chambers and arc-jets, the possibility of chemical reactions and real gas effects cannot be ignored however. Such reactions as dissociation and ionization and the converse, recombination, are but two examples. The purpose of the present investigation therefore is to extend the theory of swirling flows to include real gas effects and specifically dissociation reactions.

The equations of swirling, reacting flow are developed using the Lighthill ideal dissociating gas approximation. Initial conditions are derived from two different tangential velocity distributions. The equations are then integrated using an Adams-Moulton predictor-corrector method for various initial conditions. Results are presented both for the design and performance problems and the differences between the two are investigated. Finally, a rough estimate of boundary layer effects is made to evaluate the validity of the results.

II LITERATURE SURVEY

There have been a number of analytical studies of swirling nozzle flow done in the past. Mager⁽⁴⁾ first looked at the problem when a potential or free vortex is assumed. This type vortex is characteristic of a solid-fuel rocket with an internal burning grain as well as most arc-jets. It is characterized by a potential function that is constant with r . This leads to a tangential velocity distribution of the form $vr = \text{constant}$. Since the flow is isentropic, the solution can be derived from Bernoulli's equation and an integrated form of the continuity equation. Mager did this for various values of a non-dimensional swirl parameter given by the local tangential velocity at the wall divided by the maximum possible wall tangential velocity for a given stagnation temperature. Results are given for $\gamma = 1.4$. The results have been extended to other values of γ by Glick and Kilgore.⁽⁵⁾ Both solutions are restricted to perfect gases with constant γ .

One consequence of the free vortex model is that the tangential velocity approaches infinity as the radius goes to zero. It turns out however that the density simultaneously decreases and at a certain limiting radius becomes zero. The solution thus predicts a void area surrounding the centerline that may extend clear thru the throat to the exit plane of the nozzle. This void area is perhaps the most undesirable feature of the solution as the existence of a region of zero density in the nozzle seems unlikely. Mager points out that in an actual physical situation, the void area would be replaced by a low-density core in which viscous effects predominate. This also

raises the possibility of backpressure effects propagating upstream thru the nozzle and causing regions of reversed flow in the core, and thus further complicating the problem. Experimental results concerning this phenomenon will be discussed later.

Bastress⁽⁶⁾ looked at the problem of swirling flow with a rotational or solid-body vortex ($v/r = \text{constant}$). This type of vortex is generally produced by a solid-fuel rocket with an end-burning grain or any time a vortex breaks down due to internal viscous dissipation. The tangential velocity is directly proportional to the distance from the centerline and therefore increases continuously from zero at the centerline to a maximum value at the nozzle wall. In his solution, Bastress assumed isentropic flow and neglected the radial pressure gradient. These assumptions guarantee uniform contraction of the streamlines, i.e., the tangential velocity distribution is mathematically similar through the nozzle. The scale factor is simply the local contraction ratio. At the throat he assumed that the centerline velocity is sonic and the pressure, temperature, and density are the same as for the non-swirling problem. Since the magnitude of the velocity vector remains constant under his assumptions, the axial velocity decreases as the tangential velocity increases with increasing radial distance from the centerline. This has the same net effect as a reduction in throat area and thus the ratio A^*_{eff}/A^* can be computed as a function of the angular velocity and contraction ratio for constant γ .

One objection to Bastress' analysis is the neglect of the radial pressure gradient. While this gradient is indeed small in the chamber

area, it becomes quite large at the throat for nozzles with realistic contraction ratios. His analysis can also be shown to violate conservation of energy as pointed out by Manda⁽⁷⁾ who reworked the problem using a radial pressure gradient at the throat. Manda again assumed a solid-body rotation throughout the nozzle and $M = 1$ on the axis at the throat.

Neither analysis can satisfy all the conservation equations if the maintenance of a forced vortex is assumed throughout the nozzle. The assumption of $M = 1$ on the axis of the throat is similarly arbitrary. M. L. King⁽⁸⁾ tried to correct this by applying a "choking criterion". This was accomplished by maximizing the expression for $\dot{m}/A$. He still found $M = 1$ on the axis at the throat as a consequence of the maintenance of the forced vortex. He obtained a slightly different expression for the velocity distribution at the throat however. His results were also questioned by Manda⁽⁹⁾ who asserted that King failed to satisfy the radial momentum equation at the throat.

W. S. King^(10,11) has investigated the same problem using a "quasi-cylindrical" approach. In this he assumed that the rate of change of the nozzle cross-sectional area with length is small enough that radial velocities and their derivatives can be neglected. This is the same as the assumptions of ordinary one-dimensional gas dynamics and the equations are therefore similar except for the swirl terms. He does not however require the maintenance of any certain tangential velocity distribution throughout the flow. Only an

initial velocity distribution is specified and downstream solutions are related to the initial conditions by assuming that circulation remains constant along streamlines. The problem therefore becomes that of locating the streamlines in any downstream plane in which a solution is required. This requires a numerical integration of the governing differential equations rather than algebraic solutions such as were possible under the assumptions used by previous authors. For sufficiently small swirl, a perturbation approach can be used to give closed form solutions.

King has presented a large number of numerical solutions for both incompressible and compressible flow. He assumed a uniform axial velocity, a specified tangential velocity distribution, and a centerline pressure and temperature as initial conditions. These, along with the assumption of isentropic flow were shown to be sufficient to completely specify the problem, and no artificial choking conditions were assumed at the throat. King used both a solid-body rotation and a free vortex with a solid-body core as initial conditions. In the latter case, the solid-body core was used as an approximation to the breakdown of a real free vortex close to the axis because of internal viscous dissipation. This also avoids the difficulty of a zero-density area in the flow as in Mager's solution. The velocity, energy, density, and circulation are matched at the core boundary and the core radius becomes a free parameter.

King's solutions show regions of reversed flow near the centerline in the subsonic portion of the nozzle. In no case do the

reverse bubbles penetrate the throat region. It is also interesting to note that King's solutions show an axial Mach number greater than one on the centerline at the throat. As in previous cases, King's results are restricted to flow of a perfect gas with a constant γ .

Mager⁽¹²⁾ has further studied the viscous core problem for incompressible flow. He found regions of reversed flow in the core for values of a swirl parameter larger than a certain critical value. The swirl parameter consists of a combination of the maximum velocity, throat Reynolds number, and the circulation at the core boundary. His results are not directly applicable to compressible flow however, since the choking phenomenon is absent in incompressible flow, and thus the recirculation regions can extend through the entire nozzle.

Lewellen, Burns, and Strickland⁽¹³⁾ have also considered the swirling nozzle flow problem in a manner similar to King. Their assumptions concerning the initial conditions restrict their analysis to small spin rates since they assume P_0 and T_0 to be constant across the initial chamber. They assumed a circulation distribution of the form:

$$\Gamma = c [(1 - e^{-K\psi^*(X)}) / (1 - e^{-K\psi^*(1)})]$$

where ψ^* is a normalized stream function and X is a nondimensional radius equal to the local radial distance from the centerline divided by the nozzle throat radius. The constant K is analogous to a Reynolds number and its magnitude influences the shape of the tangential velocity profile. At $K = 0$, the profile goes to a solid-body rotation while $K \rightarrow \infty$ corresponds to a free vortex. For values of K be-

tween the two extremes, the profile goes through a transition from a free vortex in the outer annulus to a solid-body rotation near the centerline. The maximum velocity occurs at some intermediate radius whose value depends on K , as does the magnitude of the velocity. This distribution can be derived analytically and has special significance when turbulent dissipation is considered. This will be discussed further in Section III.

Only a few numerical results are given in Reference (13) and these are for the solid-body rotation ($K = 0$) case. Their results are also restricted to a perfect gas with constant γ . In addition however the authors performed an experimental study using smoke rings and concluded that there are recirculation regions in the flow but that they do not penetrate the throat area, thus confirming W. S. King's theoretical predictions.

One of the most comprehensive studies done is the work of Norton, Farquhar, and Hoffman^(14,15). The analysis due to Norton is similar to those by W. S. King and Lewellen, et al. All the initial conditions were developed in terms of a stream function. Downstream solutions are then a matter of locating the streamlines in the spacial coordinate, the radial distance from the centerline. Both solid-body rotation and a free vortex distribution similar to Lewellen's, et al, are used. The analysis again applies only to perfect gases with constant γ . Norton corrected a small error in the previous analyses by allowing for entropy gradients in the case of the rotational vortex. The necessity for this can be observed by

considering the Crocco-Vazsoni equation for steady flow:

$$\nabla h_0 - T \nabla s = \vec{V} \times \nabla \times \vec{V}$$

where $\vec{V}$ is the velocity vector. When the right-hand side does not vanish (rotational vortex), an entropy gradient is necessary in general to balance the equation. All of the previously described analyses omitted consideration of this entropy gradient. The free vortex is irrotational and thus can be considered isentropic since the r.h.s. of the Crocco-Vazsoni equation does vanish. A summary of Norton's analytical method is given in Reference (16).

Norton's analytical work is compared with some cold-gas experimental results of Farquhar's in (14) and (15) for both solid-body and free-vortex tangential velocity distributions. The results in terms of a mass flow rate parameter $\dot{m} a_0 / A^* P_0$ show fairly close agreement between the analytical and experimental results. A viscous correction added to the analytical calculations further improves the comparison at large angular velocities. Norton's approach is thus reasonably well substantiated.

Other experimental studies are relatively few in number, consisting largely of unpublished and other classified works. The majority of the experiments are concerned with gross effects such as thrust and chamber pressure of actual rocket motors rather than precise investigations of fluid mechanical effects. A series of wind-tunnel studies have been done by P. F. Massier⁽¹⁷⁻¹⁹⁾ in which argon at room temperature was injected tangentially into the stagnation chamber to produce a free vortex superimposed on a small axial

velocity. The gas was then accelerated through a supersonic nozzle into an evacuated tank. Some velocity profiles are given on the end wall of the chamber showing a solid-body rotation near the centerline and essentially a free vortex in the outer annulus. The maximum tangential velocity occurs at a radius of slightly less than one tenth of the initial chamber radius. Other data presented include such quantities as pressure and temperature as functions of axial distance down the nozzle as required for heat transfer calculations.

Batson and Sforzini⁽²⁰⁾ have performed a cold-gas study of swirling flow using a free vortex created by tangential injection of air. Velocity profiles are given at several locations in the nozzle. The basic profile is again a free vortex with a solid-body core. The peak tangential velocity is observed to decrease rather than increase with the contraction of the streamlines in the nozzle. This is probably due to viscous effects within the vortex itself. The free vortex tends to decay to solid-body rotation due to viscous effects in the core spreading outward into the free vortex. Their results for the decrease in mass flow rate are close to Mager's⁽⁴⁾ for small values of Mager's swirl parameter but show increasing deviation at larger values. This again is probably due to changes in the vortex caused by internal viscous dissipation.

Boundary Layer Effects

It is quite possible that the existence of a non-zero angular velocity component in the nozzle flowfield could affect the boundary layer flow as well as the freestream. This particular problem has

received far less attention than the inviscid analysis however. This is due to the three-dimensional character of such a boundary layer and thus the few existing analyses contain a large number of approximations intended to get around the three-dimensional nature of the boundary layer problem.

H. E. Weber⁽²¹⁾ looked at the problem of a boundary layer inside a cyclone dust separator. He made the assumption that the axial velocity at the wall would be much smaller than the tangential velocity and thus neglected the axial velocity entirely. He solved the axial and tangential momentum equations for both laminar and turbulent flow using a momentum-integral technique. The axial pressure gradient was found to induce a small axial crossflow in the boundary layer even though the axial velocity at the edge was still assumed to be zero. In addition it was found that the boundary layer could completely fill the cone toward the apex end and this invalidated the analysis in this region. Agreement with experiment was remarkably good.

Norton⁽¹⁵⁾ employed a viscous correction derived by solving only the tangential momentum equation. The reduction in the moment of momentum was then subtracted from the initial vortex strength to give a corrected inviscid solution. This method gave reasonable agreement with experimental studies.

Back⁽²²⁾ treated the swirling boundary layer problem using a similarity technique. The complete set of partial differential equations was reduced to ordinary differential equations by use of

appropriate similarity transformations for the case of a free vortex in the freestream. The similarity assumptions placed rather severe restrictions on the pressure gradient parameter and swirl parameter with the result that the solution would only be valid for low speed, small swirl flows. The solutions did predict an axial velocity overshoot in the boundary layer, i.e., an increased axial flow near the wall. The heat transfer was found to increase also due to swirl. Back used the concept of local similarity (23) to extrapolate his results to the throat area. He found only a small increase in heat transfer at the throat. Massier⁽²⁴⁾ has noted similar trends experimentally.

Scope of the Current Investigation

All of the previous analytical results have been derived using the perfect gas assumption with constant specific heats, and furthermore all previous experimental results have been for low temperature gases. However, in an actual rocket engine or arc-jet, the temperatures can be sufficiently high to cause such real gas phenomena as dissociation and ionization to take place. It seems reasonable to ask then what the effects of real gas phenomena on the swirling flow problem are.

The purpose of the present investigation is to determine the effects of gas-phase reactions, specifically dissociation, on the swirling flow phenomena. This is accomplished using the Lighthill ideal dissociating gas^(25,26) approximation. Though this is only an approximation to the actual dissociation of diatomic gases, it is

useful in predicting the trends to be expected in more complicated reacting systems. In addition the validity of the assumption of constant ratio of specific heats is investigated and its range of applicability determined. This information is of use in extrapolating previous results to the case of reacting systems. A series of curves is presented to display the influence of the various parameters of the problem. Viscous boundary layer effects are taken into account in an approximate fashion in order to indicate the relative importance of the boundary layer in swirling reacting flows.

III ANALYSIS

The governing equations of motion for a reacting swirling flow through a nozzle are written using the Lighthill ideal dissociating gas model. The analysis is therefore restricted to the case of a symmetrical diatomic gas. This is not too restrictive since many real gases have properties similar to particular diatomic gases. The ideal dissociating gas can be expected to show trends connected with dissociation phenomena even if it does not model a particular polyatomic gas exactly. The basic assumptions connected with the ideal dissociating gas model and the calculation of constants for several diatomic gases are presented in Appendix A.

The use of this particular gas model in reacting flow problems is well-established for both equilibrium and non-equilibrium conditions. Bray⁽²⁷⁾ used it to develop the "sudden freezing" theory concerning the abrupt transition from equilibrium to frozen (zero reaction rate) flow in a hypersonic nozzle. Wegner's experiments⁽²⁸⁾ with N_2O_4 confirmed the essential features of Bray's theory. Both investigators concluded that the flow in the subsonic portion of the nozzle is close to equilibrium with freezing taking place in the supersonic part of the nozzle. The following analysis is based on equilibrium flow, therefore, though the equations of frozen flow are also presented, thus establishing upper and lower bounds for the non-equilibrium problem. The analysis assumes steady, isentropic, axisymmetric, quasi one-dimensional flow. The overall approach is similar to that given by Norton⁽¹⁶⁾ for perfect gases.

A. Governing Equations

The governing equations are:

$$\text{Continuity: } \frac{\partial \rho u r}{\partial r} + \frac{\partial \rho w r}{\partial z} = 0 \quad (1)$$

$$\text{Axial Mom.: } u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial P}{\partial z} \quad (2)$$

$$\text{Angular Mom.: } u \frac{\partial v}{\partial r} + w \frac{\partial v}{\partial z} + \frac{u v}{r} = 0 \quad (3)$$

$$\text{Radial Mom.: } u \frac{\partial u}{\partial r} + w \frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} \quad (4)$$

where u , v , and w are the radial, tangential and axial components of the velocity vector and r and z are the radial and axial coordinates.

$$\text{Energy: } h_o = \text{constant on streamlines} \quad (5)$$

$$\text{Entropy: } S = \text{constant on streamlines} \quad (6)$$

$$\text{State: } P = (1 + \alpha) \rho R T \quad (7)$$

$$\text{Mass Action: } \frac{\alpha^2}{1 - \alpha} = \frac{\rho_d}{\rho} e^{-\Theta_d / T} \quad (8)$$

where α is the dissociation fraction and ρ_d and Θ_d are the characteristic density and temperature of dissociation for any particular diatomic gas. The gas constant R is the ordinary gas constant for the undissociated species of the gas.

The circulation is defined as:

$$\Gamma' = \oint_c \vec{U} \cdot d\vec{r} = 2\pi u r \quad (9)$$

The angular momentum equation can be rewritten in terms of the circulation:

$$u \frac{\partial \Gamma'}{\partial r} + w \frac{\partial \Gamma'}{\partial z} = 0 \quad (10)$$

Since the flow is steady, this can be rewritten

$$\frac{D\Gamma'}{Dt} = 0 \quad (11)$$

This means that the circulation remains constant along streamlines.

Since the circulation remains constant along streamlines, the product vr does also. Thus a new quantity Γ can be defined,

$$\Gamma = vr \quad (12)$$

and Γ is constant along streamlines. The prime was used to distinguish the circulation from the product vr since the two are not always interchangeable. The quantity Γ is used throughout the rest of the analysis for convenience since it is only needed to denote the constancy of the product vr on streamlines.

It can be shown that the three component momentum equations integrated along streamlines add to form the energy equations. Therefore only two are independent and thus the axial momentum equation need not be solved explicitly. In the quasi one-dimensional approach, the radial velocity components are neglected. This requires that the cross-sectional area of the nozzle vary slowly along the nozzle length. The same assumption is used in ordinary one-dimensional gas dynamics. This assumption is easily justified for nozzles with a well rounded throat contour since most of the pressure

variation and compressibility effects occur in the immediate vicinity and downstream of the throat.

The continuity equation can be satisfied by a function $F(r)$ where,

$$\rho w r = \frac{\partial F(r)}{\partial r} \quad (13)$$

Thus a nondimensional stream function can be defined,

$$\psi = \int_0^{\dot{m}(r)} \frac{dS}{\dot{m}_T} \quad (14)$$

and

$$\frac{\partial \psi}{\partial r} = 2\pi \rho w r / \dot{m}_T \quad (15)$$

where $\dot{m}_T$ is the total mass flow rate through the nozzle.

The neglect of the radial velocity causes the radial momentum equation to reduce to:

$$\frac{\partial P}{\partial r} = \rho \frac{r^2}{r^3} \quad (16)$$

The other equations remain the same and are summarized here for completeness:

Energy $h_o = h + \frac{w^2}{2} + \frac{u^2}{2} \quad (17)$

Entropy: $S = \text{constant along streamlines} \quad (18)$

State: $P = (1 + \kappa) \rho R T \quad (19)$

Mass Action:
$$\frac{\alpha^2}{1-\alpha} = \frac{p_d}{p} e^{-\theta_d/T} \quad (20)$$

Equations (15) through (20) represent a complete set of independent equations that must be solved simultaneously in order to describe the nozzle flow. It is useful at this point to introduce the radial component of the Crocco-Vazsoni equation explicitly. It is not an additional independent equation but rather a combination of the momentum equation and the combined first and second laws of thermodynamics. It is useful in that it relates the entropy gradient directly to the fluid rotation. This equation is:

Crocco-Vazsoni:
$$T \frac{\partial S}{\partial r} = \frac{\partial h_o}{\partial r} - \frac{U^2}{r} - \frac{\partial}{\partial r} \left(\frac{w^2 + U^2}{2} \right) \quad (21)$$

Thermodynamic Considerations

Solution of equations (11) - (16) requires the use of certain thermodynamic relationships for the enthalpy, entropy, and isentropic exponent in terms of the state variables, P, T, and α . The development of these relations is given in References (25) and (26). Therefore only the required results are summarized here.

Enthalpy :
$$h = (4 + \alpha)RT + \alpha R \theta_d \quad (22a)$$

Entropy:
$$S = 3R \ln \left(\frac{T}{\theta_d} \right) + R\alpha(1 - 2\ln \alpha) - R(1 - \alpha) \ln(1 - \alpha) - R(1 + \alpha) \ln \left(\frac{p}{p_d} \right) \quad (22b)$$

$$\text{Isentropic Exponent: } \gamma_e = 1 + \frac{\frac{2T}{\Theta_d} + \frac{2 + 3\alpha^2 - \alpha^3}{\alpha(1-\alpha^2)} \left(\frac{T}{\Theta_d}\right)^2}{1 + \frac{3(2-\alpha)}{\alpha(1-\alpha)} \left(\frac{T}{\Theta_d}\right)^2} \quad (22c)$$

In the case of frozen flow, the isentropic exponent is given by:

$$\gamma_f = \frac{4 + \alpha}{3} \quad (22d)$$

As mentioned earlier, the ideal dissociating gas is derived for a symmetrical diatomic gas in equilibrium. The equations hold for frozen flow if α is held constant and the mass action equation is omitted. It should be noted that the vibrational mode is arbitrarily assumed to be half excited and thus γ_e and γ_f reduce to 1.33 for $\alpha \rightarrow 0$ rather than 1.4 for an ordinary diatomic gas at low temperatures and thus will be somewhat in error for temperatures below which the vibrational energy is not appreciable for each gas.

Solution of Equations

The solution of equations (15) - (20) is carried out by using the fact that h_0 , s , and r remain constant on the streamlines. The influence of the initial conditions including the tangential velocity distribution is carried in these quantities so a downstream solution can be obtained by locating the streamlines. First the equations must be put into a suitable form.

The energy equation (17) can be rearranged to give:

$$w = \pm \left[2(h_0 - h) - \frac{r^2}{r^2} \right]^{1/2} \quad (23)$$

This is substituted into (15) to give:

$$\frac{\partial \psi}{\partial r} = \pm \frac{2\pi p r}{\dot{m}_T} \left[2(h_0 - h) - \frac{\Gamma^2}{r^2} \right]^{1/2} \quad (24)$$

Using the state equation (19), p can be eliminated from equations (16), (24), (20), and (21) to give:

$$\frac{\partial p}{\partial r} = \frac{p \Gamma^2}{(1+\alpha) R T r^3} \quad (25)$$

$$\frac{\partial \psi}{\partial r} = \pm \frac{2\pi p r}{(1+\alpha) R T \dot{m}_T} \left[2(h_0 - h) - \frac{\Gamma^2}{r^2} \right]^{1/2} \quad (26)$$

$$\frac{\alpha^2}{1-\alpha} = \frac{p_d (1+\alpha) R T}{p} e^{-\theta_d/T} \quad (27)$$

$$T \frac{\partial s}{\partial r} = (4+\alpha) R \frac{\partial T}{\partial r} + (T+\theta_d) R \frac{\partial \alpha}{\partial r} - \frac{\Gamma^2}{r^2} \quad (28)$$

where the expression (22a) for h has been substituted into the r - component of the Crocco-Vazsoni equation. The mass action equation is highly nonlinear and is put into differential form in order to expedite numerical solution. Equation (27) can be differentiated and rearranged to give:

$$\begin{aligned} \frac{2\alpha}{(1-\alpha^2)^2} \frac{\partial \alpha}{\partial r} &= -\frac{1}{p} \frac{p_d R T}{p} e^{-\theta_d/T} \frac{\partial p}{\partial r} + \\ &\frac{1}{T} \frac{p_d R T}{p} e^{-\theta_d/T} \frac{\partial T}{\partial r} + \\ &\frac{\theta_d}{T^2} \frac{p_d R T}{p} e^{-\theta_d/T} \frac{\partial T}{\partial r} \end{aligned} \quad (29)$$

The repeated term $\frac{P_d R T}{P} e^{-\theta_d/T}$ can be eliminated using equation (27) again to get:

$$\frac{\partial \alpha}{\partial r} = \frac{-\alpha}{2P} (1-\alpha^2) \frac{\partial P}{\partial r} + \frac{\alpha(1-\alpha^2)}{2T} \left(1 + \frac{\theta_d}{T}\right) \frac{\partial T}{\partial r} \quad (30)$$

If $\bar{r} = \frac{r}{r_c}$, where r_c = the initial chamber radius, the four differential equations (25), (26), (28), and (30) become:

$$\frac{dP}{d\bar{r}} = A_1 \quad (31)$$

$$\frac{d\psi}{d\bar{r}} = A_2 \quad (32)$$

$$\frac{d\alpha}{d\bar{r}} = -A_3 + A_4 \frac{dT}{d\bar{r}} \quad (33)$$

$$\left(\frac{4+\alpha}{T}\right) \frac{dT}{d\bar{r}} + \left(1 + \frac{\theta_d}{T}\right) \frac{d\alpha}{d\bar{r}} = \frac{1}{R} \frac{dS}{d\psi} A_2 + \frac{\Gamma^2}{RT r_c^2 \bar{r}^3} \quad (34)$$

The partial derivatives have been written as total derivatives since only derivatives with respect to $\bar{r}$ appear in the equations. The equations are therefore ordinary differential equations.

The coefficients A_1 , A_2 , A_3 and A_4 have been defined for algebraic convenience as follows:

$$A_1 = \frac{P \Gamma^2}{(1+\alpha) R T r_c^2 \bar{r}^3} \quad (35)$$

$$A_2 = \frac{2\pi P r_c^2 \bar{r}}{(1+\alpha)RT\dot{M}_T} \left[2(h_o - h) - \frac{\Gamma^2}{r_c^2 \bar{r}^2} \right]^{1/2} \quad (36)$$

$$A_3 = \frac{\alpha(1-\alpha^2)}{2P} A_1 \quad (37)$$

$$A_4 = \frac{\alpha(1-\alpha^2)}{2T} \left(1 + \frac{\Theta_d}{T} \right) \quad (38)$$

Substituting (33) into (34) yields:

$$\left[\frac{1+\alpha}{T} + \left(1 + \frac{\Theta_d}{T} \right) A_4 \right] \frac{dT}{d\bar{r}} = \frac{1}{R} \frac{ds}{d\psi} A_2 + A_3 \left(1 + \frac{\Theta_d}{T} \right) + \frac{\Gamma^2}{RT r_c^2 \bar{r}^3} \quad (39)$$

Equation (39) can be rearranged to give:

$$\frac{dT}{d\bar{r}} = \frac{\frac{A_2}{R} \frac{ds}{d\psi} + A_3 \left(1 + \frac{\Theta_d}{T} \right) + \frac{\Gamma^2}{RT r_c^2 \bar{r}^3}}{\frac{1+\alpha}{T} + \left(1 + \frac{\Theta_d}{T} \right) A_4} \quad (40)$$

The other three equations (31), (32) and (33) are,

$$\frac{dP}{d\bar{r}} = A_1 \quad (31)$$

$$\frac{d\psi}{d\bar{r}} = A_2 \quad (32)$$

$$\frac{d\alpha}{d\bar{r}} = -A_3 + A_4 \frac{dT}{d\bar{r}} \quad (33)$$

The equations (31), (32), (33) and (40) are then the four differential equations that must be solved for the variables T , P , α , and ψ as

functions of r . All the other important flow parameters can be deduced from these four.

These equations have been derived for the case of flow in thermodynamic equilibrium. For the case of frozen flow, the dissociation fraction α is assumed to remain constant at some initial value. The development of the analysis is identical to the equilibrium case up to equations (31), (32), (33) and (34) except that the mass action equation is omitted and thus equation (33) is omitted. In equation (34) the term containing $\frac{d\alpha}{dF}$ is zero since α is assumed constant. Therefore, the frozen analog to equations (31), (32), (33) and (34) is:

$$\frac{dP}{dF} = A_1 \quad (41)$$

$$\frac{d\psi}{dF} = A_2 \quad (42)$$

$$\left(\frac{1+\alpha}{T}\right) \frac{dT}{dF} = \frac{1}{R} \frac{dS}{d\psi} A_2 + \frac{\Gamma^2}{RT r_c^2 F^3} \quad (43)$$

These three equations are solved for P , ψ and T since α is a specified constant throughout the flowfield.

The sets of equations (31)-(33), (40) and (41)-(43) are both sets of first-order, ordinary, linear differential equations. They are amenable to solution by standard numerical methods if boundary conditions are given for P , T , α and ψ and the functions $h_o(\psi)$, $\Gamma(\psi)$, and $\frac{dS}{d\psi}(\psi)$ are known. The development of these functions is the subject of the next section.

B. Initial Conditions

In order to compute the solution to equations (40)-(43), h_0 , Γ , and $\bar{B}$ must be known as functions of ψ . The required initial distributions of these quantities can be derived once the initial tangential velocity distribution is given. The most commonly used tangential velocity distributions are v proportional to r (solid body) and v proportional to $1/r$ (free vortex). Though a real vortex can be expected to deviate somewhat from these distributions due to viscous effects, they are nonetheless useful for analytical work and should approach real vortex velocity distributions fairly closely in many cases.

Solid Body Rotation

The solid-body or rotational vortex arises naturally in the case of the end-burning rocket motor. The rotational vortex is characterized by a constant angular velocity and the tangential velocity proportional to r :

$$v = \omega r \quad (44)$$

Therefore Γ is given by,

$$\Gamma = r^2 \omega \quad (45)$$

The radial pressure gradient is given by,

$$\frac{dP}{dr} = \frac{\rho \Gamma^2}{r^3} \quad (46)$$

The treatment of the thermodynamic variables depends on the assumed source of the gas. Some authors have assumed various relations

between the pressure, temperature, and mass rate based on burning rate models. Others have chosen more general conditions that do not depend on particular burning rate models. In the interest of generality, the latter course is chosen here and thus the results are not restricted to solid fuel rocket motors entirely.

If the gas is assumed to be produced by a uniform source, e.g., burning at some flame temperature T_f , the stagnation enthalpy can be reasonably assumed constant except for the contribution of the angular velocity. The initial axial velocity w is also assumed constant over the area of the chamber. The stagnation enthalpy is therefore given by:

$$h_o = h_{c1} + \frac{v^2}{2} = h_{c1} + \frac{r^2 \omega^2}{2} \quad (47)$$

In order to evaluate ψ as a function of r , the density and thus the pressure and temperature distribution must be known across the initial plane. Equation (47) can be used to derive the temperature distribution as follows. Using equations (17) and (22a), h_o is given by:

$$h_o = R(4 + \alpha)T + R\alpha\theta_d + \frac{w^2 + v^2}{2} \quad (48)$$

Substituting this into equation (47) and differentiating yields:

$$(4 + \alpha)R \frac{dT}{dr} + R(T + \theta_d) \frac{d\alpha}{dr} + \frac{d}{dr} \left(\frac{v^2 + w^2}{2} \right) = \frac{d}{dr} \left(\frac{v^2}{2} \right) \quad (49)$$

Remembering that w is uniform over the initial plane, this reduces to,

$$(4 + \alpha)R \frac{dT}{dr} + R(T + \theta_d) \frac{d\alpha}{dr} = 0 \quad (50)$$

An expression for $\frac{d\alpha}{d\bar{r}}$ has been derived previously from the mass action equation. This is given by equation (33),

$$\frac{d\alpha}{d\bar{r}} = -A_3 + A_4 \frac{dT}{d\bar{r}} \quad (51)$$

where A_3 and A_4 are given by equations (37) and (38).

Substituting (51) into (50) and using the nondimensional radius $\bar{r}$ in equation (46) gives the following three equations specifying the distribution of T , α , and P in terms of $\bar{r}$:

$$\frac{dT}{d\bar{r}} = \frac{A_3 R (T + \theta_d)}{R(4 + \alpha) + A_4 (T + \theta_d) R} \quad (52)$$

$$\frac{d\alpha}{d\bar{r}} = A_4 \frac{dT}{d\bar{r}} - A_3 \quad (53)$$

$$\frac{dP}{d\bar{r}} = \frac{\rho r^2}{(1 + \alpha) R T r_c^2 \bar{r}^3} \quad (54)$$

$\psi(r)$ is then obtained from the continuity equation:

$$\frac{d\dot{m}}{d\bar{r}} = 2\pi \rho \omega r_c^2 \bar{r} \quad (55)$$

$$\psi = \frac{\dot{m}}{\dot{m}_T} \quad (56)$$

Thus if w, ω, P_{cl} , and T_{cl} are specified, then P, T, α, h_0 , and ψ can all be obtained as functions of r using equations (48), (52-56), and the equation of state. The entropy gradient is determined directly from the r -component of the Crocco-Vazsoni equation (21):

$$T \frac{ds}{dr} = \frac{dh_0}{dr} - \frac{v^2}{r} - v \frac{dv}{dr} \quad (57)$$

$$= -r\omega^2 = -r_c \bar{r} \omega^2 \quad (58)$$

Once all the state variables and ψ are known as functions of r , the results can be easily correlated to give these variables as functions of ψ . In the frozen case, α is a constant and thus by equation (50), T is a constant with respect to r . The other expressions (54) - (57) for P , ψ and the entropy gradient are identical.

Free Vortex

The free vortex is characterized by a distribution of velocity that is inversely proportional to r . A true free vortex should therefore have an infinite velocity and velocity gradient at the centerline. However, viscous effects become large close to the centerline and break down the free vortex. Therefore close to the axis, the velocity profile takes on more the characteristics of solid body rotation. A useful approximation to the real free vortex is then a true irrotational vortex with a solid body core:

$$\left. \begin{aligned} v &= r\omega & 0 \leq r \leq r_v \\ v &= \frac{r_v^2 \omega}{r} & r_v \leq r \leq r_c \end{aligned} \right\} \quad (59)$$

Where r_v is the core radius and r_c is the chamber radius. The tangential velocity, enthalpy, pressure, and temperature are matched at r_v though their gradients are of course discontinuous. The magnitude of r_v must also be chosen as will be described later.

Since the equations for the solid-body core have been derived previously, only the free vortex portion is considered here. The free vortex can be assumed homenergetic and therefore isentropic since all the gas is assumed to enter the chamber with a uniform tangential velocity such as in a solid fuel rocket motor with an internal burning grain. Such a vortex could also be created by injecting a stream of gas tangentially into the chamber of a vortex tube or arc-jet. The temperature distribution in the initial plane is derived by differentiating the enthalpy equation (48):

$$\frac{d}{dr} (h_o) = R(1+\alpha) \frac{dT}{dr} + R(T+\theta_d) \frac{d\alpha}{dr} + \frac{d}{dr} \left(\frac{v^2 + w^2}{2} \right) = 0 \quad (60)$$

$$R(1+\alpha) \frac{dT}{dr} + R(T+\theta_d) \frac{d\alpha}{dr} = \frac{\Gamma^2}{r^3} \quad (61)$$

where w has again been assumed to be uniform over the initial plane.

Using equation (33) for $\frac{d\alpha}{d\bar{r}}$ and introducing $\bar{r}$ gives:

$$\frac{dT}{d\bar{r}} = \frac{A_3 R(T+\theta_d) + \frac{\Gamma^2}{r_c^2 \bar{r}^3}}{R(1+\alpha) + A_4 (T+\theta_d) R} \quad (62)$$

$$\frac{d\alpha}{d\bar{r}} = A_4 \frac{dT}{d\bar{r}} - A_3 \quad (63)$$

where equation (63) is identical to (33).

The expressions for the pressure gradient and stream function ψ are identical to the solid-body case:

$$\frac{dP}{d\bar{r}} = \frac{P \Gamma^2}{(1+\alpha) R T r_c^2 \bar{r}^3} \quad (64)$$

$$\frac{d\dot{m}}{d\bar{r}} = 2\pi\rho w r_c \bar{r} \quad (65)$$

$$\psi = \frac{\dot{m}}{\dot{m}_T} \quad (66)$$

For the irrotational vortex the entropy gradient is zero and h_0 and $\bar{r}$ are constants with respect to r .

The choice of r_v is the most troublesome aspect of the free vortex problem. Since the creation of a solid-body core depends on viscous phenomena, some estimate of the viscous forces must be made. Deissler and Perlmutter⁽²⁹⁾ performed a simplified analysis of vortex flow in a Ranque-Hilsch vortex tube for both laminar and turbulent flow. Their results are correlated in terms of a Reynolds number based on the initial radial velocity. For turbulent flow, the ratio of the radial to tangential velocity at a reference radius is also important. Their results matched fairly well with limited experimental results when turbulence was included in the model.

Application of their analysis to spinning solid-fuel rockets yields the result that the vortex is basically of solid-body character even for internal burning grains due to the small ratio of radial to tangential velocity. They used a highly simplified model for the eddy diffusivity however. Ragsdale⁽³⁰⁾ has more recently published a correlation for the eddy diffusivity based on experiments with an actual vortex tube. He found the value of χ to be about one tenth of that used by Deissler and Perlmutter and thus predicted eddy diffusivities approximately one hundredth of those used in Ref. (29).

Using Ragsdale's value for the eddy diffusivity still yields a vortex with a large degree of solid-body character. The fact that many internal burning grains actually have a star or slotted configuration further impedes the formation of a free vortex. Finally, the effect of the additional turbulence created by the actual combustion process is as yet uninvestigated. In view of the above arguments only solid-body rotations were considered in the computations for cases typical of rocket motors.

In the case of arc-jets however, the radial velocity will be quite a bit larger since injection usually takes place over a small area of the chamber. This larger radial velocity should create a velocity distribution that more closely resembles a free vortex. Massier⁽¹⁸⁾ measured the velocity profile in an arc-jet at room temperature and found r_v to be slightly smaller than $1/10 r_c$. Therefore $r_v = 1/10 r_c$ has been chosen here for the computations of typical arc-jet problems.

C. Solution Procedure

In order to solve the sets of equations derived above, numerical techniques must be employed. The initial conditions are specified by integrating equations (52) - (56) and (62) - (66) using a fourth-order Runge-Kutta integration scheme. A centerline pressure and temperature are specified and the solution proceeds outward until the given chamber radius is reached. In the case of solid-body rotation the entropy gradient is also computed using equation (58).

The downstream solution plane is determined by specifying the

centerline pressure. The centerline temperature and dissociation fraction are then found using the entropy equation (22b) and the law of mass action (20). The non-linear nature of these equations necessitates the use of a Newton-Raphson iteration procedure. The solution of the differential equations is computed using an Adams-Moulton predictor-corrector method with variable step size. This method was chosen because it is possible to estimate the error and adjust the step size far more easily than with the Runge-Kutta method. Some savings in computing time is realized also. A Runge-Kutta routine was used to start the procedure and to change the step size. Since the initial conditions $\Gamma(\psi)$, $h_o(\psi)$, and $\frac{dS}{d\psi}(\psi)$ were in tabular form, the required values were found at each point in the solution plane using an interpolation based on the current value of ψ . The quantities Γ , h_o , and ψ are all either quadratic functions of r in the initial plane or constants. Therefore Γ and h_o are very nearly linear functions of ψ . The entropy gradient is nearly constant with r . Therefore a two-point linear interpolation proved sufficiently accurate and saved considerable computing time over more elaborate schemes.

The solution proceeds outward from the centerline until the streamline $\psi = 1$ is reached. This is then the nozzle wall and the value of r at this point is the nozzle radius r_n . It is not possible to predict the centerline pressure, P_{cl} , at the throat a priori. However $\frac{dr_n}{dP_{cl}}$ must be zero at the throat since this is the minimum radius. Thus the throat can be found as follows. A series of 6-8 solutions are carried out for various P_{cl} 's thought to be near the

throat, thus giving values of r_n for corresponding P_{cl} 's. The derivatives dr_n/dP_{cl} are then found at intermediate points using central differences. One then has dr_n/dP_{cl} as a tabular function of P_{cl} . If one could then pass a polynomial through the points dr_n/dP_{cl} , then the value of P_{cl} corresponding to $dr_n/dP_{cl} = 0$ could be found. This can all be accomplished using an iterated inverse interpolation scheme similar to Aitken's interpolation routine. This method becomes progressively more accurate as the number of points increases and an estimate of the error is available by evaluating the change in the value of the throat P_{cl} predicted by adding one more point. This method was found to be very accurate with only 7 or 8 points.

It should be noted that the initial choice of P_{cl} 's for the downstream solution is entirely arbitrary and thus sometimes more than one set must be run to get in the neighborhood of the throat. This problem is especially difficult for the free vortex cases when the centerline pressure at the throat is often extremely low. Another difficulty is that some centerline pressures will not yield a solution to the problem at all. As pointed out by Norton, this is probably due to the existence of a recirculation bubble in the flow in which case the whole analysis breaks down. This problem seems to occur only in the low subsonic part of the nozzle. For sufficiently low P_{cl} 's, solutions could always be found that led to a throat. The whole numerical procedure has been coded in Fortran IV and solutions computed on a Burroughs B-5500 computer.

D. Boundary Layer

As pointed out earlier, the boundary layer problem for swirling flows is not well developed. The boundary layer equations are naturally three-dimensional in this case and do not yield readily to analysis. The addition of reaction effects to the problem precludes almost any approach except a direct numerical integration of the full set of partial differential equations. This is a formidable problem even for the laminar case. The turbulent problem is more difficult yet due to the lack of information on the eddy diffusivity for reacting swirling flows. These problems are beyond the scope of the present investigation.

A rough estimate of the magnitude of the boundary layer effects can be made however if certain approximations are made. The problem is greatly simplified if the streamwise derivatives are neglected in the governing equations. The validity of the locally similar solutions obtained has been reasonably well substantiated by Back⁽²³⁾ and Marvin and Sinclair⁽³¹⁾ for two-dimensional boundary layers including pressure gradients such as nozzle flow and stagnation point flow. Their results for such parameters as the heat transfer rate, skin friction, and momentum thickness have been found to be quite accurate as compared to experiment and numerical solution of the corresponding nonsimilar problem. The values of the displacement thickness are found to be reasonably good for the values of the velocity gradient parameter β less than 4 or 5 where β is defined as $d \ln w_e / d \ln \xi$. This condition can be met by nozzles with slowly varying cross-sectional areas and well-rounded throats. This is the same as required for the

quasi one-dimensional inviscid solution as described earlier.

In order to apply the local similarity concept, the equations must be two-dimensional. This is most easily accomplished by neglecting the tangential velocity component. Thus the equations are identical to those for the non-swirling flow problem and are therefore invalid strictly speaking for swirling flows. The most critical boundary layer parameter as far as the effect on the freestream is concerned is the displacement thickness at the throat since the presence of the boundary layer produces an effective area reduction for the swirling freestream flow. For many cases the swirl angle $\phi = \arctan (v/w)$ is less than 45° and thus is a predominantly axial flow. Therefore an order-of-magnitude estimate of the boundary layer thickness can be made using the two-dimensional locally similar approach. This approach can not be expected to give very accurate results upstream where the angle of the velocity vector at the wall can be greater than 80° .

Finally, laminar flow is assumed due to the lack of information on the nature of the turbulence in reacting swirling flows. It has been shown by Schoenman and Block⁽³²⁾ that laminar boundary layer calculations often give more accurate results than turbulent ones for non-swirling flow in small rocket nozzles. This is probably due to a relaminarization phenomenon caused by the large favorable pressure gradient present in such nozzles. The extrapolation of this result to swirling flows must be done with the realization that the results can only be tentative until further experimentation in swirling flows can be undertaken.

In view of these assumptions the boundary layer problem can be treated in the same manner as Fay and Riddell⁽³³⁾ and Rose, Kemp, and Detra⁽³⁴⁾ did for stagnation point flow and thus reaction effects can be retained in the analysis. The following analysis is an adaptation of Rose, Kemp, and Detra's analysis using the ideal dissociating gas. The governing equations are:

$$\text{Continuity: } (\rho w r)_x + (\rho u r)_y = 0 \quad (70)$$

$$\text{Momentum: } \rho w w_x + \rho u w_y = -P_x + (\mu w_y)_y \quad (71)$$

$$\text{Species: } \rho w c_{i,x} + \rho u c_{i,y} = \{D_i \rho c_{i,y} + D_i^T \rho c_i T_y / T\}_y + w_i \quad (72)$$

$$\begin{aligned} \text{Energy: } \rho w h_{o,x} + \rho u h_{o,y} = & \{(k/\bar{C}_p) h_{o,y}\}_y + \left\{\frac{1}{2}(\mu - \frac{k}{\bar{C}_p})(w^2)_y\right\}_y + \left\{\sum_{i=1}^n (D_i \rho - \frac{k}{\bar{C}_p})(h_i - h_i^0) c_{i,y} + \sum_{i=1}^n D_i^T \rho c_i (h_i - h_i^0) \frac{T_y}{T}\right\}_y \end{aligned} \quad (73)$$

where the subscripts x, y denote partial differentiation with respect to x, the coordinate along the wall, and y, the coordinate normal to the wall respectively.

$$h = \sum_{i=1}^n c_i (h_i - h_i^0) \quad (74)$$

where h_i^0 = heat of formation of species i at 0°K per unit mass.

$$h_o = h + \frac{w^2}{2} \quad (75)$$

$$\bar{C}_p = \sum_{i=1}^n c_i (dh_i / dT) \quad (76)$$

If equilibrium flow is assumed, one can omit the species equation entirely. If thermal diffusion is also neglected, the set of equations reduces to:

$$\text{Cont.: } (\rho w r)_x + (\rho u r)_y = 0 \quad (77)$$

$$\text{Mom.: } \rho w w_x + \rho u w_y = -P_x + (\mu w)_y \quad (78)$$

$$\begin{aligned} \text{Energy: } \rho w h_{0x} + \rho u h_{0y} = & \left\{ (k/\bar{c}_p) h_{0y} \right\}_y \\ & + \left\{ \frac{1}{2} \left(\mu - \frac{k}{\bar{c}_p} \right) (w^2)_y \right\}_y \\ & + \left\{ \sum_{i=1}^n \left(D_i \rho - \frac{k}{\bar{c}_p} \right) (h_i - h_i^0) c_{iy} \right\} \end{aligned} \quad (79)$$

At this point the Howarth and Mangler transformations are introduced:

$$\xi(x) = \int_0^x \rho_e \mu_e w_e r^2 dx, \quad \frac{d\xi}{dx} = \rho_e \mu_e w_e r^2 \quad (80)$$

$$\eta(x, y) = \frac{w_e r}{\sqrt{2\xi}} \int_0^y \rho dy, \quad \frac{d\eta}{dx} = \frac{\rho w_e r}{\sqrt{2\xi}} \quad (81)$$

In addition a stream function is defined as follows:

$$\frac{w}{w_e} = f_2 \quad f = \int_0^\eta f_2 d\eta \quad (82)$$

The continuity equation then becomes:

$$\rho u = -\frac{1}{r} \left[\xi_x (\sqrt{2\xi} f)_\xi + \eta_x \sqrt{2\xi} f_2 \right] \quad (83)$$

The following non-dimensional variables are also defined:

$$g = h_o / h_{oe} \quad (84)$$

$$S_i = c_i / c_{ie} \quad (85)$$

$$N_{Pr} = \frac{\bar{c}_p \mu}{k} = \text{the Prandtl number} \quad (86)$$

$$N_{Le} = \frac{D_i \rho \bar{c}_p}{k} = \text{the Lewis number} \quad (87)$$

$$l = \frac{\rho \mu}{\rho_e \mu_e} \quad (88)$$

$$\beta = \frac{d \ln w_e}{d \ln \xi} = \frac{\xi}{w_e} \frac{d w_e}{d \xi} \quad (89)$$

Application of the transformations and use of the above non-dimensional variables yields the following forms of the governing equations:

Mom: $(l f_{zz})_z + f f_{zz} + 2\beta [(p_e / p) = \frac{\xi^2}{f_z}] \quad (90)$

$$= 2\xi (f_z f_{z\xi} - f_\xi f_{zz})$$

Energy: $(l g_z / N_{Pr})_z + f g_z + \left\{ (l / N_{Pr}) \sum_{i=1}^n c_{ie} \times \right. \quad (91)$

$$\left. [(h_i - h_i^o) / h_{oe}] [(N_{Le} - 1) S_{iz}] \right\}_z$$

$$+ (w_e^2 / h_{oe}) \times \left\{ (1 - \frac{1}{N_{Pr}}) l f_z f_{zz} \right\}_z$$

$$= 2\xi (f_z g_\xi - f_\xi g_z)$$

The boundary conditions for this problem are:

$$\left. \begin{aligned} @ y=0 : w=0, h_0=h_{0w}, C_i=C_{iw} \\ @ y \rightarrow \infty : w \rightarrow w_e, h_0 \rightarrow h_{0e}, C_i \rightarrow C_{ie} \end{aligned} \right\} \quad (92)$$

The transformed boundary conditions are:

$$\left. \begin{aligned} @ \eta=0 : f=f'=0, g=g_w, S=S_w \\ @ \eta \rightarrow \infty : f' \rightarrow 1, g \rightarrow 1, S \rightarrow S_e \end{aligned} \right\} \quad (93)$$

The assumption of local similarity means that all terms on the right hand side of the two equations are neglected as their influence is assumed small. The equations then take the form:

$$\text{Mom: } (\ell f_{\eta\eta})_{\eta} + f f_{\eta\eta} + 2\beta [(p_e/p) - f_{\eta}^2] = 0 \quad (94)$$

$$\text{Energy: } (\ell g_{\eta} / N_{Pr})_{\eta} + f g_{\eta} + \left\{ (\ell / N_{Pr}) \sum_{i=1}^n C_{ie} \times \right. \quad (95)$$

$$\left. \begin{aligned} & [(h_i - h_i^0) / h_{0e}] [(N_{Le} - 1) S_{i\eta}] \} \}_{\eta} \\ & + (w_e^2 / h_{0e}) \times \left\{ \left(1 - \frac{1}{N_{Pr}}\right) \ell f_{\eta} f_{\eta\eta} \right\}_{\eta} = 0 \end{aligned}$$

Transport & Thermodynamic Properties

The viscosity is calculated from the Chapman- Enskog relation⁽³⁵⁾:

$$\mu = 2.6693 \times 10^{-5} \frac{\sqrt{M T}}{\sigma^2 \Omega_{\mu}} \quad (96)$$

where M is the molecular weight, T is the temperature in $^{\circ}\text{K}$, σ is the collision diameter of the molecule, Ω_{μ} is the collision integral, and μ is the viscosity in gm/cm-sec.

This data was fitted to a Sutherland law of the form:

$$\frac{\mu}{\mu_w} = \frac{T_w + C_1}{T + C_1} \left(\frac{T}{T_w} \right)^{1.5} \quad (97)$$

where the subscript w denotes the wall value of the parameters μ and T. Lighthill has noted that the Sutherland law predicts the viscosity within 10% for gases where there is appreciable dissociation. Therefore the above relation was deemed adequate for this investigation.

The Prandtl number was taken as 0.70 since the ideal dissociating gas approximation predicts a variation of only 0.71 to 0.67 for dissociation fractions from 0 to 1.0. The Lewis number was taken to be 1.0 for convenience since it can be shown to vary from 1.4 for $\alpha = 0$ to about 0.7 for $\alpha = 1.0$.

The state equation, enthalpy, and dissociation fraction are again given by the ideal dissociating gas relations:

State: $P = (1 + \alpha) \rho R T$ (98)

Enthalpy: $h = R [(4 + \alpha) T + \alpha \theta_d]$ (99)

Mass Action: $\frac{\alpha^2}{1 - \alpha} = \frac{p_d}{p} e^{-\theta_d/T}$ (100)

The density can be eliminated from the mass action equation using the state equation to give:

$$\frac{\alpha^2}{1 - \alpha^2} = \frac{p_d R T}{P} e^{-\theta_d/T} \quad (101)$$

Since values of α and T are needed at each step of the calculation and because of the nonlinearity of the mass action equation, it is convenient to put this equation into the form of a differential equation in terms of the transformed variables f and g .

The mass action equation (101) can be differentiated with respect to ζ holding P constant to give:

$$\frac{2\alpha}{(1-\alpha^2)^2} \frac{d\alpha}{d\zeta} = \frac{P_d R T}{P} e^{-\theta_d/T} \frac{1}{T} \frac{dT}{d\zeta} + \frac{P_d R T}{P} \frac{\theta_d}{T^2} e^{-\theta_d/T} \frac{dT}{d\zeta} \quad (102)$$

substituting using (101) again:

$$\frac{2\alpha}{(1-\alpha^2)^2} \frac{d\alpha}{d\zeta} = \frac{1}{T} \frac{\alpha^2}{1-\alpha^2} \frac{dT}{d\zeta} + \frac{\alpha^2}{1-\alpha^2} \frac{\theta_d}{T^2} \frac{dT}{d\zeta} \quad (103)$$

Equation (103) can be rearranged to give:

$$\frac{2}{1-\alpha^2} \frac{d\alpha}{d\zeta} = \frac{\alpha}{T} \left(1 + \frac{\theta_d}{T}\right) \frac{dT}{d\zeta} \quad (104)$$

The total enthalpy is given by :

$$h_o = h + \frac{w^2}{2} = R(4+\alpha)T + R\alpha\theta_d + \frac{w^2}{2} \quad (105)$$

Differentiating gives:

$$\frac{dh_o}{d\zeta} = R(4+\alpha) \frac{dT}{d\zeta} + R(T+\theta_d) \frac{d\alpha}{d\zeta} + w \frac{dw}{d\zeta} \quad (106)$$

This can be rearranged using the transformed variables to give:

$$R(4+\alpha) \frac{dT}{d\zeta} = h_{oe} g_{\zeta} - w_e^2 f_{\zeta} f_{\zeta\zeta} - R(T+\theta_d) \frac{d\alpha}{d\zeta} \quad (107)$$

Substituting for $\frac{d\alpha}{dk}$ using (104) gives:

$$R(4+\alpha)\frac{dT}{dk} = h_{oe} g_k - \omega_e^2 f_k f_{kk} - \frac{R(T+\theta_d)\alpha(1-\alpha^2)}{2T} \left(1 + \frac{\theta_d}{T}\right) \frac{dT}{dk} \quad (108)$$

This can be rearranged to give:

$$\frac{dT}{dk} = \frac{h_{oe} g_k - \omega_e^2 f_k f_{kk}}{R \left[(4+\alpha) + \frac{\alpha(1-\alpha^2)}{2} \left(1 + \frac{\theta_d}{T}\right)^2 \right]} \quad (109)$$

From Equation (105),

$$\alpha = \frac{h_{oe} g - \frac{1}{2} \omega_e^2 f_k^2 - 4RT}{RT + R\theta_d} \quad (110)$$

Solution Procedure

Equations (94), (95), and (109) must be integrated numerically.

In order to do this they must be broken down into a set of first order differential equations. The following set of variables is defined:

$$\begin{aligned} f_1 &= f \\ f_2 &= f_k \\ f_3 &= k f_{kk} \\ f_4 &= g \\ f_5 &= k g_k \end{aligned} \quad (111)$$

Substituting these into (94), (95) and (101) and using these definitions gives the following set of equations:

$$\frac{df_1}{dk} = f_2 \quad (112)$$

$$\frac{df_2}{dk} = \frac{f_3}{l} \quad (113)$$

$$\frac{df_3}{dk} + \frac{f_1 f_3}{l} + 2\beta \left[(p_e/p) - f_2^2 \right] = 0 \quad (114)$$

$$\frac{df_4}{dk} = \frac{f_5}{l} \quad (115)$$

$$\frac{1}{N_{pr}} \frac{df_5}{dk} + \frac{f_1 f_5}{l} + \frac{\omega_e^2}{h_{oe}} \left(1 - \frac{1}{N_{pr}} \right) \left(\frac{f_3^2}{l} + f_2 \frac{df_3}{dk} \right) = 0 \quad (116)$$

$$\frac{dT}{dk} = \frac{\frac{h_{oe} f_5}{l} - \frac{\omega_e^2 f_2 f_3}{l}}{R \left[(4 + \alpha) + \frac{\alpha(1 - \alpha^2)}{2} \left(1 + \frac{\theta_d}{T} \right)^2 \right]} \quad (117)$$

Note that the assumption of $N_{1e} = 1$ causes the diffusion term to drop from the energy equation. The set of equations (112)-(117) is a set of 6 first order differential equations. These are then integrated using a fourth-order Runge-Kutta technique. The initial conditions required consist of:

$$\begin{aligned} f_{1w} &= f_w = 0 & f_{4w} &= g_w = h_{ow}/h_{oe} \\ f_{2w} &= f_{kw} = 0 & T_w &= \text{given} \end{aligned} \quad (118)$$

Values of f_{3w} and f_{5w} must be guessed as f_{kw} and g_{kw} are not known a priori. At $k = 5.0$, $f_2 = f_k$ and $f_4 = g$ should approach 1.0. The deviation from 1.0 is used to correct the values of f_{3w} and f_{5w} iteratively.

The displacement thickness is defined as:

$$\delta^* = \int_0^{\infty} \left(1 - \frac{\rho w}{\rho_e w_e}\right) dy \quad (119)$$

This is in terms of the transformed variables,

$$\delta^{*'} = \int_0^{h_e} \left(1 - \frac{\rho w}{\rho_e w_e}\right) \frac{\sqrt{2\xi}}{\rho_e w_e r} dz \quad (120)$$

IV RESULTS

A number of numerical solutions to the reacting swirling flow problem have been carried out. Since only symmetrical diatomic gases could be considered, oxygen and fluorine were chosen to approximate real reacting systems. Fluorine was chosen since it dissociates over the temperature range of $3000^{\circ} - 6000^{\circ}\text{R}$ at pressures of several hundred psia. These are typical combustion chamber conditions for many solid-fuel rocket motors. Furthermore, its dissociation characteristics and molecular weight closely approximate those of such combustion products as CO_2 , H_2O and NO . Oxygen was used to extend the results to higher temperatures including the arc-jet cases since oxygen dissociates in the range $6000^{\circ} - 9000^{\circ}\text{R}$ for low pressures of 10 - 20 psia and is appreciably dissociated at the higher pressures.

Solid-Body Rotation

The first series of calculations was performed using the rotational vortex as the initial condition. These calculations have been made for several stagnation pressures and temperatures. The results show only a slight variation (less than 3%) in P_0 and T_0 across the initial plane in the combustion chamber for even the highest angular velocities. Therefore the values of P_0 and T_0 used to label and discuss the figures are the centerline P_0 and T_0 and these are identical for each curve in any one figure. Suitable average values of P_0 and T_0 were used in plotting the curves themselves however.

The results are presented using the parameter $\frac{\dot{m} \sqrt{T_0}}{A^* P_0}$. This is the mass flow rate parameter which is a constant for any given gas in

nonreacting, nonswirling, perfect gas theory. This parameter can not be derived directly from the governing equations for reacting swirling flow since both A^* and $\dot{m}$ depend on the angular velocity and reaction effects for a specific problem. It still represents a convenient combination of parameters however and is useful in comparing results to nonswirling perfect gas flow.

In each figure, the mass flow rate parameter is plotted against ω , the angular velocity of the gas in the chamber. If a solid-fuel rocket is considered, ω would be the angular velocity of the motor casing including the solid-fuel charge. The peak tangential velocity in the chamber is simply $r_c \omega$. The chamber radius r_c was taken to be 3 in. for every computation as this is a typical dimension for small motors and arc-jets.

The throat area determines the contraction ratio C.R.,

$$C.R. = \left(\frac{r_c}{r_t} \right)^2 \quad (121)$$

where r_c and r_t are the chamber radius and throat radius respectively. The throat area is controlled by varying the initial mass rate since r_t is determined as part of the solution. This problem can be treated in two different ways. One way is to determine the throat area required to pass a certain mass flow rate. This corresponds to the design problem where it is desired to minimize the effects of spinning the motor on the thrust and mass flow rate. Here $\dot{m}$ is constant while C.R. varies with ω . The second possibility is the performance problem where the effect of the spin on the mass flow rate is sought for a particular nozzle design. In this case $\dot{m}$ is varied to keep

C.R. constant with ω . Both problems are treated here to illustrate the differences in the two results.

The first eight figures illustrate the design problem. In each case, a mass flow rate was chosen to give a C.R. of 78, again a typical number, for equilibrium flow with $\omega = 0$. In subsequent calculations at increasing values of ω , the throat area required to pass this mass flow rate was determined. The same mass flow rate was used for both the equilibrium and frozen flow cases. Figures 1-4 show the effect of temperature on swirling flows for the design problem with values of ω up to 2000 rad./sec. (approximately 20,000 rpm). The higher temperatures are observed to decrease the effect of the swirl on the flow, a trend observed elsewhere for perfect gases. The effect of temperature is exaggerated in reacting flow since higher temperatures produce larger dissociation fractions, thus decreasing the effective molecular weight. Both increasing temperature and decreasing molecular weight increase the axial velocity of the gas at the throat.

The curve corresponding to the frozen flow cases lies above the equilibrium curve in all the figures. This is due to the smaller A^* required to pass a given mass flow rate in frozen flow than is required in equilibrium flow. That this should be so can be observed by examining the mass flow rate parameter as a function of γ for perfect gases:

$$\frac{\dot{m} \sqrt{T_0}}{A^* P_0} = \sqrt{\frac{\gamma}{R} \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{\gamma-1}}} \quad (122)$$

The ratio of specific heats, γ , is the controlling factor in this formula and the quantity under the square root sign increases for

Fig. 1 DESIGN PROBLEM -- SOLID BODY ROTATION

Fig. 2 DESIGN PROBLEM -- SOLID BODY ROTATION

Fig. 3 DESIGN PROBLEM -- SOLID BODY ROTATION

Fig. 4 DESIGN PROBLEM -- SOLID BODY ROTATION

increasing γ . Since the frozen γ_f is always larger than the equilibrium γ_e , the mass flow rate parameter will always be larger for frozen flow than for equilibrium flow at the same stagnation conditions.

In swirling reacting flow, the throat pressure and temperature for equilibrium flow are always higher than for frozen flow. The energy released by recombination seems to go into increasing the temperature of the gas in the equilibrium case rather than increasing the velocity. Thus recombination effects do not appear to reduce greatly the effects of the swirl. The temperature variation across the throat plane is much smaller for equilibrium flow than for frozen flow however. Some typical profiles are plotted in figures 9 thru 12.

Figures 5 and 6 display the effects of stagnation pressure on the results. The two curves are reasonably close to the curves for 300 psia. This is due to the fact that α and γ_e are relatively weak functions of pressure. Figures 7 and 8 extend the results to higher temperatures using oxygen. A considerable difference is noted between the results for fluorine at 6000°R and oxygen at 6000°R. Fluorine is essentially completely dissociated into the monatomic species with a molecular weight of 19 and $\gamma_e = 1.297$. Oxygen on the other hand is only slightly dissociated with a molecular weight close to 32 and $\gamma_e = 1.191$. The frozen flow results follow as before.

For each case described in figures 1-8, a calculation using perfect gas assumptions was made. The properties of the gas were determined by the equilibrium flow calculations in the initial chamber as follows:

Fig. 5 DESIGN PROBLEM -- SOLID BODY ROTATION

Fig. 6 DESIGN PROBLEM -- SOLID BODY ROTATION

Fig. 7 DESIGN PROBLEM -- SOLID BODY ROTATION

Fig. 8 DESIGN PROBLEM -- SOLID BODY ROTATION

Fig. 9 PRESSURE DISTRIBUTION AT THROAT

Fig. 10 TEMPERATURE DISTRIBUTION AT THROAT

Fig. 11 AXIAL VELOCITY DISTRIBUTION AT THROAT

Fig. 12 DISSOCIATION FRACTION

1. The dissociation fraction was assumed frozen at the equilibrium value for the particular P_0 and T_0 being considered.
2. A new gas constant $\bar{R}$ was defined as

$$\bar{R} = (1 + \alpha)R \quad (123)$$

3. The ratio of specific heats was taken as the equilibrium value of γ_e in the chamber.
4. The enthalpy was given by:

$$h = C_p T \quad (124)$$

where C_p is

$$C_p = \bar{R} \frac{\gamma}{\gamma - 1} \quad (125)$$

This expression for C_p is the well-known perfect gas relation where $\bar{R}$ is used in place of R . This case is therefore analogous to the frozen flow cases except a different value of γ is used and the frozen heat capacity $(4 + \alpha)R$ is replaced by C_p .

It can be observed that the perfect gas results lie very close to the equilibrium flow curve in every case. This is considered fortuitous since the equilibrium temperature at the throat is usually larger than that calculated from the perfect gas approximation. On the other hand, the dissociation fraction and sonic velocity are slightly larger for the perfect gas case. The values of γ at the throat are in general different for the two methods. However, all these effects appear to cancel each other and the values of A^* predicted are very close making the values of the mass flow rate parameter nearly equal. This is indeed fortunate since it appears that simple reactions in swirling flow can be easily treated simply by modifying

the values of γ and R in perfect gas theory. The only equilibrium calculation required is the finding of the initial α and γ . A full equilibrium calculation is still required however if accurate values of other parameters such as temperature, velocity, etc. are desired at the throat or at other points in the nozzle.

One other feature of interest in figures 1-8 is the reverse curvature noted for large values of ω . This is especially noticeable for the lower stagnation temperature cases. This is gratifying in that it indicates that the curves probably approach the axis asymptotically. Thus it requires an infinite ω to produce an infinite A^* . If this were not true, there would exist a finite angular velocity that could contain the gas in the chamber without any wall. This is physically unrealistic.

Figures 13, 14, 15 show some results for the performance problem for several different stagnation temperatures. It is not possible to scale the values of $\dot{m}$ directly from the mass flow rate parameter as one can for nonswirling perfect gas theory. The size of the throat directly affects the magnitude of the tangential velocity at the throat which in turn controls the mass flow rate that can be passed. For a constant mass flow rate, the throat area required increases with increasing ω . Thus to hold the throat area constant, $\dot{m}$ must decrease with increasing ω . The effect is nonlinear since holding the throat area constant increases the tangential velocity at the throat relative to the constant $\dot{m}$ case with the same initial ω . From a computational standpoint, the problem is more difficult since values of both $\dot{m}$ and the pressure in the vicinity of the throat must be guessed at the

Fig. 13 PERFORMANCE PROBLEM -- SOLID BODY ROTATION

Fig. 14 PERFORMANCE PROBLEM -- SOLID BODY ROTATION

Fig. 15 PERFORMANCE PROBLEM -- SOLID BODY ROTATION

start and thus the solution becomes a double trial and error process. This can be quite time consuming as the pressure on the centerline of the throat can drop to $.001 P_0$ at the highest values of ω . The solution also becomes extremely sensitive to centerline pressure.

The results show an increasing deviation from the design problem curve as ω increases. The deviation seems to be almost directly proportional to T_0 as is the whole swirl effect. The deviation does not become appreciable until ω becomes greater than 800-1000 rad./sec. Thus it is possible to scale the mass flow rate and A^* directly for values of ω less than 800 rad./sec. The relative insensitivity of the results to pressure suggest that the pressure rise due to spin in certain solid-fuel rockets could be estimated directly from the mass flow rate parameter and the reduction in $\dot{m}$ due to swirl if a suitable relation between P_0 and $\dot{m}$ is provided. One such simplified relation is:

$$\frac{\dot{m}}{A_c} = a P_0^n \quad (126)$$

where P_0 is the local stagnation pressure and a and n are constants that must be determined for a particular propellant and grain configuration.

Free Vortex Results

A few cases were computed using the initial conditions of a free vortex with a solid-body core. The core radius was chosen to be $0.1 r_c$ based on experimental data presented in Refs. 18 and 20. The results are shown in fig. 16 for an arc-jet using oxygen at a stag-

Fig. 16 FREE VORTEX

nation temperature of 8000°R and pressure of 15 psia. A comparable cold-gas case of oxygen at a stagnation temperature of 530°R and pressure of 15 psia is also shown. The results are again correlated in terms of the mass flow rate parameter. In the cold-gas case there is considerable variation (approx. 50%) in the pressure and temperature across the initial plane. The high-temperature cases show variations on the order of 5%. The mass flow rate parameter is again plotted against ω . In this case ω varies across the flowfield and thus the ω shown on the graph is the value of ω at the chamber wall. Therefore a value of ω of 280 implies a tangential velocity at the wall of 70 fps. The peak tangential velocity on the other hand occurs at $0.1 r_c$ and has a value of 700 fps. This is the largest value noted in any previous experimental results.

The results show very little effect due to swirl at the higher stagnation temperature for the range of tangential velocities normally used. The pressure in the core at the throat can drop to 25% of the value it has for the non-swirling case however. The perfect gas correlation appears to hold also for the free vortex. The cold-gas flow case shows a much greater effect on the mass flow rate parameter due to swirl. This demonstrates the unreliability of low temperature studies as a guide to the results that can be expected in high-temperature applications. The centerline pressure at the throat is much lower for the low temperature case than for the high temperature case also.

Boundary Layer Results

Boundary layer solutions have been performed for several of the inviscid flow situations described above using a wall temperature of 2000°R . and assuming a $37\frac{1}{2}^{\circ}$ conical nozzle. In all cases the boundary layer thickness was less than $0.001''$ compared to a throat radius of $0.3400''$ or larger. Thus the boundary layer thickness is on the order of 0.3% . Even if the actual boundary layer should prove to be an order of magnitude larger due to turbulence and the influence of the tangential velocity, its displacement thickness would still be only 3% of the throat radius. Thus one can expect boundary layer effects to be small for all cases considered in this investigation for throat swirl angles less than 45° (ω approx. 1000 rad./sec.). It should be emphasized that the boundary layer analysis is probably considerably in error for swirl angles greater than 45° (ω larger than 1000 rad./sec.). This case requires the solution of the complete three-dimensional reacting boundary layer problem.

Norton's solution for the tangential momentum equation shows considerable effect due to boundary layer growth for values of ω greater than 1000 rad./sec. However, results are presented only for cold-flow problems and these cannot be applied directly to high temperature cases for the reasons cited earlier. Therefore, modification of the solution to account for swirl effects at large values of ω must await future work in three-dimensional, reacting boundary layers.

V CONCLUSIONS

The theory of swirling flow has been extended to include real gas effects based on the dissociation reaction. A method of calculation has been presented that incorporates reaction effects in a simple and convenient way using the ideal dissociating gas approximation. Numerical results show the effects of swirl on reacting nozzle flows to be very sensitive to temperature but relatively insensitive to pressure. This comes about because the dissociation reaction itself is highly temperature dependent. The temperature and the change in effective molecular weight in turn affect the ratio of the tangential to the axial velocity component at the throat.

It is further shown that the gross effects of dissociation can be lumped into two parameters, the isentropic exponent and the effective gas constant. This leads to a useful approximation to the full equilibrium calculation based on certain modifications to the perfect gas equations as summarized below:

1. A γ and α based on equilibrium calculations at the stagnation temperature and pressure are used.
2. $\bar{R} = (1 + \alpha)R$
3. $C_p = \bar{R} \frac{\gamma}{\gamma - 1}$

This approximation is shown to predict the gross effects of swirl on reacting flows quite well over a large range of pressures and temperatures though individual flow parameters may be somewhat in error.

The differences in the design and performance problems have been

investigated and the results compared to show a large difference in results for values of ω greater than 1000 rad./sec. The design problem is far less severe due to the increase in throat area and resulting decrease in contraction ratio for increasing values of ω . Thus the tangential velocity at the throat is smaller than for the performance problem corresponding to the same initial conditions.

Finally, boundary layer effects have been shown to be small, at least for values of ω less than 1000 rad./sec.

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APPENDIX A

CALCULATION OF CONSTANTS FOR THE IDEAL DISSOCIATING GAS

The ideal dissociating gas approximation is derived from the equation expressing the law of mass action for a symmetrical diatomic (homonuclear) gas. This expression is:

$$\frac{\alpha^2}{1-\alpha} = \frac{e^{-\Theta_d/T}}{\rho} \left[m \left(\frac{\pi m k_b}{h_p^2} \right)^{3/2} \Theta_r \sqrt{T} (1 - e^{-\Theta_r/T}) \frac{(Q_{el}^a)^2}{Q_{el}^{aa}} \right] \quad (A-1)$$

The approximation is based on assuming the term in brackets to be a constant compared with the large variation of the exponential term with temperature. Values of this constant, ρ_d , are given in References (25) and (26) for oxygen and nitrogen. The purpose of this appendix is to evaluate this constant for several other gases. These gases are hydrogen, fluorine, and chlorine.

In order to evaluate the term in brackets for various temperatures, values of the various physical constants are required for each gas. Values of Θ_r and Θ_d , the characteristic temperatures for rotation and vibration are taken from Reference (36). The electronic partition functions are evaluated from the spectroscopic data of Hertzberg⁽³⁷⁾ for the diatomic gas and Moore⁽³⁸⁾ for the monatomic species of each gas. These values are listed in table A-1.

Using this data, ρ_d can be evaluated (table A-2) for various temperatures. This variation is observed to be of the same order of magnitude as that for oxygen and nitrogen as observed by Lighthill. Suitable average values have been chosen for each gas as shown in table A-3. Values of Θ_d as calculated from Hertzberg's data are also given. In the case of fluorine, there is some dispute as to the

correct value of the dissociation energy. The largest and oldest value has been chosen here since the primary purpose of this work is to investigate the largest possible effects of dissociation rather than to concentrate on the exact properties of fluorine itself.

TABLE A-1

Gas	M	$\Theta_v(^{\circ}\text{K})$	$\Theta_r(^{\circ}\text{K})$	Q_{el}^{aa}	Q_{el}^a
Hydrogen	1.00	5987	85.38	1	2
Fluorine	19.00	1283	1.270	1	$4 + 2e^{-582/T}$
Chlorine	35.45	798	0.3495	1	$4 + 2e^{-1268/T}$

TABLE A-2

T ($^{\circ}\text{K}$)	$\rho_d \text{ (gm/cm}^3\text{)}$		
	H ₂	F ₂	Cl ₂
1000	1.19	131.98	104.45
1500	1.43	140.86	108.86
2000	1.60	140.97	108.75
2500	1.71	138.01	106.80
3000	1.78	134.01	104.11
3500	1.83	129.77	101.20
4000	1.85	125.59	98.28
4500	1.86	121.62	95.46
5000	1.86	117.89	92.79
5500	1.85	114.41	90.26
6000	1.84	111.18	87.89
6500	1.83	108.17	85.67
7000	1.81	105.38	83.59
7500	1.79	102.77	81.64
8000	1.78	100.33	79.80

TABLE A-3

Gas	$\Theta_d(^{\circ}\text{K})$	$\rho_d (\text{gm/cm}^3)$
O_2	59018	150
N_2	113253	130
H_2	51960	1.85
F_2	31924	125
Cl_2	28731	100