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AT MODERATE SUPERSONIC SPEEDS

by

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Thesis Director's signature:

A handwritten signature in black ink, appearing to read "Augusto M. ...", written over a horizontal line.

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Abstract

BASE DRAG EFFECTS ON MAXIMUM LIFT-TO-DRAG RATIO AIRFOILS
AT MODERATE SUPERSONIC SPEEDS

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ROBERT E. PRITCHARD

Maximum lift-to-drag ratio airfoils at moderate supersonic speeds are determined using Ackeret's linear theory for the forebody pressure coefficient and Chapman's experimental results for the base pressure coefficient. Two cases are investigated for both laminar flow and turbulent flow: (i) given length and thickness and (ii) given length and enclosed area.

For case (i), it is shown that the maximum lift-to-drag ratio airfoil is identical with the minimum drag airfoil. Furthermore, for turbulent flow and given Mach number and Reynolds number, two critical values of the thickness ratio exist, τ_1 and τ_2 . For $\tau \leq \tau_1$, the optimum airfoil is a diamond shape with a closed trailing edge and exhibits maximum thickness at midchord; for $\tau_1 \leq \tau \leq \tau_2$, the optimum airfoil is a diamond shape with a blunt trailing edge and exhibits maximum thickness between midchord and the trailing edge; finally, for $\tau \geq \tau_2$, the optimum airfoil is a wedge and exhibits maximum thickness at the trailing edge.

For case (ii), it is shown that the maximum lift-to-drag ratio airfoil is identical with the minimum drag airfoil. Furthermore, for turbulent flow and given Mach number and Reynolds number, a critical value of the enclosed area ratio exists,

$(A/\ell^2)_1$. For $A/\ell^2 \leq (A/\ell^2)_1$, the optimum airfoil is biconvex with a closed trailing edge and exhibits maximum thickness at midchord; for $A/\ell^2 \geq (A/\ell^2)_1$, the optimum airfoil is biconvex with a blunt trailing edge and exhibits maximum thickness between midchord and the trailing edge.

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1. Introduction

In Ref. 1, Miele and Damoulakis considered the maximum lift-to-drag ratio problem for airfoils at moderate supersonic speeds using linear theory or second-order theory. They investigated (i) the case of given length and thickness and (ii) the case of given length and enclosed area, neglecting base drag effects on the optimum shape. In this thesis, the maximum lift-to-drag ratio problem is reinvestigated for cases (i) and (ii) including base drag effects on the optimum shape. The following hypotheses are employed: (a) the airfoil is two-dimensional and slender; (b) the distribution of pressure coefficients on the forebody is determined by Ackeret's linear theory; (c) the base pressure coefficient is determined according to Chapman's experiments (Ref. 2); and (d) the surface-averaged friction coefficient is constant.

2. Formulation of the Problem

Consider a two-dimensional, slender airfoil in supersonic flow, and let X denote a chordwise coordinate, Y a coordinate perpendicular to X (positive upward), Z a coordinate perpendicular to X (positive downward), and ℓ the length. Introduce the dimensionless coordinates (Fig. 1)

$$x = X/\ell, \quad y = Y/\ell, \quad z = Z/\ell \quad (1)$$

and observe that the drag coefficient C_D and the lift coefficient C_L can be written in the form

$$C_D = D/q\ell = \int_0^1 [2C_f + C_p(\dot{y} - \alpha) + \tilde{C}_p(\dot{z} + \alpha)] dx - C_{pb}(y_f + z_f) \quad (2)$$

$$C_L = L/q\ell = \int_0^1 (\tilde{C}_p - C_p) dx$$

where D is the drag per unit span, L the lift per unit span, q the free-stream dynamic pressure, $y(x)$ the function describing the upper contour, $z(x)$ the function describing the lower contour, C_p the forebody pressure coefficient at points on the upper contour, $\tilde{C}_p$ the forebody pressure coefficient at points on the lower contour, C_{pb} the base pressure coefficient, C_f the surface-averaged friction coefficient (assumed constant), and α the angle of attack. The dot denotes derivative with respect to x . The functions $y(x)$ and $z(x)$ satisfy the boundary conditions

$$x_i = 0, \quad y_i = 0, \quad z_i = 0 \quad (3)$$

$$x_f = 1, \quad y_f = z_f \quad (4)$$

If Ackeret's linear theory is employed, the distribution of pressure coefficients on the forebody is given by

$$C_p = C_1(\dot{y} - \alpha) \quad , \quad \tilde{C}_p = C_1(\dot{z} + \alpha) \quad (5)$$

where

$$C_1 = 2/\sqrt{(M^2 - 1)} \quad (6)$$

and where M is the free-stream Mach number. In the light of Eqs. (3)-(5), the drag and lift coefficients can be rewritten in the form

$$C_D = C_1 \int_0^1 (\dot{y}^2 + \dot{z}^2) dx + 2C_f + 2C_1\alpha^2 - C_{pb}(y_f + z_f) \quad (7)$$

$$C_L = 2C_1\alpha$$

Coordinate Transformation. In linear theory, it is convenient to employ the camber u and the thickness w as dependent variables in place of the upper and lower coordinates y and z . Therefore, after the following definitions are introduced:

$$u = (y - z)/2 \quad , \quad w = (y + z)/2 \quad (8)$$

the integrals (7) become

$$C_D = 2C_1 \int_0^1 [\dot{u}^2 + \dot{w}^2] dx + 2C_f + 2C_1\alpha^2 - 2C_{pb}w_f \quad (9)$$

$$C_L = 2C_1\alpha$$

while the boundary conditions (3)-(4) are rewritten as

$$x_i = 0 \quad , \quad u_i = 0 \quad , \quad w_i = 0 \quad (10)$$

$$x_f = 1 \quad , \quad u_f = 0 \quad (11)$$

According to Chapman's experiments (Ref. 2), the base pressure coefficient is primarily a function of the form

$$C_{pb} = f(M, Re, w_f) \quad (12)$$

where Re is the free-stream Reynolds number and w_f is the base thickness. For the purposes of this thesis, it is appropriate to define a base parameter $C_{pb} w_f$ and observe that it has the functional form (see Figs. 2-3)

$$C_{pb} w_f = \varphi(M, Re, w_f) \quad (13)$$

Minimal Problem. In the following sections, we discuss the maximization of the lift-to-drag ratio

$$E = L/D = C_L/C_D \quad (14)$$

that is, the minimization of the drag-to-lift ratio

$$1/E = D/L = C_D/C_L \quad (15)$$

with respect to the functions $u(x)$, $w(x)$ and the parameter α which satisfy the boundary conditions (10)-(11), with the understanding that C_D and C_L are

defined by Eqs. (9).

In order to rule out the trivial solution $w = 0$, we consider two combinations of constraints: (i) given length and thickness and (ii) given length and enclosed area. For case (i), the function $w(x)$ is required to satisfy the inequality constraint

$$w \leq \tau/2 \quad (16)$$

($\tau = t/\ell$ is the thickness ratio, the ratio of the total thickness t to the length ℓ) everywhere in the interval of integration. It is understood that Ineq. (16) becomes an equality for, at least, one station x_0 ; that is,

$$w_0 = \tau/2 \quad , \quad 0 \leq x_0 \leq 1 \quad (17)$$

For case (ii), the function $w(x)$ is required to satisfy the isoperimetric constraint

$$A/\ell^2 = 2 \int_0^1 w dx \quad (18)$$

where A is the enclosed area.

3. Given Length and Thickness

Standard variational methods (see, for instance, Chapter 2 of Ref. 3) show that the extremal arc must minimize the integral

$$I = \int_0^1 F(\dot{u}, \dot{w}) dx + G(w_f, \alpha, \mu) \quad (19)$$

where the functions F and G are given by

$$\begin{aligned} F &= 2C_1(\dot{u}^2 + \dot{w}^2) \\ G &= 2C_f + 2C_1\alpha^2 - 2C_{pb}w_f - 2C_1\mu\alpha \end{aligned} \quad (20)$$

In accordance with Ref. 4, the undetermined, constant Lagrange multiplier μ satisfies the relation

$$\mu = 1/E \quad (21)$$

Euler Equations. From calculus of variations, it is known that the extremal arc must satisfy the Euler equations

$$dF_{\dot{u}}/dx = 0 \quad , \quad dF_{\dot{w}}/dx = 0 \quad (22)$$

whose explicit form is the following

$$\ddot{u} = 0 \quad , \quad \ddot{w} = 0 \quad (23)$$

thereby showing that the extremal arc is composed of straight lines.

Camber Distribution. We note that the relation

$$u = 0 \quad (24)$$

satisfies the differential equation (23-1) and the boundary conditions (10-2) and (11-2).

Therefore, the camber line is a straight line, and the associated airfoil is symmetric with respect to the chord, that is,

$$y = z = w \quad (25)$$

Thus, the maximum lift-to-drag ratio airfoil is identical with the minimum drag airfoil (see Refs. 5-7).

Thickness Distribution. While the particular solution $w = 0$ satisfies the initial condition (10-3), it does not satisfy the thickness requirement (17). As a consequence, we postulate a discontinuous solution composed of two straight line subarcs joined at a corner point of coordinates x_0, w_0 consistent with (17). If $\dot{w}_i$ denotes the initial slope and $\dot{w}_f$ denotes the final slope, the optimum thickness distribution is described by the relations

$$w = \dot{w}_i x \quad , \quad 0 \leq x \leq x_0 \quad (26)$$

$$w = w_f + \dot{w}_f (x - 1) \quad , \quad x_0 \leq x \leq 1 \quad (27)$$

in which the terminal slopes and the base thickness must be determined.

Continuity and Thickness Requirements. Since the thickness distribution is continuous, the value of w predicted with either Eq. (26) or Eq. (27) must be identical at x_o . This yields the relation

$$\dot{w}_i x_o = w_f + \dot{w}_f (x_o - 1) \quad (28)$$

Furthermore, upon combining (17) and (26), we rewrite the thickness constraint in the form

$$\dot{w}_i x_o = \tau/2 \quad (29)$$

Corner Conditions. At a corner point, the condition

$$\Delta(F - \dot{u}F_{\dot{u}} - \dot{w}F_{\dot{w}})\delta x_o + \Delta F_{\dot{u}}\delta u_o + \Delta F_{\dot{w}}\delta w_o = 0 \quad (30)$$

must be satisfied identically for every set of variations consistent with the constraint (17). In Eq. (30), the symbol $\Delta(\dots)$ denotes the difference of quantities evaluated after the corner and before the corner. From (17) and (30), we deduce that the following relations hold:

$$\Delta(F - \dot{u}F_{\dot{u}} - \dot{w}F_{\dot{w}}) = 0 \quad , \quad \Delta F_{\dot{u}} = 0 \quad (31)$$

and their explicit form is

$$\Delta(\dot{u}^2 + \dot{w}^2) = 0 \quad , \quad \Delta \dot{u} = 0 \quad (32)$$

Equation (32-2) shows that the slope of the camber distribution must be continuous,

a conclusion consistent with Eq. (24). Equation (32-1) shows that the slope of the thickness distribution can be discontinuous providing

$$\Delta \dot{w}^2 = 0 \quad (33)$$

that is, providing the initial and final slopes satisfy the relation

$$\dot{w}_i = -\dot{w}_f \quad (34)$$

Optimum Angle of Attack. The Euler equations (22) must be completed by the condition

$$G_\alpha = 0 \quad (35)$$

relative to the optimization of the angle of attack. The explicit form of this relation is the following:

$$\alpha = 1/2E \quad (36)$$

Transversality Condition. Since the base thickness w_f is unspecified, the transversality condition

$$G_{w_f} \delta w_f + [(F - \dot{u} F_{\dot{u}} - \dot{w} F_{\dot{w}}) \delta x + F_{\dot{u}} \delta u + F_{\dot{w}} \delta w]_i^f = 0 \quad (37)$$

must be satisfied for every system of variations consistent with the boundary conditions (10)-(11). After accounting for Eqs. (20), Eq. (37) leads to

$$(K + 2C_1 \dot{w}_f) \delta w_f = 0 \quad (38)$$

where, by definition,

$$K \approx -\partial(C_{pb} w_f)/\partial w_f \quad (39)$$

Since the base thickness is free, Eq. (38) implies that

$$K + 2C_1 \dot{w}_f = 0 \quad (40)$$

While Eq. (40) does not supply explicitly the optimum base thickness, we note that w_f must be in the range

$$0 \leq w_f \leq \tau/2 \quad (41)$$

If the solutions of Eq. (40) violate either the lower bound or the upper bound in Ineq. (41), then the final condition (40) must be replaced by either $w_f = 0$ or $w_f = \tau/2$.

Maximum Lift-to-Drag Ratio. After accounting for Eqs. (24), (26), (27),

Eqs. (9) can be rewritten as

$$\begin{aligned} C_D &= 2C_1[\dot{w}_i^2 x_o + \dot{w}_f^2(1 - x_o)] + 2C_f + 2C_1\alpha^2 - 2C_{pb}w_f \\ C_L &= 2C_1\alpha \end{aligned} \quad (42)$$

Substitution of (42) into (14) leads to the relation

$$C_1[\dot{w}_i^2 x_o + \dot{w}_f^2(1 - x_o)] + C_1(\alpha^2 - \alpha/E) + C_f - C_{pb}w_f = 0 \quad (43)$$

which determines the maximum lift-to-drag ratio.

4. Turbulent Flow Solutions

In the previous section, a general theory was presented for both laminar flow and turbulent flow. Here, detailed solutions are derived for turbulent flow. Analytically, this is equivalent to assuming that K is constant for each given Mach number and Reynolds number (see Fig. 3). We note that Eqs. (28), (29), (34), (36), (40), (43) form a system of six equations in the six unknowns $\dot{w}_i$, $\dot{w}_f$, w_f , x_o , α , E . After simple manipulations, the following solutions are obtained:

$$\begin{aligned}
 \dot{w}_i &= K/2C_1 \\
 \dot{w}_f &= -K/2C_1 \\
 w_f &= \tau - K/2C_1 \\
 x_o &= C_1 \tau / K \\
 \alpha &= \sqrt{(C_f/C_1 - C_{pb} w_f / C_1 + K^2 / 4C_1^2)} \\
 E &= 1/2 \sqrt{(C_f/C_1 - C_{pb} w_f / C_1 + K^2 / 4C_1^2)}
 \end{aligned} \tag{44}$$

These solutions are valid subject to the inequality (41), that is, providing

$$\tau_1 \leq \tau \leq \tau_2 \tag{45}$$

where

$$\tau_1 = K/2C_1, \quad \tau_2 = K/C_1 \tag{46}$$

If the lower bound of Ineq. (45) is violated, Eq. (40) must be replaced by

$$w_f = 0 \quad (47)$$

In this case, Eqs. (28), (29), (34), (36), (43) admit the solutions

$$\begin{aligned} \dot{w}_i &= \tau \\ \dot{w}_f &= -\tau \\ x_o &= 1/2 \\ \alpha &= \sqrt{(C_f/C_1 + \tau^2)} \\ E &= 1/2\sqrt{(C_f/C_1 + \tau^2)} \end{aligned} \quad (48)$$

If the upper bound of Ineq. (45) is violated, Eqs. (34) and (40) must be replaced by

$$\begin{aligned} w_f &= \tau/2 \\ x_o &= 1 \end{aligned} \quad (49)$$

Under these conditions, Eqs. (28), (29), (36), (43) admit the solutions

$$\begin{aligned} \dot{w}_i &= \tau/2 \\ \alpha &= \sqrt{(C_f/C_1 - C_{pb} w_f/C_1 + \tau^2/4)} \\ E &= 1/2\sqrt{(C_f/C_1 - C_{pb} w_f/C_1 + \tau^2/4)} \end{aligned} \quad (50)$$

Summarizing, three kinds of solutions are possible, depending on the value of the thickness ratio. If $\tau \leq \tau_1$, the optimum airfoil is a diamond shape with a closed trailing edge and exhibits maximum thickness at midchord. If $\tau_1 \leq \tau \leq \tau_2$, the optimum airfoil is a diamond shape with a blunt trailing edge and exhibits maximum thickness between midchord and the trailing edge. Finally, for $\tau \geq \tau_2$, the optimum airfoil is a wedge and exhibits maximum thickness at the trailing edge.

Example. In order to illustrate the previous theory, an example is given for an airfoil at $M = \sqrt{5}$ and $Re = 10^7$. For these conditions, the relations $C_1 = 1$ and $K = 0.2$ hold. The assumed surface-averaged friction coefficient is $C_f = 10^{-3}$. As a consequence, the critical values of the thickness ratio are given by

$$\tau_1 = 0.10 \quad , \quad \tau_2 = 0.20 \quad (51)$$

The initial slope, the base thickness, the abscissa of the maximum thickness point, the angle of attack, and the maximum lift-to-drag ratio are plotted in Figs. 4-8 versus the thickness ratio. The dashed line in Fig. 8 represents the lift-to-drag ratio of a diamond shape airfoil with maximum thickness at midchord and closed trailing edge. The difference of ordinates between the solid line and the dashed line represents the gain to be made with blunt trailing edge airfoils. Finally, the optimum thickness distribution is given in Fig. 9 for several values of the thickness ratio. From this example, the most important conclusion is that airfoils with closed trailing edge are desirable in the range of thickness ratios used in practice, that is, $0.02 \leq \tau \leq 0.08$.

5. Given Length and Enclosed Area

Standard variational methods (see, for instance, Chapter 2 of Ref. 3) show that the extremal arc must minimize the integral

$$I = \int_0^1 F(\dot{u}, w, \dot{w}, \lambda) dx + G(w_f, \alpha, \mu) \quad (52)$$

where the functions F and G are defined as

$$\begin{aligned} F &= 2C_1(\dot{u}^2 + \dot{w}^2) - \lambda w \\ G &= 2C_f + 2C_1\alpha^2 - 2C_{pb}w_f - 2C_1\mu\alpha \end{aligned} \quad (53)$$

and where λ, μ are undetermined, constant Lagrange multipliers. In accordance with Ref. 4, the multiplier μ is given by

$$\mu = 1/E \quad (54)$$

Euler Equations. From calculus of variations, it is known that the extremal arc must satisfy the Euler equations

$$dF_{\dot{u}}/dx = 0 \quad , \quad dF_{\dot{w}}/dx - F_w = 0 \quad (55)$$

whose explicit form is the following:

$$\ddot{u} = 0 \quad , \quad \ddot{w} + \lambda/4C_1 = 0 \quad (56)$$

Camber Distribution. We note that the relation

$$u = 0 \quad (57)$$

satisfies the differential equation (56-1) and the boundary conditions (10-2) and (11-2). Therefore, the camber line is a straight line, and the associated airfoil is symmetric with respect to the chord, that is,

$$y = z = w \quad (58)$$

Thus, the maximum lift-to-drag ratio airfoil is identical with the minimum drag airfoil (see Refs. 5-7).

Thickness Distribution. The Euler equation (56-2) admits the first integral

$$\dot{w} = -(\lambda/4C_1)x + C_2 \quad (59)$$

and the second integral

$$w = -(\lambda/8C_1)x^2 + C_2x + C_3 \quad (60)$$

in which C_2 , C_3 are constants. Because of the initial conditions (10-1), (10-3), the relation

$$C_3 = 0 \quad (61)$$

holds and Eq. (60) reduces to

$$w = -(\lambda/8C_1)x^2 + C_2x \quad (62)$$

If Eq. (59) is applied at the terminal points, the relations

$$C_2 = \dot{w}_i, \quad \lambda = 4C_1(\dot{w}_i - \dot{w}_f) \quad (63)$$

are obtained. In the light of Eqs. (63), the thickness distribution (62) becomes

$$w = \dot{w}_i x + (\dot{w}_f - \dot{w}_i)x^2/2 \quad (64)$$

which, at the final point, implies that

$$w_f = (\dot{w}_i + \dot{w}_f)/2 \quad (65)$$

Optimum Angle of Attack. The Euler equations (55) must be completed by the condition

$$G_\alpha = 0 \quad (66)$$

relative to the optimization of the angle of attack. The explicit form of this relation is the following:

$$\alpha = 1/2E \quad (67)$$

Transversality Condition. Since the base thickness w_f is unspecified, the transversality condition

$$G_{w_f} \delta w_f + [(F - \dot{u}F_{\dot{u}} - \dot{w}F_{\dot{w}})\delta x + F_{\dot{u}}\delta u + F_{\dot{w}}\delta w]_i^f = 0 \quad (68)$$

must be satisfied for every system of variations consistent with the boundary

conditions (10)-(11). After accounting for Eqs. (53), Eq. (68) leads to

$$(K + 2C_1 \dot{w}_f) \delta w_f = 0 \quad (69)$$

where K is given by Eq. (39). Since the base thickness is free, Eq. (69) implies that

$$K + 2C_1 \dot{w}_f = 0 \quad (70)$$

While Eq. (70) does not supply explicitly the optimum base thickness, we note that w_f must satisfy the inequality

$$w_f \geq 0 \quad (71)$$

If the solutions of Eq. (70) violate this inequality, the final condition (70) must be replaced by $w_f = 0$.

Enclosed Area Constraint. After accounting for Eq. (64), the isoperimetric constraint (18) becomes

$$A/\ell^2 = (2\dot{w}_i + \dot{w}_f)/3 \quad (72)$$

Maximum Lift-to-Drag Ratio. In the light of Eq. (64), Eqs. (9) can be rewritten as

$$C_D = (2C_1/3)[\dot{w}_i^2 + \dot{w}_i \dot{w}_f + \dot{w}_f^2] + 2C_f + 2C_1 \alpha^2 - 2C_{pb} w_f \quad (73)$$

$$C_L = 2C_1 \alpha$$

Substitution of (73) into (14) leads to the relation

$$(C_1/3)[\dot{w}_i^2 + \dot{w}_i \dot{w}_f + \dot{w}_f^2] + C_1(\alpha^2 - \alpha/E) + C_f - C_{pb} w_f = 0 \quad (74)$$

which determines the maximum lift-to-drag ratio.

6. Turbulent Flow Solutions

In the previous section, a general theory was presented for both laminar flow and turbulent flow. Here, detailed solutions are derived for turbulent flow. Analytically, this is equivalent to assuming that K is constant for each given Mach number and Reynolds number (see Fig. 3). We note that Eqs. (65), (67), (70), (72), (74) form a system of five equations in the five unknowns $\dot{w}_i$, $\dot{w}_f$, w_f , α , E . After simple manipulations, the following solutions are obtained:

$$\begin{aligned}
 \dot{w}_i &= 3A/2\ell^2 + K/4C_1 \\
 \dot{w}_f &= -K/2C_1 \\
 w_f &= 3A/4\ell^2 - K/8C_1 \\
 \alpha &= \sqrt{(C_f/C_1 - C_{pb}w_f/C_1 + 3A^2/4\ell^4 + K^2/16C_1^2)} \\
 E &= 1/2\sqrt{(C_f/C_1 - C_{pb}w_f/C_1 + 3A^2/4\ell^4 + K^2/16C_1^2)}
 \end{aligned} \tag{75}$$

These solutions are valid subject to Ineq. (71), that is, providing

$$A/\ell^2 \geq (A/\ell^2)_1 \tag{76}$$

where

$$(A/\ell^2)_1 = K/6C_1 \tag{77}$$

If Ineq. (76) is violated, Eq. (70) must be replaced by

$$w_f = 0 \tag{78}$$

In this case, Eqs. (65), (67), (72), (74) admit the solutions

$$\begin{aligned}
 \dot{w}_i &= 3A/\ell^2 \\
 \dot{w}_f &= -3A/\ell^2 \\
 \alpha &= \sqrt{(C_f/C_1 + 3A^2/\ell^4)} \\
 E &= 1/2\sqrt{(C_f/C_1 + 3A^2/\ell^4)}
 \end{aligned} \tag{79}$$

Summarizing, two kinds of solutions are possible, depending on the value of the enclosed area ratio. If $A/\ell^2 \leq (A/\ell^2)_1$, the optimum airfoil is biconvex with a closed trailing edge and exhibits maximum thickness at midchord. For $A/\ell^2 \geq (A/\ell^2)_1$, the optimum airfoil is biconvex with a blunt trailing edge and exhibits maximum thickness between midchord and the trailing edge.

Example. In order to illustrate the previous theory, an example is given for an airfoil at $M = \sqrt{5}$ and $Re = 10^7$. For these conditions, the relations $C_1 = 1$ and $K = 0.2$ hold. The assumed surface-averaged friction coefficient is $C_f = 10^{-3}$. As a consequence, the critical value of the enclosed area is given by

$$(A/\ell^2)_1 = 0.0333 \tag{80}$$

The initial slope, the final slope, the base thickness, the angle of attack, and the maximum lift-to-drag ratio are plotted in Figs. 10-14 versus the enclosed area ratio. The dashed line in Fig. 14 represents the lift-to-drag ratio of a biconvex airfoil with maximum thickness at midchord and closed trailing edge. The difference of ordinates between the solid line and the dashed line represents the

gain to be made with blunt trailing edge airfoils. Finally, the optimum thickness distribution is given in Fig. 15 for several values of the enclosed area ratio.

In engineering applications, the values of the enclosed area ratio are in the range $0.01 \leq A/\ell^2 \leq 0.06$, and the associated values of the thickness ratio are in the range $0.015 \leq \tau \leq 0.08$. As a consequence, both the solution with a closed trailing edge and the solution with a blunt trailing edge may occur in practice, depending on the value of A/ℓ^2 .

7. Discussion and Conclusions

In the previous sections, maximum lift-to-drag ratio airfoils at moderate supersonic speeds are determined using Ackeret's linear theory for the forebody pressure coefficient and Chapman's experimental results for the base pressure coefficient. Two cases are investigated for both laminar flow and turbulent flow: (i) given length and thickness and (ii) given length and enclosed area.

For case (i), it is shown that the maximum lift-to-drag ratio airfoil is identical with the minimum drag airfoil. Furthermore, for turbulent flow and given Mach number and Reynolds number, two critical values of the thickness ratio exist, τ_1 and τ_2 . For $\tau \leq \tau_1$, the optimum airfoil is a diamond shape with a closed trailing edge and exhibits maximum thickness at midchord; for $\tau_1 \leq \tau \leq \tau_2$, the optimum airfoil is a diamond shape with a blunt trailing edge and exhibits maximum thickness between midchord and the trailing edge; finally, for $\tau \geq \tau_2$, the optimum airfoil is a wedge and exhibits maximum thickness at the trailing edge.

For case (ii), it is shown that the maximum lift-to-drag ratio airfoil is identical with the minimum drag airfoil. Furthermore, for turbulent flow and given Mach number and Reynolds number, a critical value of the enclosed area ratio exists, $(A/\ell^2)_1$. For $A/\ell^2 \leq (A/\ell^2)_1$, the optimum airfoil is biconvex with a closed trailing edge and exhibits maximum thickness at midchord; for $A/\ell^2 \geq (A/\ell^2)_1$, the optimum airfoil is biconvex with a blunt trailing edge and exhibits maximum thickness between midchord and the trailing edge.

The verification of the Legendre condition shows that local minimality with respect to weak variations of the slopes and the angle of attack is ensured for every value of the parameter τ or A/ℓ^2 (Refs. 8-9). Furthermore, the verification of the Weierstrass condition shows that local minimality with respect to strong variations of the slopes and the angle of attack is ensured for every value of the parameter τ or A/ℓ^2 (Refs. 8-9).

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Fig. 1 Coordinate system.

Fig. 2 Base pressure parameter versus base thickness for several Mach numbers (laminar flow, $Re = 10^6$).

Fig. 3 Base pressure parameter versus base thickness for several Mach numbers (turbulent flow, $Re = 10^7$).

Fig. 4 Initial slope.

Fig. 5 Base thickness.

Fig. 6 Abscissa of the maximum thickness point.

Fig. 7 Angle of attack.

Fig. 8 Maximum lift-to-drag ratio.

Fig. 9 Optimum airfoils.

Fig. 12 Base thickness.

Fig. 13 Angle of attack.

Fig. 14 Maximum lift-to-drag ratio.

Fig. 15 Optimum airfoils.