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STEADY-STATE THERMAL DESIGN OF SPACE RADIATORS

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NOMENCLATURE

C	arbitrary constant
D	factor in approximate solutions - $\left[\left(2\phi E k t (1-\phi) \right)^{1/2} \right]$
E	the emissivity if heat is radiated from one side of fin and twice the emissivity if heat is radiated from both sides of fin
H	overall heat transfer coefficient based on tube length - BTU/hr-ft-R
L	fin length - ft
N_u	Nusselt Number - hD/k_f
P	factor in approximate solutions - $\frac{1}{(1-\phi)^{1/2}} \left[\frac{5(1-\phi)}{1-\phi^5} \right]^{1/8}$
Q	heat radiated by half fin - BTU/hr
S	factor in approximate solutions - $\left[\frac{1-\phi^5}{5(1-\phi)} \right]$
T	temperature - R
W	fin half width - ft
k	thermal conductivity - BTU/hr-ft-R
l	stringer length - ft
q	heat transfer - BTU/hr
t	fin thickness - ft
x,y,z	length coordinates - ft
C_p	specific heat at constant pressure - BTU/lb _m -R
M	mass flow rate - lb _m /hr
$\tilde{W}$	dimensionless fin half width - $(10\phi E T_{f1}^3 / kt)(W)$
$\tilde{Q}$	dimensionless heat radiated by half fin - $(5/kt_{f1} t)(Q)$
$\tilde{L}$	dimensionless fin length - $(10\phi E T_{f1}^3 / kt)^{1/2} L$

α	dimensionless parameter - kt/MC_p
$\beta_z(M,N)$	incomplete Beta Function
γ	dimensionless parameter - $H(5/2\sigma Ekt T_{fl}^3)^{1/2}$
δ	factor in approximate solutions - $(2.5\sigma Ekt(1-\varphi))^{1/2}$
σ	Stefan-Boltzmann Constant - BTU/hr-ft ² -R ⁴
ψ	$\left(\frac{5(1-\varphi)}{1-\varphi^5}\right)\left(\frac{T_s}{T_b}\right)^4$
φ	constant used in approximating temperature distribution in approximate solutions
ζ	T_{mp}/T
ξ	$(T_{mp}/T)^5$
ΔL	width of fin strip
ΔQ	heat radiated by a fin strip
τ	dimensionless temperature T/T_{fl}
θ_s	T_s/T_{mp}
$\lambda(\zeta, \theta_s)$	solution to conduction Equation for $\tau_s < 1$

Subscripts

b	fin base
f	fluid
fl	fluid at inlet conditions
f2	fluid at outlet conditions
mp	fin mid point (adiabatic point)
s	sink
	no subscript refers to the fin

CHAPTER I

INTRODUCTION

A space radiator is one component of a system used to reject heat from a spacecraft. The complete system consists of a closed coolant loop, several heat exchangers where heat is added to the coolant, coolant pumps, control valves, etc. After the coolant pumps, control valves, etc. After the coolant has passed through the heat exchangers, it is pumped to the space radiator where heat is taken from the coolant and radiated to space. Systems of this type are now being used, and will be used in the future, as part of the spacecraft's environmental control system, power generating system, etc. It is the design of space radiators which will receive attention in this thesis.

Figure 1 shows a section from a typical space radiator. The

FIGURE 1

dimension L , is the length of the radiator, and the dimension W is the fin half width. The coolant flows in the tubes where it is cooled by transferring heat to the fins. The fins radiate into space. A common operating condition for space radiators is one in which the radiator is irradiated by external sources such as the sun, earth, etc. For convenience, the incident heat flux is usually expressed in terms of an equivalent sink temperature. The definition of the equivalent sink temperature can best be given by considering the Stefan-Boltzmann equation for heat radiation. Consider a body having a uniform temperature T , a surface area A , an emissivity ϵ , incident heat flux q_s , and absorptivity α . The net heat rejected by the body (Q) is given by:

$$Q = \sigma \epsilon A T^4 - \alpha A q_s$$

where σ is the Stefan-Boltzmann constant. The equivalent sink temperature, T_s , is defined to be that temperature which, when used in the Stefan-Boltzmann equation, produces a heat flux equal to q_s , i.e.,

$$Q = \sigma \epsilon A (T^4 - T_s^4) .$$

This means that $T_s = (\alpha q_s / \sigma \epsilon)^{1/4}$.

If the radiator's tubes are connected in parallel, if the mass flow rate is the same in all of the tubes, and if the incident heat flux distribution is symmetric about the line $x = W$, the fluid temperature distributions are identical in all the tubes. This symmetry means that no heat is transferred across the point midway between the tubes. This point is called the "adiabatic mid-point" and the temperature occurring at this point is called the "adiabatic mid-point temperature"

or the "mid-point temperature". If all of the fins in a radiator have adiabatic mid-points, the various tubes and their half fins are thermally isolated from the adjoining tubes and their half fins. For these cases, only one tube and its half fins need be analyzed since the results for any tube are valid for all of the tubes.

The point at which the fin is attached to the tube is referred to as the "fin base", and the temperature at this point is called the "fin base temperature".

A factor often used in discussing the heat transfer properties of a fin is the "fin effectiveness". This term is usually reserved for fins having a constant base temperature along the fins entire length, i.e., a fin having one-dimensional conduction and no longitudinal temperature gradient. the fin effectiveness is defined to be the ratio of the heat rejected by the actual fin to the heat which would be rejected if the fin were at a uniform temperature equal to the fin base temperature. Fin effectiveness is dependent on the material properties, incident heat flux, fin geometry, and fin base temperature.

There are two basic problems associated with space radiators. The first of these is the evaluation of the performance of a given radiator. The statement of this problem specifies the material properties, incident radiation, inlet fluid temperature, fluid mass flow rate, and all radiator dimensions. The performance calculations yield the heat rejection rate. The second problem is the problem of radiator design. In designing a radiator the heat rejection rate is given, and the radiator length is unspecified. The design calculations yield the

length of radiator which is required to obtain the specified heat rejection rate.

Most of the published work on space radiators is directed at analyzing the performance of a given radiator. These analyses are not, in general, well suited for design. There exists some analyses which deal with the fin alone and which consider one-dimensional conduction. The results of these analyses are usually presented as curves of fin effectiveness and are useable for design, but they completely neglect the temperature variations along the tube.

There exists a definite need for design techniques which consider the longitudinal temperature variations. It is this need with which this thesis is concerned.

If one limits the analysis to realistic configurations, certain assumptions can be made which have little effect on the final results, yet produce a great simplification of the mathematics involved. Assumptions of this type are an integral part of the analysis which follow. These analyses take two separate approaches to the problem. The first approach is to obtain computer programs which allow the rapid design of space radiators. The second approach is to make an additional assumption based on the computer results which results in solutions in terms of well-known functions and are, therefore, in a form suitable for hand calculations.

CHAPTER II

REVIEW OF THE LITERATURE

In this section existing literature dealing with space radiators will be reviewed. This is not intended to be a complete bibliography of the existing literature, but rather is intended to establish the state of the art by discussing some of the most significant publications as well as to establish those publications which influenced the present work.

The majority of the literature concerns itself with the one-dimensional fin. Typical of these are the works of MacKay and Bacha¹, Bartas and Sellers², and Lieblein³. Mackay and Bacha considered one-dimensional fins having rectangular and trapesoidal profiles as well as fins having a constant temperature gradient. Their results are presented in terms of curves of fin effectiveness. MacKay and Bacha also considered the analysis of radiators by using a heat transfer coefficient to couple a fluid energy balance to the fin analysis. They also considered the problem of a duct (no extended surfaces) having radiating walls and obtained an analytic solution. They were the first to recognize that the solution to the fin equation for the one-dimensional rectangular fin radiating to a zero temperature sink is a Beta Function. Because tabulated values for the particular Beta Function pertinent to the problem were not available in the literature, they present a graphical technique for approximating the required values. Bartas and Sellers, and Lieblein considered the one-dimensional fin with constant material properties. The governing differential equation was solved numerically and the results presented in the form of

curves of fin effectiveness.

Stockman and Kramer⁴ studied the effect of variable thermal properties on the results of the fin analysis. For an aluminum fin having a base temperature of 1260 R and a 200 R drop in temperature from fin base to fin mid-point they found the error from assuming constant thermal conductivity to be 0.76%. For the same fin but with a temperature drop of 382 R the error due to considering a constant conductivity was 1.3%. They also considered the effects of variable emissivity. Several coatings were considered and although for some coatings the error due to assuming constant emissivity was as large as 9%, the authors conclude that for the realistic coatings considered the effect was quite small. The combined effect of neglecting the variations in both thermal conductivity and emissivity was found, for realistic cases, to be small. Only for those cases particularly selected to produce a large discrepancy and which were not of a practical nature as far as application to environmental control system radiators, did the authors find a significant effect due to assuming constant thermal properties. It should be noted that Stockman and Kramer considered high temperature radiators. In environmental control radiators the temperature differences which occur along the fin are much smaller than those reported by Stockman and Kramer and the effects discussed should be even less pronounced.

Sparrow, Jonsson, and Minkowycz⁵ studied the effect of neglecting longitudinal heat conduction by considering radiators typical of those proposed for use in space-vehicle power systems. They showed that neglecting longitudinal heat conduction produced, for the cases

studied, a maximum error of slightly over 0.01%. This was for a ratio of fin half width to fin length of 1/30. Decreasing the ratio to 1/60 reduced the error by a factor of 4.

While Sparrow, et al. considered radiators typical of those used in space-vehicle power systems. French and Shield⁶ considered environmental control radiators and showed that the results presented by Sparrow et al. remain valid. French and Shield considered six cases and showed that the effect of neglecting longitudinal conduction was less than 1% for ratios of fin half width to fin length of up to 1/15. These results were found using a program, developed by French and Shield, which is based on an explicit method of finite difference solution. This program allows the consideration of variable thermal properties, variable incident thermal radiation, and three-dimensional conduction. The calculations were made once with longitudinal conduction suppressed in the computations. By using the same computer program the results could be shown to be a property of the physics of the problem and not of the numerical techniques used. The complete program has as an option a method of analysis in which the radiator is divided into isothermal lumps. The heat transferred from the tube to the fin is determined from precalculated tabulations of fin base thermal gradients. This option gives results very close to the results obtained with the nodal solution with longitudinal conduction suppressed. The authors attribute the slight discrepancy to errors introduced by interpolating in the table of fin base thermal gradients. The results of the numerical work is verified by experiments.

MacKay and Bacha¹ studied the effect of radiant interaction be-

tween the fin and the tube. They considered the radiating surface to be the fin plus a strip having a width equal to the diameter of the tube and a length equal to the tube length. Considering practical values of material properties they find that for a tube diameter equal to the fin half width a maximum multiplying correction factor of 1.07 must be applied to the calculated heat rejection rate. For "relatively small tubes" as are used in actual practice, they conclude that no correction factor need be applied.

All of the work mentioned has been aimed at analyzing a given radiator configuration. Results presented in the form of fin effectiveness curves permit designing radiators with a small amount of effort, but this completely neglects the temperature variations in the longitudinal direction. Other design techniques using programs such as the one used by French and Shield⁶ require a trial and error approach which is very time consuming. The need for a program better suited to design led Saule, Krebs, and Auer⁷ to develop a computer program for the design of sensible-heat space radiators. The radiator panel is divided into a number of elemental strips, each of which conducts heat in only one direction and each of which has an adiabatic mid-point. Each strip radiates heat at the same rate, and therefore each strip has a different length. The fin equation for each strip is solved numerically. The program calculates meteoroid protection, designs manifolds so as to produce equal flows in all of the parallel tubes and so as to produce a prescribed pressure drop in the tubes.

A considerable number of papers have been published which are directed at finding the fin which gives the largest ratio of heat re-

jected to fin weight. These optimization studies have dealt almost exclusively with the one-dimensional fin.

Liu^{8,9} investigated one-dimensional rectangular fins in order to determine the optimum fin thickness. The resulting fin for materials and temperatures encountered in environmental control radiators is so thin that in practice structural and manufacturing considerations dictate a thicker fin. As an example, an aluminum fin having an emissivity of 0.90, a 2.08 inch fin half width and a 560 R base temperature, has an optimum fin thickness of 0.002 inch.

Wilkins^{10,11,12}, Liu¹³, and Karlefar and Chao¹⁴ investigated fins having variable fin thicknesses of various types. Their work indicates that a considerable reduction in fin weight can be realized by proper shaping of the fin. In practice, however, such fins are not used because of difficulties in manufacturing.

While all of the optimization studies above considered only the fin, other portions of an actual radiator such as the tubes, headers, and fluid constitute a major portion of the total weight. Walker, Smith, and Gritton¹⁵, and Reynolds¹⁶ studied the optimization of the complete finned tube radiator, i.e., they considered the weight of the tubes, headers, and fluid. Walker et al. included a statistical analysis for calculating meteoroid protection. The analyses considered one-dimensional fins. Their analyses yield an optimum fin which is thicker than the fins which result from optimizing the fin alone. The resulting fin is still quite thin and in actual practice the fin thickness will be governed by structural and manufacturing considerations.

The equations which describe the operation of space radiators are quite complex in their complete form, and their solution requires

lengthy numerical techniques. It would be useful, at least for preliminary design purposes, if the equations could be simplified by making reasonable assumptions which would lead to a closed form solution. With this in mind Anderson, Jameson, Johnson, Stake, and Wright¹⁷ carried out a simplified radiator design procedure in which they neglect longitudinal conduction and assume a constant temperature drop from the bulk fluid temperature drop from the bulk fluid temperature to the fin base temperature. By assuming a constant fin effectiveness for the entire fin they are able to integrate the resulting equations along the tube axis in closed form. The analysis has one weakness in that the assumption of a constant fin effectiveness can have a considerable effect on the accuracy of the results.

CHAPTER III

DESCRIPTION OF MODEL

The analyses which are to follow will consider essentially the same model. In this section the model will be described and the assumptions which will be made during the analyses will be discussed. Additional assumptions or exceptions to the assumptions made in this section will be noted when they become necessary.

Figure 2 shows the basic geometric configuration for the finned tube radiator.

FIGURE 2

For purposes of discussion the terms fin strip or strip will be used to refer to a section of the fin such as the sections labeled "A" and "B" in Figure 2. The term fin will refer to the complete fin which is formed by all of the fin strips taken together.

Based on the work of Sparrow et al.⁵ and French et al.⁶, longitudinal conduction will be neglected, i.e., each fin strip is assumed to be thermally isolated from the adjoining fin strip. It will also be assumed that at any point in a given strip the temperature is constant across the fin thickness. Because of these assumptions the treatment of the fin reduces to solving a one-dimensional heat conduction equation point by point along the fin base.

The radiation coming from space will be introduced into the fin equation by means of an equivalent sink temperature. The sink temperature will be assumed constant for the entire fin. This means that the rate of radiant energy striking the fin per unit area is a constant over the entire fin.

The radiator will be assumed to be composed of a series of tubes connected in parallel, and all the tubes will be assumed to have the same fluid flow rate. This assumption in conjunction with the assumption of a constant sink temperature implies, as a result of symmetry, that an adiabatic point exists midway between the tubes. Because of this only one tube is considered since the results for one tube hold for all the other tubes.

Based on the work of Stockman et al.⁴, the fin thermal conductivity and emissivity will be assumed constant. Radiant interaction between the tube and the fin will be neglected. The work of MacKay et al.¹ shows this assumption to be valid for environmental control

radiators having a configuration such as is shown in Figure 2. Another configuration which is used for environmental control radiators is shown in Figure 3. In such a radiator the top surface radiates and the bottom surface is thermally insulated. For this configuration there is no radiant interaction between the

FIGURE 3

fin and tube.

Longitudinal conduction is neglected in the fluid, and the thermal

properties are assumed to be constant. The convective heat transfer coefficient is assumed to be constant.

The analyses will consider only heat rejection systems, and will not be applicable to energy collection systems.

CHAPTER IV

DERIVATIONS AND SOLUTIONS OF THE EQUATIONS
DESCRIBING THE FINED TUBE RADIATOR

As was discussed in the model description, the treatment of the fin involves solving a one-dimensional heat conduction equation point by point along the fin base. In order to carry out this procedure it is necessary to obtain the differential equation governing the one-dimensional fin, and to obtain the solution of this equation. The analysis of the complete radiator will require, in addition to the solution of the fin equation, an equation to account for the drop in temperature from the fluid to the fin base, and a heat balance on the fluid which will introduce the variation in fluid temperature in the longitudinal direction. These equations, alone with an application of Fourier's law, will yield four equations which allow the determination of the radiator length required to reject a given amount of heat. These equations, therefore, constitute the solution of the space radiator design problem.

In order to derive the differential equation governing one-dimensional conduction in the fin strip, consider a fin strip having length ΔL , thickness t , and thermal conductivity k .

FIGURE 4

Consider the small parallelepiped having length ΔL and width Δx

FIGURE 5

From Fourier's law of heat conduction:

$$q_1 = -kt\Delta y \left. \frac{dT}{dx} \right|_x$$

$$q_2 = -kt\Delta y \left. \frac{dT}{dx} \right|_{x + \Delta x}$$

where the subscript after the derivative specifies the point at which the derivative is evaluated.

From the Stefan-Boltzmann equation for thermal radiation one has:

$$q_3 = \Delta x \Delta L \sigma E (T^4 - T_s^4)$$

where T is the mid-point temperature for the block, T_s is the equivalent sink temperature, and σ is the Stefan-Boltzmann constant. The term E is used, rather than ϵ , to signify that heat can be radiated from both sides of the fin, in which case $E = 2\epsilon$.

Writing a heat balance for the parallelepiped, one obtains:

$$q_1 = q_2 + q_3$$

$$-kt\Delta L \left. \frac{dT}{dx} \right|_x = -kt\Delta L \left. \frac{dT}{dx} \right|_{x+\Delta x} + \Delta x \Delta L \sigma E (T^4 - T_s^4)$$

Expanding $\left. \frac{dT}{dx} \right|_{x+\Delta x}$ in a Taylor series about the point x

and dropping the x subscript one obtains:

$$-kt\Delta L \frac{dT}{dx} = -kt\Delta L \left(\frac{dT}{dx} + \frac{d^2T}{dx^2} \Delta x + \frac{d^3T}{dx^3} (\Delta x)^2 + \dots \right) + \Delta x \Delta L \sigma E (T^4 - T_s^4)$$

$$-\frac{d^2T}{dx^2} = \frac{d^3T}{dx^3} \Delta x + \frac{d^4T}{dx^4} (\Delta x)^2 + \dots + \left(-\frac{\sigma E}{kt} (T^4 - T_s^4) \right).$$

Letting Δx approach zero one has:

$$\frac{d^2T}{dx^2} = \frac{\sigma E}{kt} (T^4 - T_s^4) \quad (1)$$

Equation 1 is the governing differential equation for the one-dimensional radiating fin. In order to obtain the first integral of this equation, let $P = dT/dx$. Substituting this expression in Equation 1, using the fact that $dx = dT/P$, separating variables, and inte-

grating one obtains:

$$P^2 = \frac{2\alpha E}{kt} \left(\frac{T^5}{5} - T_s^4 T + C_1 \right) \quad (2)$$

where C_1 is the constant of integration.

$$\frac{dT}{dx} = - \left(\frac{2\alpha E}{kt} \left[\frac{T^5}{5} - T_s^4 T + C_1 \right] \right)^{1/2} . \quad (3)$$

The minus sign is taken since the temperature gradient is known to be negative.

The existence of an adiabatic fin mid-point gives the following boundary condition:

$$\frac{dT}{dx} = 0 \quad \text{at } x = W \quad (4)$$

$$T = T_{mp} \quad \text{at } x = W$$

where W is the fin half width. Using this boundary condition in Equation 3, one obtains;

$$C_1 = T_{mp} T_s^4 - \frac{T_{mp}^5}{5} . \quad (5)$$

Substituting this expression for C_1 in Equation 3, one obtains:

$$\frac{dT}{dx} = - \left(\frac{2\alpha E}{kt} \left[\frac{T^5}{5} - T_s^4 T - \frac{T_{mp}^5}{5} + T_{mp} T_s^4 \right] \right)^{1/2} . \quad (6)$$

At this point it is possible to obtain an equation which relates the fin base temperature, the fluid bulk temperature, and the fin mid-

point temperature. This equation will be determined before finding the integral of Equation 6. From Fourier's law of heat conduction one has:

$$\Delta Q = -kt\Delta L \left. \frac{dT}{dx} \right|_{x=b} \quad (7)$$

where ΔQ is the heat transferred across the base of a fin strip, and ΔL is the width of the fin strip. Using Equation 6 (evaluated at $T = T_b$) in Equation 7, one obtains:

$$T_s^4 T_{mp} - \frac{T_{mp}^5}{5} = T_s^4 T_b - \frac{T_b^5}{5} + \frac{1}{2\sigma Ekt} \left(\frac{\Delta Q}{\Delta L} \right)^2 \quad (8)$$

For a given sink temperature and ratio of heat rejection rate to fin length, Equation 8 relates the fin base temperature and fin mid-point temperature. It would be useful to eliminate $(\Delta Q/\Delta L)$ in favor of the bulk fluid temperature, as it is the bulk fluid temperature which is known. In order to do this an overall heat transfer coefficient is introduced to account for the temperature drop from the fluid bulk temperature to the fin base temperature. The overall heat transfer coefficient accounts for both the tube wall resistance and the convective heat transfer coefficient.

$$2\Delta Q = HAL (T_f - T_b) \quad (9)$$

The two multiplying ΔQ occurs from considering two fins to be attached to each tube, and from defining ΔQ to be the heat rejected by each fin. The overall heat transfer coefficient (H) is based on tube length rather than on the tube surface area which is the more

common definition. Using Equation 9 to eliminate $\Delta Q/\Delta L$ in Equation 8, one obtains:

$$T_s^4 T_{mp} - \frac{T_{mp}^5}{5} = T_s^4 T_b - \frac{T_b^5}{5} + \frac{H^2(T_f - T_b)^2}{8(\alpha E k t)} \quad (10)$$

Equation 12 exhausts the information which can be extracted from Equation 3. It is now necessary to obtain the second integral of Equation 1.

Lieblein³ points out that if the sink temperature is low, it can be assumed to be zero without much error. He states that if the sink temperature is less than 40 percent of the fin base temperature, the assumption of a zero sink temperature will produce less than a 5 percent error in the calculated heat rejection rate. Because of this, the zero sink temperature case can have fairly wide application. Because the assumption of a zero sink temperature results in a significant simplification of the governing equations, it seems useful to treat this as a special case.

Special Case: The Sink Temperature Equal to Zero

For a sink temperature equal to zero Equation 10 becomes Equation 11 and Equation 3 becomes Equation 12.

$$\frac{T_b^5}{5} - \frac{T_{mp}^5}{5} = \frac{H^2(T_f - T_b)^2}{8(\alpha E k t)^2} \quad (11)$$

$$\frac{dT}{dx} = - \left(\frac{2\alpha E}{k t} \left(\frac{T_b^5}{5} - \frac{T_{mp}^5}{5} \right) \right)^{1/2} \quad (12)$$

The substitution $C_1 = -\frac{T_{mp}^5}{5}$ has been made in Equation 12.

One must now obtain the integral of Equation 12. Equation 12 can be written as a quadrature, as follows:

$$W = - \int_{T_b}^{T_{mp}} \frac{dT}{\left(\frac{2\alpha E}{kt}\right)^{1/2} \left(\frac{T^5}{5} - \frac{T_{mp}^5}{5}\right)^{1/2}}$$

$$W = - \left(\frac{5kt}{2\alpha E}\right)^{1/2} \int_{T_b}^{T_{mp}} \frac{dT}{T^{5/2} \left(1 - \left(\frac{T_{mp}}{T}\right)^5\right)^{1/2}} \quad (13)$$

where W is the fin half-width.

$$\text{Define } \xi = \left(\frac{T_{mp}}{T}\right)^5$$

$$\text{then } d\xi = -5\xi^{1.2} \frac{dT}{T_{mp}}$$

Making these substitutions in Equation 13

$$W = \left(\frac{kt}{10\alpha E T_{mp}^3}\right)^{1/2} \int_{\xi_b}^1 \frac{d\xi}{\xi^{0.7} (1 - \xi)^{1/2}} \quad (14)$$

Equation 14 can be written as the difference of two integrals as follows:

$$W = \left(\frac{kt}{10\alpha E T_{mp}^3}\right)^{1/2} \left[\int_0^1 \frac{d\xi}{\xi^{0.7} (1 - \xi)^{1/2}} - \int_0^{\xi_b} \frac{d\xi}{\xi^{0.7} (1 - \xi)^{1/2}} \right] \quad (15)$$

The first integral is the complete Beta Function $\beta(.3,.5)$ and the second integral is the incomplete Beta Function $\beta_{\xi_b}(.3,.5)$, where the incomplete Beta Function $\beta_z(M,N)$ is defined to be:

$$\beta_z(M,N) = \int_0^z x^{(M-1)} (1-x)^{(N-1)} dx$$

and the complete Beta Function results when Z equals 1.

In the notation above equation 15 becomes:

$$W = \left(\frac{kt}{100ET_{mp}^3} \right)^{1/2} \left[\beta(.3,.5) - \beta_{\xi_b}(.3,.5) \right] \quad (16)$$

Equation 16 was first found by Mackay et al.¹, but was not used because $\beta_z(.3,.5)$ is not tabulated in the literature. Mackay et al.¹ presented a graphical technique for approximating the Beta function. An analytic technique is presented in Appendix A which allows calculating Beta functions to any desired accuracy. The results of that analysis were programmed in Fortran IV for the IBM 7040. This program is listed in Appendix C and is called BETAF. For convenience a table of $\beta_z(.3,.5)$ is presented in Appendix B.

All of the equations necessary to analyze a single fin strip, for a zero sink temperature environment, have now been obtained. Before the complete radiator can be designed, however, one more equation is necessary. In order to account for the variation in fluid temperature along the tube, a heat balance on the fluid is introduced. Neglecting axial heat conduction in the fluid and noting that ΔQ represents the heat radiated by the fin strip, which is one half the heat transferred from the fluid, one obtains:

$$\Delta T_f = \frac{2\Delta Q}{\dot{M}C_p} \quad (17)$$

where ΔT_f is the temperature drop from inlet to outlet of a strip.

For convenience in explaining the design procedure, Equations 11, 16, 9, and 17 are restated below.

$$\frac{T_b^5}{5} - \frac{T_{mp}^5}{5} = \frac{1}{8} \left(\frac{H^3}{\sigma E k t} \right) (T_f - T_b)^2 \quad (11)$$

$$W = \left(\frac{kt}{10\sigma E T_{mp}^3} \right)^{1/2} \left[B(.3, .5) - B_{\xi_b}(.3, .5) \right] \quad (16)$$

$$\Delta L = \frac{2\Delta Q}{H(T_f - T_b)} \quad (9)$$

$$\Delta T_f = \frac{2\Delta Q}{\dot{M}C_p} \quad (17)$$

In specifying the design problem the fluid flow rate, fluid inlet temperature, fluid properties, fin properties, fin half width, heat rejection rate, and tube inner and outer diameters would be given. It is required to find L , i.e., the length of radiator required to obtain the specified heat rejection rate. The first step in the design procedure is the determination of the heat transfer coefficient. This can be determined by using any of the equations available in the literature, e.g., by assuming a constant Nusselt number. The next step is to divide the radiator into a number of strips. The actual number is at the discretion of the designer. Table number 2, presented in the discussion of results at the end of this chapter, gives some indication as to the number of strips which should be used. Once the number of strips has been selected, the heat to be rejected by each strip

is determined by dividing the total heat rejection by the number of strips. This quantity is ΔQ and is the same for all strips.

Starting with the first strip, one solves Equations 11 and 16 simultaneously to determine T_b and T_{mp} . Knowing T_b and T_f , Equation 9 yields the length of the first strip, i.e., ΔL for the first strip. ΔL is the length of strip required in order to reject ΔQ . Equation 17 is now used to determine the fluid temperature for the second strip. Equations 11 and 16 are then solved for T_b and T_{mp} using the new value of T_f . Equation 9 yields the length of the second strip. Equation 17 is used to find the fluid temperature for the third strip. This procedure is continued until the lengths of all of the strips have been determined. Note that as the fluid temperature decreases, the length of the strip must increase in order to produce the same ΔQ . Summing all of the fin strip lengths yields the total radiator length.

The procedure just described was incorporated into a computer program. Before this, the equations were made dimensionless as follows:

$$\text{Dimensionless length} = \left[\frac{100\epsilon T_f^3}{kt} \right]^{1/2} (\text{length})$$

$$\text{Dimensionless heat} = \frac{5}{ktT_{f1}} (\text{heat})$$

$$\text{Dimensionless Temperature} = \frac{1}{T_{f1}} (\text{temperature})$$

Using these expressions, Equations 9, 16, 11, and 17 become respectively:

$$\tau_b^5 - \tau_{mp}^5 = \frac{1}{4} \gamma^2 (\tau_f - \tau_b)^2 \quad (18)$$

$$\tilde{W} = \frac{1}{\tau_{mp}^{3/2}} \left[\beta(.3, .5) - \beta_{\xi_0}(.3, .5) \right] \quad (19)$$

$$\Delta \tilde{L} = 2 \left[\frac{\Delta \tilde{Q}}{\gamma(\tau_f - \tau_b)} \right] \quad (20)$$

$$\Delta \tau_f = \frac{2}{5} \alpha \Delta \tilde{Q} \quad (21)$$

where the forms for the dimensionless constants α and γ are:

$$\alpha = \frac{kt}{MC_p}$$

$$\gamma = H \left(\frac{5}{2\alpha Ekt T_{f1}^3} \right)^{1/2} .$$

Equations 18, 19, 20, and 21 were programmed for the IBM 7040. This Program, FTUBE0, is listed in Appendix C. Inputs to the program are α , γ , $\tilde{W}$, the number of strips into which the radiator is to be divided, and the heat to removed from the fluid i.e., $2\tilde{Q}$. The principal output is $\tilde{L}$.

The operation of the program is as follows:

- I The heat to be rejected is divided by the number of strips to obtain the heat rejected per strip
- II Equation 21 is used to obtain the difference in fluid bulk temperature from inlet to outlet of a strip. One half of this difference is subtracted from the fluid inlet temperature ($\tau_f = 1$) to get the average fluid temperature for the

first strip.

III An estimate for τ_b is made and Equation 18 is used to find τ_{mp} . With this starting information, entry is made into a Newton-Raphson iteration scheme using Equation 19. Each iteration yields a new value of τ_b which is used in Equation 18 to obtain τ_{mp} . The iteration continues until the absolute value of the difference between the left and right side of Equation 19 is less than 10^{-7} .

IV Once convergence has occurred, Equation 21 is used to find $\Delta\tilde{L}$, Equation 21 is used to find a new fluid temperature, and the process above is repeated. The temperature drop from fluid to fin base, for each strip, is used to obtain the first guess at the fin base temperature for the next strip. As each $\Delta\tilde{L}$ is calculated it is summed to the previously calculated $\Delta\tilde{L}$'s to give the total length of the fin.

At this point in the analysis, the equations required to design a space radiator, for a zero sink temperature, have been determined. These equations have been incorporated into a computer program which designs a space radiator. It is now necessary to return to the more general problem, i.e., the design problem when the sink temperature is not equal to zero.

Solution for Sink Temperatures Not Equal to Zero

In order to obtain the design solution for space radiators when $T_s \neq 0$, it is necessary to consider Equations 6, 9, 10, and 17. These equations are valid regardless of the value of T_s . Equations 9,

10, and 17 are in the form required by the design problem. Equation 6 must be integrated before the design problem can be solved. Equation 6 is restated as Equation 22.

$$\frac{dT}{dx} = - \left(\frac{2\sigma E}{kt} \left[\frac{T^5}{5} - T_s^4 T - \frac{T_{mp}^5}{5} + T_s^4 T_{mp} \right] \right)^{1/2} \quad (22)$$

Equation 22 can be written as a quadrature, as follows:

$$W = - \int_{T_b}^{T_{mp}} \frac{dT}{\left(\frac{2\sigma E}{kt} \right)^{1/2} \left[\frac{T^5}{5} - T_s^4 T - \frac{T_{mp}^5}{5} \right]^{1/2}} \quad (23)$$

$$\text{Let } \zeta = \frac{T_{mp}}{T} ; \quad dT = -T_{mp} \frac{d\zeta}{\zeta^2}$$

$$\text{and } \theta_s = \frac{T_s}{T_{mp}}$$

Using these transformations, Equation 23 becomes

$$W = \frac{1}{\left(\frac{2\sigma E}{5kt} \right)^{1/2} T_{mp}^{3/2}} \int_{\zeta_b}^1 \frac{1/2 d\zeta}{\left[1 + (5\theta_s^4 - 1) \zeta^5 - 5\theta_s^4 \zeta^4 \right]^{1/2}} \quad (24)$$

For convenience in writing, let $5\theta_s^4 - 1 = B$, and $-5\theta_s^4 = C$. Consider the denominator of the integrand and expand it in a Binomial series.

$$\begin{aligned} \left[1 + (B\zeta^5 + C\zeta^4) \right]^{1/2} &= \left[1 - \frac{1}{2} (B\zeta^5 + C\zeta^4) + \frac{1}{2} \left(\frac{3}{2} \right) (B\zeta^5 + C\zeta^4)^2 \right. \\ &\quad \left. - \frac{1}{2} \frac{3}{2} \frac{5}{2} (B\zeta^5 + C\zeta^4)^3 + \dots \right] \end{aligned} \quad (25)$$

Multiplying the series by $\zeta^{1/2}$ one obtains:

$$\begin{aligned} \zeta^{1/2} [1 + (B\zeta^5 + C\zeta^4)] &= \left[\zeta^{1/2} - \frac{1}{2} (B\zeta^5 + C\zeta^4) \right. \\ &+ \frac{\frac{1}{2} \frac{3}{2} \zeta^{1/2} (B\zeta^5 + C\zeta^4)^2}{2!} \\ &\left. - \frac{\frac{1}{2} \frac{3}{2} \frac{5}{2} \zeta^{1/2} (B\zeta^5 + C\zeta^4)^3}{3!} + \dots \right] \quad (26) \end{aligned}$$

This series is the expansion of the integrand in Equation 24.

Consider the integral

$$F(\zeta) = \int_0^\zeta \frac{\zeta^{1/2} d\zeta}{[1 + (B\zeta^5 + C\zeta^4)]^{1/2}} \quad (27)$$

This is the same integral as Equation 24, except the limits of integration are different. In order to obtain this integral, one integrates Equation 26 term by term. This gives:

$$\begin{aligned} F(\zeta) &= \left(\frac{2}{3} \zeta^{3/2} - \frac{1}{2} \left[\frac{2}{13} B \zeta^{13/2} + \frac{2}{11} C \zeta^{11/2} \right] \right. \\ &+ \frac{\frac{1}{2} \binom{3}{2}}{2!} \left[\frac{2}{23} B^2 \zeta^{23/2} + \frac{4}{21} BC \zeta^{21/2} + \frac{2}{19} C^2 \zeta^{12/2} \right] \\ &- \frac{\frac{1}{2} \binom{3}{2} \binom{5}{2}}{3!} \left[\frac{2}{33} B^3 \zeta^{33/2} + \frac{6}{31} B^2 C \zeta^{31/2} + \frac{6}{29} BC^2 \zeta^{29/2} \right. \\ &\left. + \frac{2}{27} C^3 \zeta^{27/2} \right] + \dots \end{aligned}$$

$$F(\zeta) = \sum_{j=1}^{\infty} \left\{ \left(\prod_{n=1}^j \frac{\left(\frac{3}{2} - n\right)}{1/2} \right) \sum_{k=1}^j \left[\frac{2B^{(j-k)}C^{(k-1)}}{(10j-2k-5)(k-1)!(j-k)!} \zeta^{\frac{10j-2k-5}{5}} \right] \right\} \quad (2)$$

where $B^{(j-k)} = 0$, for $B = 0$, and $C^{(k-1)} = 0$ for $C = 0$. These two conditions make Equation 28 valid for all sink temperatures such that $\theta_s < 1$, i.e., heat rejection systems. Series 28 is valid for $0 \leq \zeta < 1$ and $0 \leq \theta_s < 1$; however, the value of ζ should be limited to a maximum of approximately 0.8 in order to insure rapid convergence. Because the ultimate goal is to find the integral of Equation 25. A series must now be found which is valid in the range $0.8 \leq \zeta \leq 1$. Note that the cut off point $\zeta = .8$ was selected merely for convenience.

Again consider the integral

$$F(\zeta) = \int_{\zeta}^1 \frac{\zeta^{1/2} d\zeta}{\zeta^5 [1 + (B\zeta^5 + C\zeta^4)]^{1/2}} \quad (29)$$

Let $u = 1 - \zeta$; $du = -d\zeta$, then

$$H(u) = \int_0^u \frac{(1-u)^{1/2} du}{u^{1/2} (b + cu + du^2 + eu^3 + fu^4)^{1/2}} \quad (30)$$

Where:

$$\begin{aligned} b &= -(4C + 5B) \\ c &= (6C + 10B) \\ d &= -(4C + 10B) \\ e &= (C + 5B) \\ f &= -B \end{aligned}$$

For convenience in writing, let

$$P = (b + cu + du^2 + eu^3 + fu^4)^{-1/2}$$

Expanding $(1 - u)^{1/2}P$ in a Taylor Series about $u = 0$, i.e., about $\zeta = 1$, one obtains:

$$(1 - u)^{1/2}P = N(u) = (N(0) + N'(0)u + \frac{N''(0)u^2}{2!} + \dots) \quad (31)$$

Using this expansion in Equation 30, one has:

$$H(u) = \int_0^u u^{-1/2} \left\{ N(0) + N'(0)u + \frac{N''(0)u^2}{2!} + \frac{N'''(0)u^3}{3!} + \dots \right\} du .$$

Integrating term by term one obtains:

$$H(u) = \left(2N(0)u^{1/2} + \frac{2}{3} N'(0)u^{3/2} + \frac{2}{5} \left(\frac{N''(0)}{2!} \right) u^{5/2} + \dots \right) \quad (32)$$

The form of the constants $N(0)$, $N'(0)$, $N''(0)$, etc., must now be determined. These terms can be found by differentiating $(1 - u)^{1/2}P$.

This yields, for the first four terms:

$$N(0) = P(0)$$

$$N'(0) = -\frac{1}{2} P(0) + P'(0)$$

$$N''(0) = -\frac{1}{4} P(0) - P'(0) + P''(0)$$

$$N'''(0) = -\frac{3}{8} P(0) - \frac{3}{4} P'(0) - \frac{3}{2} P''(0) + P'''(0)$$

Where:

$$P(0) = [-(4C + 5B)]^{-1/2}$$

$$P'(0) = -\frac{1}{2} [-(4C + 5B)]^{-3/2} [6C + 10B]$$

$$P''(0) = \frac{3}{4} [-4C + 5B]^{-5/2} [6C + 10B]^2 + [-4C + 5B]^{-3/2} [4C + 10B]$$

$$P'''(0) = -\frac{15}{8} [- (4C + 5B)]^{-7/2} [6C + 10B]^3 - \frac{9}{2} [- (4C + 5B)]^{-5/2}$$

$$[6C + 10B][4C + 10B] - 3 [-(4C + 5B)]^{-3/2}$$

$$[C + 5B]$$

(33)

Equations 33 indicate the increasing complexity of the constants in Equation 32. It should not be too surprising to note, in view of this, that all attempts at finding the general germ for the series failed.

At this point it is possible to return to the actual integral in question i.e., Equation 24. Using $H(u)$ and $F(\zeta)$, one can write

$$W = \frac{1}{\left(\frac{2\phi E}{5kt}\right) T_{mp}^{3/2}} \left[\lambda(\zeta, \theta_s) \right] \quad (34)$$

where for any given θ_s :

$$\lambda(\zeta, \theta_s) = [F(0.8) + H(0.2) - F(\zeta)]$$

For $0 \leq \zeta < .8$

and

$$\lambda(\zeta, \theta_s) = H(1-\zeta)$$

For $0.8 \leq \zeta \leq 1$.

The expressions for $\lambda(\zeta, \theta_s)$ have been programmed in Fortran IV for the IBM 7040. Eleven terms of the series $H(u)$ are included. Values of $\lambda(\zeta, \theta_s)$ are returned which are accurate up to and including the sixth decimal place. This subroutine, INTGRA, is listed in Appendix C.

Equations 9, 10, 17, and 34 are sufficient to design a space radiator. The design procedure parallels the design procedure for the zero sink temperature case. The only differences are that the sink temperature is specified in the statement of the problem, Equation 34 is used in place of Equation 16, and Equation 10 is used in place of Equation 11.

Using the dimensionless variables and dimensionless parameters listed on page 24. Equations 9, 10, 17, and 34 become, respectively,

$$\Delta \tilde{L} = 2 \left(\frac{\Delta \tilde{Q}}{\gamma(\tau_f - \tau_b)} \right) \quad (35)$$

$$\tau_b^5 - \tau_{mp}^5 = 5\tau_s^4 \tau_b - 5\tau_s^4 \tau_{mp} + \frac{1}{4} \gamma^2 (\tau_f - \tau_b)^2 \quad (36)$$

$$\Delta \tau_f = \frac{2}{5} \alpha \Delta \tilde{Q} \quad (37)$$

$$\tilde{W} = \frac{5\lambda(\zeta, \theta_s)}{\tau_{mp}^{3/2}} \quad (38)$$

The design procedure, using the dimensionless equations, was incorporated into a computer program written in Fortran IV for the IBM 7040. This program, FTBSNC, is listed in Appendix C. Inputs to the program are α , γ , $\tilde{W}$, τ_s , the number of strips into which the

radiator is to be divided, and the heat to be removed from the fluid (i.e., $2\tilde{Q}$). The principal output is the dimensionless length of the radiator, $\tilde{L}$.

The operation of the Program is as follows:

- I The heat to be rejected is divided by the number of strips to obtain the heat rejected per strip.
- II Equation 37 is used to obtain the difference in fluid bulk temperature from inlet to outlet of a strip. One half of this temperature difference is subtracted from the fluid inlet temperature ($\tau_f = 1$), to get the average fluid temperature for the first strip.
- III An estimate for τ_b and τ_{mp} is made. With this information entry is made into a Newton-Raphson Iteration in two variables, which solves Equations 36 and 38 simultaneously for τ_b and τ_{mp} . The convergence criterion for the iteration is that the absolute value of the difference between the left and right sides of both Equations 36 and 38 be less than 10^{-5} . Results from the program indicate that for this convergence criterion the values found for τ_b and τ_{mp} are accurate to at least 10^{-6} .
- IV Once convergence has occurred Equation 35 is used to find $\Delta\tilde{L}$. Equation 37 is used to find a new fluid temperature, and the process above is repeated. The temperature drops

from the fluid to the fin base and from the fin base to the fin midpoint, for any strip, are used to obtain the first guesses at τ_b and τ_{mp} for the next strip. As each $\Delta\tilde{L}$ is calculated it is summed to the previously calculated $\Delta\tilde{L}$'s to give the total length of the fin.

RESULTS

The results obtained using the computer programs developed in this chapter were compared with the results obtained using the computer program developed by French and Shield⁶. The program developed by French and Shield is a performance program which calculates the heat rejection rate for a given radiator. Since the two programs developed in this chapter are design programs, the heat rejection rate must be known. The design programs calculate the radiator length. For comparison purposes, the heat rejection rate calculated using the performance program were taken as a specified quantity for the design calculations. The design program calculates the radiator length required to obtain this specified heat rejection rate. These calculated lengths were then compared to the lengths specified for the performance program. The degree of agreement between these two lengths were taken as the measurement of accuracy of the design programs.

A total of six cases were compared, of which four were for a zero sink temperature. These six cases are described below. The description and dimensions given are based on the model shown in Figure 6.

FIGURE 6

Cases 1 and 2

These cases were taken from French and Shield⁶, who selected these configurations so as to "involve operating conditions that impose the most severe longitudinal temperature gradients, yet which are within practical environmental control system operating ranges."

The fluid inlet temperature for both radiators is 250 F. A fluid flow rate of 0.6 lb_m/hr of a 60 percent ethylene glycol and 40 percent water solution is used. The radiator material is 3¹/₂ stainless steel and has a constant surface emissivity of 0.83. The dimensions and heat rejections are:

	Radiator 1	Radiator 2
W (inches)	1.8	3.6
t (inches)	0.020	0.020
l (inches)	0.27	0.27
tube inner diameter (inches)	0.18	0.18
tube wall thickness (inches)	0.035	0.035
heat rejections rate (BTU/hr)	101.44	113.86
outlet fluid temperature (°F)	31.22	0.06
L (inches)	36.0	36.0

Case 3

This case was taken from Russell¹⁹. Constant material properties are assumed. The fin thermal conductivity is 118.0 BTU/hr-ft-F and the emissivity is 0.95. The fluid conductivity is 0.225 BTU/hr-ft-F,

the fluid heat capacity is $0.7 \text{ BTU/lb}_m\text{-F}$, and the fluid flow rate is $10.9 \text{ lb}_m/\text{hr}$. The fluid inlet temperature is 104.0 F . The dimensions and heat rejection rate are:

$$W = 2.41 \text{ in.}$$

$$t = 0.030 \text{ in.}$$

$$l = 0.374 \text{ in.}$$

$$\text{tube inner diameter} = 0.18 \text{ in.}$$

$$\text{tube wall thickness} = 0.035 \text{ in.}$$

$$\text{heat rejection rate} = 471.867 \text{ BTU/hr}$$

$$L = 10.5 \text{ ft.}$$

This radiator is patterned after the first proposed Apollo radiator as described in Reference 20.

The calculations for the remaining three cases were made using the computer program developed by French and Shield⁶.

Case 4

The specifications for the fourth case are:

Fin Specifications

$$\text{thermal conductivity} = 100.0 \text{ BTU/hr-ft-F}$$

$$\text{emissivity} = 0.9$$

$$t = 0.04 \text{ in.}$$

$$W = 4.20 \text{ in.}$$

$$L = 25.0 \text{ ft.}$$

Tube Specifications

inner diameter = 0.24 in.

$l = 0.547$ in.

tube wall thickness = 0.04 in.

Fluid Specifications

Mass flow rate = $46.54 \text{ lb}_m/\text{hr}$

$C_p = 1.0 \text{ BTU/lb}_m\text{-hr}$

thermal conductivity = $0.065 \text{ BTU/hr-ft-R}$

fluid inlet temperature = 80.0 F

heat rejection rate = 1257.75 BTU/hr

Case 5

Case number five is identical to case number three, except the sink temperature is 347 R and the heat rejection rate is 400.368 BTU/hr .

Case 6

Case number six is identical to case number four, except the sink temperature is 352 R and the heat rejection rate is 1011.35 BTU/hr .

The following procedure was used to set up the six cases just mentioned for use in the design programs developed in this chapter.

A constant Nussult Number ($N_u = 3.66$) was used to determine the convective heat transfer coefficient. The tube was assumed to have an outer radius equal to the distance between the tube center line and the fin. These factors were combined into an overall heat transfer coefficient.

Constant material properties were used. For cases numbers one

and two, all of the material properties were evaluated at the average of inlet and outlet temperatures.

Table 1 is a comparison of results. "Percent Error" was calculated assuming the results from the performance program to be correct. The values under "L" are the results obtained using the design programs developed in this chapter. The fins were divided into 100 strips.

The very small discrepancies indicated by the percent error entries, indicate the validity of the approach taken in this work. It should be noted that cases one and two were specifically designed so as to make the longitudinal temperature gradients large and thus bring into question the assumption which neglected longitudinal heat conduction. The assumption that the heat transfer coefficient

CASE	SPECIFIED RADIATOR LENGTH	L (CALCULATED)	% ERROR
1	36.0 in	36.948 in	+2.633
2	36.0 in	36.110 in	+0.305
3	10.5 ft	10.550 ft	+0.476
4	25.0 ft	24.920 ft	-0.320
5	10.5 ft	10.450 ft	-0.476
6	25.0 ft	24.900 ft	-0.400

TABLE 1

is a constant also appears to introduce negligible effects. Part of the results generated by the program developed by French and Shield were values of the convective heat transfer coefficient along the tube. For case 3, the heat transfer coefficient was equal to the value given by $N_u = 3.66$ along 90 percent of the tube. For case 4 the heat trans-

fer coefficient was equal to the value given by $N_u = 3.66$ along only 4 percent of the tube and was three times higher at the entry to the tube. Although these values may not be accurate, they do introduce a variable heat transfer coefficient. If the variation in the heat transfer coefficient has a significant effect on the results, case 4 should result in a large error. In view of the close agreement between results, the assumption that the heat transfer coefficient is constant appears to be a valid approximation.

In order to study the effect which the number of strips has on the results, case number one was run for differing number of strips. Table 2 shows the results. The resulting lengths are shown in dimensionless form. They may be converted into feet by dividing $\tilde{L}$ by 18.548642.

STRIPS	$\tilde{L}$
2	56.993468
10	58.154469
25	58.196735
100	58.204323
150	58.204597
200	58.204713
250	58.204763
300	58.204780

TABLE 2

Increasing the number of strips from 100 to 300 increases L by 0.00076 percent. This indicates that calculations using 100 strips should be adequate.

CHAPTER V

APPROXIMATE CLOSED FORM SOLUTIONS

A large number of radiators were designed using the programs developed in Chapter IV. These design cases considered a wide range of design parameters. A result of these design calculations was that the ratio of fin mid-point temperature to fin base temperature was approximately constant along the fin base. This observation suggests that the fin mid-point temperature can be approximated by Equation 39.

$$\tau_{mp} = \phi \tau_b \quad (39)$$

Figure 7 shows a plot of the results for one of the cases considered. The solid lines are the results obtained from the design programs, and the dotted line is the mid-point distribution generated by Equation 39.

FIGURE 7

For purposes of discussion, the temperature distribution generated by the design program will be called the "actual distributions", while the temperature distribution obtained from Equation 39 will be called the "approximate distribution". It should be mentioned that the case illustrated in Figure 7 is, of the many cases considered, the one which showed the poorest agreement between the actual and approximate distributions.

With reference to Figure 7, note that the actual curves get closer together as one moves along the base towards the lower base temperatures. This property is preserved by the approximation. Further, the general shape of the curve for τ_{mp} is preserved. With a simple assumption such as Equation 39 it can be seen that the approximate mid-point temperature distribution can be made to coincide with the actual distribution at only one point. If they are made to coincide at the inlet, the resulting temperature difference between the fin base and fin mid-point will be too large everywhere except at the inlet. Similarly if the actual and approximate distributions are made to coincide at the outlet conditions the opposite will occur. This would suggest that neither location is the best at which to determine ϕ , for the former would result in too short a fin, and the latter in too long a fin. A parametric study will be undertaken later which will establish the best location at which to evaluate ϕ .

Using the approximation given by Equation 39, one can find analytic solutions to the design problem. As was done in the previous chapter, the case where the sink temperature is zero will be considered first. The design solution for those cases where the sink temperature is not zero will then be found.

Special Case: The Sink Temperature Equal to Zero

From Equation 12, the temperature gradient at the fin base is

$$\left. \frac{dT}{dx} \right|_{x=0} = - \left(\frac{2\sigma E}{5kt} (T_b^5 - T_{mp}^5) \right)^{1/2} \quad (40)$$

Using Fourier's Law, Equation 39, Equation 40, and collecting terms one has:

$$dq = \left[\frac{2\sigma Ekt(1-\phi^5)}{5} \right]^{1/2} T_b^{5/2} dy \quad .$$

Setting

$$\delta = \left[\frac{2\sigma Ekt(1-\phi^5)}{5} \right]^{1/2}$$

$$dq = \delta T_b^{5/2} dy \quad . \quad (41)$$

Considering two fins to be connected to each tube and neglecting longitudinal conduction in the fluid, a simple heat balance yields

$$2dq = -\dot{M}C_p dT_f \quad . \quad (42)$$

Introducing an overall heat transfer coefficient, the following Equation can be written to relate the heat transferred from the fluid to the fluid bulk temperature and fin base temperature;

$$2dq = H(T_f - T_b) dy \quad . \quad (43)$$

Eliminating dq between Equation 41 and Equation 43 one obtains:

$$\frac{2\delta}{H} T_b^{5/2} = T_f - T_b \quad . \quad (44)$$

Eliminating dq between Equation 48 and Equation 49, and using Equation 44 to eliminate dT_f one obtains:

$$dy = - \frac{\dot{M}C_p}{2\delta} \left[\frac{5\delta}{H} \left(\frac{1}{T_b} \right) + \frac{1}{T_b^{5/2}} \right] dT_b \quad (45)$$

Equations 51 and 52 are now put in dimensionless form. These equations become:

$$dy = - \frac{5}{2\alpha(1-\phi^5)^{1/2}} \left[\frac{5(1-\phi^5)^{1/2}}{\gamma} \frac{1}{\tau_b} + \frac{1}{\tau_b^{5/2}} \right] d\tau_b \quad (46)$$

$$\frac{2(1-\phi^5)^{1/2}}{\gamma} \tau_b^{5/2} + \tau_b = \tau_f \quad . \quad (47)$$

Integrating Equation 46 one obtains:

$$\tilde{L} = - \frac{5}{2\alpha(1-\phi^5)^{1/2}} \left[\frac{5(1-\phi^5)^{1/2}}{\gamma} \ln \tau_b - \frac{2}{3} \left(\frac{1}{\tau_b^{3/2}} \right) \right]_{\tau_{b1}}^{\tau_{b2}} \quad . \quad (48)$$

Note that in Equation 48 the limits of integration are based on the fin base temperatures, while the information available for design is the fluid inlet and outlet temperatures. Equation 47 relates the fluid temperature and the required fin base temperatures.

Equation 48 is a solution to the radiator design problem for the zero sink temperature case. The statement of the design problem

supplies all the information required to find α , γ , τ_{f1} , and τ_{f2} . It will be shown later that sufficient information exists to determine ϕ . Once the method for finding ϕ is established, Equation 47 can be used to find τ_{b1} , and τ_{b2} . This establishes all the information required to find $\tilde{L}$, which is the dimensionless radiator length and thus is the solution to the design problem. Before discussing the method for finding ϕ , the analysis above will be extended to those cases where the radiator is irradiated, i.e., $T_s \neq 0$. The approximation that $T_{mp} = \phi T_b$ will again be used.

Solution For Sink Temperatures Not Equal to Zero

From Equation 22 one has the following expression for the temperature gradient at the fin base:

$$\frac{dT}{dx} = - \left(\frac{2\sigma E}{kt} \right)^{1/2} \left[\frac{T_b^5}{5} - T_s^4 T_b - \frac{T_{mp}}{5} + T_s^4 T_{mp} \right]^{1/2} . \quad (49)$$

Using Fourier's Law, the assumption that $T_{mp} = \phi T_b$, and collecting terms one obtains:

$$dq = \left[2\sigma Ekt(1-\phi) \right]^{1/2} T_s^2 T_b^{1/2} \left[\frac{1-\phi^5}{5(1-\phi)} \left(\frac{T_b}{T_s} \right)^4 - 1 \right]^{1/2} dy .$$

For convenience in writing let

$$D = [2\sigma Ekt(1-\phi)]^{1/2} T_s^2$$

and

$$S = \frac{1-\phi^5}{5(1-\phi)} .$$

Then

$$dq = DT_b^{1/2} \left[S\left(\frac{T_b}{T_s}\right)^4 - 1 \right]^{1/2} dy \quad (50)$$

Neglecting longitudinal heat conduction in the fluid, and writing a simple heat balance, one obtains:

$$2dq = -\dot{M}C_p dT_f \quad (51)$$

The factor 2 comes from considering two fins to be attached to each tube. The heat transferred from the fluid can also be expressed in terms of an overall heat transfer coefficient from fluid to fin base.

$$2dq = H(T_f - T_b) dy \quad (52)$$

where H is based on tube length.

Combining Equations 52 and 50

$$\frac{2D}{H} \left\{ T_b \left[S\left(\frac{T_b}{T_s}\right)^4 - 1 \right] \right\}^{1/2} = T_f - T_b \quad (53)$$

Eliminating dq between Equations 50 and 51, using Equation 53 to eliminate dT_b in favor of dT_f , and integrating from inlet to outlet one obtains:

$$L = - \int_{\frac{T_{b1}}{T_s}}^{\frac{T_{b2}}{T_s}} \frac{\dot{M}C_p}{2H} \frac{5S\left(\frac{T_b}{T_s}\right)^4 - 1}{\left(S\left(\frac{T_b}{T_s}\right)^5 - \frac{T_b}{T_s}\right)} d\left(\frac{T_b}{T_s}\right)$$

$$- \int_{\frac{T_{b1}}{T_s}}^{\frac{T_{b2}}{T_s}} \frac{\dot{M}C_p}{2D} \frac{T_s^{1/2} d\left(\frac{T_b}{T_s}\right)}{\left(\frac{T_b}{T_s}\right)^{1/2} \left(s \left(\frac{T_b}{T_s}\right)^4 - 1\right)^{1/2}} \quad (54)$$

The first integral on the right of the equal sign is:

$$\frac{\dot{M}C_p}{2H} \ln \left[s \left(\frac{T_b}{T_s}\right)^5 - \frac{T_b}{T_s} \right]_{\frac{T_{b1}}{T_s}}^{\frac{T_{b2}}{T_s}} .$$

In order to perform the second integration we make the following transformations:

$$\psi = \frac{1}{s} \left(\frac{T_s}{T_b}\right)^4 ; \quad d\left(\frac{T_b}{T_s}\right) = - \frac{1}{4s^{1/4}} \frac{d\psi}{\psi^{5/4}} .$$

This transforms the second integral into:

$$- \frac{\dot{M}C_p T_s^{1/2}}{\delta D S^{1/8}} \int_{\psi_1}^{\psi_2} \frac{d\psi}{\psi^{5/8} (1-\psi)^{1/2}} = - \frac{\dot{M}C_p T_s^{1/2}}{\delta D S^{1/8}} \left[\int_0^{\psi_2} \frac{d\psi}{\psi^{5/8} (1-\psi)^{1/2}} - \int_0^{\psi_1} \frac{d\psi}{\psi^{5/8} (-\psi)^{1/2}} \right] \quad (55)$$

These last two integrals are in the form of incomplete Beta Functions. Before it can be said that they are indeed Incomplete Beta Functions, it must be shown that the upper limit of integration is less

than or equal to one, for it is only in this range that the Beta Function is defined.

Consider the limit of integration, which will be signified by Ψ without a subscript since it could be at inlet or outlet conditions.

$$\Psi = \frac{1}{S} \left(\frac{T_s}{T_b} \right)^4 = \left(\frac{T_s}{T_b} \right)^4 \left[\frac{5(1-\phi)}{1-\phi^5} \right]$$

From the basic assumption i.e., $T_{mp} = \phi T_{fb}$, one has:

$$\Psi = \frac{T_s^4}{T_b^4} \left[\frac{5 \left(1 - \frac{T_{mp}}{T_b} \right)}{1 - \left(\frac{T_{mp}}{T_b} \right)^5} \right] \quad (56)$$

Because the analysis is being limited to the design of heat rejection systems, the following inequalities can be stated.

$$T_b \geq T_{mp} \geq T_s > 0 \quad .$$

This means that $\Psi \geq 0$. It must now be shown that $\Psi \leq 1$. Using the inequalities above, the following can be written.

$$\Psi = \frac{T_s^4}{T_b^4} \left[\frac{5 \left(1 - \frac{T_{mp}}{T_b} \right)}{1 - \left(\frac{T_{mp}}{T_b} \right)^5} \right] \leq \frac{T_{mp}^4}{T_b^4} \left[\frac{5 \left(1 - \frac{T_{mp}}{T_b} \right)}{1 - \left(\frac{T_{mp}}{T_b} \right)^5} \right] = \bar{\Psi} \quad .$$

If it can be shown that $\bar{\Psi} \leq 1$, then $\Psi \leq 1$, and it is established that we are dealing with Beta Functions.

$$\bar{\Psi} = 5 \frac{\frac{T_b}{T_{mp}} - 1}{\left(\frac{T_b}{T_{mp}} \right)^5 - 1}$$

Consider the case where the system is at a uniform temperature such that $T_b = T_{mp}$. This case results in an indeterminate form for $\bar{\Psi}$. Using L'Hospital's Rule, it can be shown that the limit is 1. For all other cases the ratio $T_b/T_{mp} > 1$, and therefore the quantity $((T_b/T_{mp})^5 - 1)$ increases faster than the quantity $(T_b/T_{mp} - 1)$, as T_b/T_{mp} increases. Therefore, $\bar{\Psi} \leq 1$.

It has now been established that the last two integrals in Equation 56 are Beta Functions. The solution to Equation 61 can now be written.

$$L = \frac{\dot{M} C_p T_s^{1/2}}{8 D S^{1/8}} \left[\beta_{\psi_2} \left(\frac{3}{8}, \frac{1}{2} \right) - \beta_{\psi_1} \left(\frac{3}{8}, \frac{1}{2} \right) \right] - \frac{\dot{M} C_p}{2 H} \ln \left[S \left(\frac{T_b}{T_s} \right)^5 - \frac{T_b}{T_s} \right]_{\frac{T_b}{T_s} = 1}^{\frac{T_b}{T_s} = \frac{T_{b2}}{T_{b1}}} \quad (57)$$

Again note that, as in the zero sink temperature case, Equation 57 involves fin base temperatures, while it is the fluid temperatures which are known. Equation 53 serves to relate fluid temperatures and fin base temperatures. Equations 53 and 57 can be placed in the following dimensionless forms:

$$\tau_f - \tau_b = \frac{2(5)^{1/2}}{\gamma} (1-\phi)^{1/2} \tau_s \left\{ \tau_b \left[S \left(\frac{\tau_b}{\tau_s} \right)^5 - 1 \right] \right\}^{1/2} \quad (58)$$

$$\tilde{L} = \frac{(5)^{1/2}}{8\alpha} \left[\frac{1}{\tau_s^{3/2}} \right] P \left[\beta_{\psi_2} \left(\frac{3}{8}, \frac{1}{2} \right) - \beta_{\psi_1} \left(\frac{3}{8}, \frac{1}{2} \right) \right] - \frac{5}{2\alpha\gamma} \ln \left[S \left(\frac{\tau_b}{\tau_s} \right)^5 - \frac{\tau_b}{\tau_s} \right]_{\left(\frac{\tau_b}{\tau_s} \right)_1}^{\left(\frac{\tau_b}{\tau_s} \right)_2} \quad (59)$$

$$\text{Where } P = \frac{1}{(1-\phi^5)} \left[\frac{5(1-\phi)}{1-\phi^5} \right]^{1/8}.$$

All of the dimensionless variables and parameters were defined in Chapter IV.

Equation 59 is a solution to the radiator design problem for those cases where the sink temperature is not zero. The statement of the design problem supplies all the information required to find α , γ , τ_{f1} , and τ_{f2} . It will be shown later that sufficient information exists to determine ϕ . Once the method for finding ϕ is established, Equation 58 can be used to find τ_{b1} and τ_{b2} . This establishes all of the information required to find $\tilde{L}$, which is the solution to the design problem.

In order to find the value of ϕ , one selects a value for τ_f at some point between the fluid inlet and outlet conditions. Using this value for τ_f Equations 18 and 19 are solved for τ_b and τ_{mp} , if $T_s = 0$. If $T_s \neq 0$, Equations 37 and 38 are solved for τ_b and τ_{mp} . Knowing τ_b and τ_{mp} , ϕ is calculated ($\phi = \tau_{mp}/\tau_b$). The point at which the value of τ_f is selected has not been determined. In order to determine this point, case number 3 (Chapter IV) was taken as the bases for a parametric study. The parameters are α , γ , $\tilde{L}$, $\tilde{W}$, and τ_s . Four of the five parameters were held constant while the other was varied. Values for τ_s were 0.0, 0.25, 0.50, and 0.75. Values for α , γ , and $\tilde{Q}$ were obtained by both increasing and decreasing, by a factor of five, the values in case number three. Values for $\tilde{W}$ were determined by both increasing and decreasing by factors of two and five the value of $\tilde{W}$ in case number three. Some of the cases which

resulted from this procedure were rejected because they resulted in trying to design a radiator which gave a fluid outlet temperature lower than the sink temperature. This parametric study resulted in many cases which were in operating ranges considerably outside the range of environmental control applications, but these cases were none the less considered. In addition to the cases mentioned above, some thirty other cases were also considered. These were cases which fell into the range of environmental control applications, and which had originally been used in debugging the programs discussed in Chapter IV.

With this set of cases established, solutions were found using the Equations 48 and 58, in which ϕ was found for ten equally spaced values of fluid temperature, beginning with the inlet and ending with the outlet conditions. In obtaining the solutions, two sets of integration limits were considered. One set assumed that the fluid temperatures at inlet and outlet to the radiator were equal to the fin base temperatures at these points, i.e., the temperature drop from the fluid to the fin base was neglected. This does not mean that the temperature drop from fluid to fin base is completely neglected in the solution. This temperature drop is included in ϕ as well as in Equations 48 and 59. It means that the temperature drop was neglected only in finding the limits of integration. A second set of integration limits was found using Equations 47 and 48. The programs discussed in Chapter IV were used to obtain the radiator lengths for the radiators used in the parametric study. These results were taken as correct and compared to the approximate solutions to obtain values of percent error. All of the results were plotted and it was found

that if the fluid temperatures were used as the limits of integration, the computer results could be duplicated within $\pm 2.3\%$ if ϕ is evaluated at a fluid temperature equal to the temperature after 57% of the total fluid temperature change has occurred. If Equations 47 and 58 are used, the computer results were duplicated to within $\pm 2.1\%$ if ϕ is evaluated at a fluid temperature equal to the temperature after 58.5% of the total fluid temperature change has occurred.

RESULTS

The 6 cases described in Chapter IV were compared with the approximate solutions. Two sets of integration limits were used. One set assumed the fluid temperature to be equal to the base temperature. The other set used Equations 47 and 58 to find the limits of integration.

For the first set of integration limits, ϕ was evaluated at a fluid temperature equal to the temperature at the point where 57% of the heat had been rejected. For the second set of integration ϕ was evaluated at a fluid temperature equal to the temperature at the point where 58.5% of the heat had been rejected. The comparison procedure was the same as the procedure discussed in Chapter IV. The results of these comparisons are presented in Table 3.

Case	Percent Error	
	Using Eqs. 47 and 58	Assuming $\tau_b = \tau_f$
1	+1.75	+1.50
2	-0.46	-0.51
3	+1.26	+1.25
4	-0.35	+1.45
5	+1.16	+1.05
6	-0.10	+0.80

TABLE 3

For convenience in using Equations 47 and 48, and Equations 58 and 59, tables of ϕ are presented in Appendix A. These tables were generated using Equations 18, 19, 37, and 38, and cover a range of parameters pertinent to environmental control radiators. A table of $\beta(3/8, 1/2)$ is also included in Appendix A. This table was generated using program BETAF, which is included in Appendix B.

In using the tables of ϕ , one first calculates α and γ . Since the heat rejection rate is a specified quantity, one can calculate the temperature drop from the inlet to the outlet of the radiator. This is done by writing a heat balance on the fluid, i.e.,

$$\Delta\tau_f|_{\text{total}} = (1 - \tau_{f2}) = \frac{2}{5} \alpha \tilde{Q}$$

where the substitution $\tau_{f1} = 1$ has been made. One now determines which set of integration limits are to be used. Depending on which set is selected, one subtracts 57 percent or 58.5 percent of the total temperature drop from the fluid inlet temperature. The resulting temperature, along with γ , $\tilde{W}$, and τ_s , is used to enter the ϕ tables.

Linear interpolation is adequate.

CHAPTER VI

CONCLUSION

It has been the purpose of the writer to accomplish two goals. The first goal was to obtain computer programs which could be used to design space radiators rapidly. The six cases presented in Chapter IV indicate that the computer programs developed in that chapter are sufficiently accurate for this purpose. Some indication of the relative speed of the programs can be obtained by considering case number three. This case was taken from Russell¹⁹. Russell states that the explicit method of finite difference solution required approximately 30 minutes on the IBM 7094. In comparison approximately 1.2 minutes were required on the IBM 7040 when using the programs developed in Chapter IV. Approximately 0.2 minutes of this time was used for actual computation. The remainder of the time was used in reading the programs. In judging relative speed it should be noted that the IBM 7094 is approximately five times faster than the IBM 7040. It can therefore be stated, that the first goal of this work has been obtained in so far as those cases where an adiabatic mid-point exists.

The second goal of this work was to obtain analytic solutions suitable for hand calculations. The six cases discussed in Chapter V indicate that this goal has also been accomplished.

It seems appropriate at this point to mention some useful modifications and extensions of the work presented in this thesis. A modification of the computer programs developed in this thesis which would be useful, and yet require but minor changes, would be to allow differ-

ing sink temperatures for each strip. In this way, one could account for some cases of variable incident radiation.

This thesis has been limited to cases where an adiabatic mid-point exists. It would be useful to extend the work to cases where the adiabatic mid-point does not exist. It appears that this extension is possible, provided one is considering the analysis of a given radiator, rather than the design. The first extension would be to consider cases where an adiabatic point does exist on the fin, but not necessarily at the mid-point. Such cases can occur in series flow radiators and in parallel flow radiators when the flow rates in the various tubes are not all equal. Subroutine BETAF and subroutine INTGRA could still be used although the main program would require major changes. The speed and accuracy of these subprograms in integrating the differential equation describing the fin, makes this a useful extension.

The next extension which should be made is one to allow analysis of series and parallel flow radiators for which there is no adiabatic point in some or all of the fin strips. One technique which would allow using subroutine INTGRA and subroutine BETAF for these cases requires the definition of a fictitious adiabatic temperature. This fictitious temperature can be defined in the following way.

Consider Equations 3, 7, and 9

$$\frac{dT}{dx} = - \left(\frac{2\sigma E}{kt} \left[\frac{T^5}{5} - T_s^4 T + C_1 \right] \right)^{1/2} \quad (3)$$

$$\Delta Q = - kt \Delta L \left. \frac{dT}{dx} \right|_{x=b} \quad (7)$$

$$\frac{\Delta Q}{\Delta L} = \frac{H}{2} (T_f - T_b) \quad (9)$$

Combining these equations one obtains:

$$C_1 = \frac{H^2}{\delta \sigma E k t} (T_f - T_b)^2 + T_s^4 T_b - \frac{T_b^5}{5} \quad (60)$$

Equation 60 states that C_1 must be such that the heat transferred from the fluid must be equal to the heat transferred across the fin base. It is perfectly valid regardless of the existence of an adiabatic point. One now defines a fictitious adiabatic point temperature, T_{ad} , such that

$$T_s T_{ad} - \frac{T_{ad}^5}{5} = T_s^4 T_b - \frac{T_b^5}{5} + \frac{H^2 (T_f - T_b)^2}{\delta \sigma E k t} \quad (61)$$

From Equations 60 and 61 one can write

$$C_1 = T_s^4 T_{ad} - \frac{T_{ad}^5}{5} \quad (62)$$

The fin equation is now in the same form as in Chapter IV, except that T_{ad} replaces T_{mp} . In terms of the variables ξ and ζ , the limits of integration change such that the upper limit of integration is no longer 1, but instead is based on the temperature of the fin base at the adjoining tube. The upper limit of integration is now less than one.

The procedure just described allows the use of BETAF and INTGRA. The main program must be completely rewritten.

APPENDIX A

An Analytic Technique For The Generation of Beta Functions

AN ANALYTIC TECHNIQUE FOR THE GENERATION OF BETA FUNCTIONS

Consider the definition of the Beta function.

$$\beta_z(M, N) = \int_0^z x^{(M-1)} (1-x)^{(N-1)} dx \quad A.1$$

The analysis which follows will be limited to those cases where $0 < M < 1$ and $0 < N < 1$. For these cases a singularity exists in the integrand for both $Z = 0$ and $Z = 1$. These singularities present no difficulties in the method to be described. Consider the portion of the integrand, $(1-x)^{N-1}$. This can be expanded in a binomial to obtain:

$$(1-x)^{N-1} = \left\{ 1 - (N-1)x + \frac{(N-1)(N-2)}{2!} x^2 - \frac{(N-1)(N-2)(N-3)}{3!} x^3 + \dots \right\} \quad A.2$$

The binomial series converges for $0 \leq x < 1$, so the analysis is limited to this range. Using expansion 64 in the definition of the Beta function one obtains:

$$\begin{aligned} \beta_z(M, N) &= \int_0^z \left[x^{(M-1)} \left\{ 1 - (N-1)x + \frac{(N-1)(N-2)}{2!} x^2 - \frac{(N-1)(N-2)(N-3)}{3!} x^3 + \dots \right\} \right] dx \\ &= \int_0^z \left(x^{M-1} - (N-1) x^M + \frac{(N-1)(N-2)}{2!} x^{M+1} - \frac{(N-1)(N-2)(N-3)}{3!} x^{M+2} + \dots \right) dx \end{aligned}$$

The power series converges uniformly for $x < 1$ and can be integrated term by term to yield the value of the Beta function.

$$\begin{aligned} \beta_z(M, N) &= \frac{z^M}{M} = \left(\frac{N-1}{M+1}\right) z^{M+1} + \frac{(N-1)(N-2)}{2!(M+2)} z^{M+2} \\ &\quad - \frac{(N-1)(N-2)(N-3)}{3!(M+3)} z^{M+3} + \dots \end{aligned}$$

Because this series is not valid for $Z = 1$ and because the convergence becomes slow for Z near 1, $Z = .5$ is selected as the maximum value of Z to be used and one writes

$$H(z) = \frac{z^M}{M} + \sum_{I=1}^{\infty} \left\{ \left[(-1)^I \frac{z^{M+I}}{(M+I)I!} \right] \prod_{J=1}^I (N-J) \right\} \quad z \leq .5 \quad A.3$$

where $H(Z)$ is used rather than $\beta_z(M, N)$, to indicate that Equation A.3 is to be used only for $Z \leq .5$.

A series must now be found which converges for $Z = 1$ and which is rapidly convergent near $Z = 1$. Equation A.1 can be decomposed as follows:

$$\begin{aligned} \beta_z(M, N) &= \int_0^a x^{M-1} (1-x)^{N-1} dx + \int_a^1 x^{M-1} (1-x)^{N-1} dx \\ &\quad + \int_1^z x^{M-1} (1-x)^{N-1} dx \dots \end{aligned} \quad A.4$$

where a is some arbitrary value of x (0.5 in this case) which is selected to be such that equation A.3 converges rapidly. Consider

$$f(z) = \int_1^z x^{M-1} (1-x)^{N-1} dx \quad .$$

Let $U = (1-x)$ then $dU = -dx$

$$f(z) = q(V) = - \int_0^V (1-U)^{M-1} U^{N-1} dU \quad .$$

Proceeding as before, expand $(1-U)^{M-1}$ in a binomial series

$$(1-U)^{M-1} = \left(1 - (M-1)U + \frac{(M-1)(M-2)}{2!} U^2 - \frac{(M-1)(M-2)(M-3)}{3!} U^3 + \dots \right) \quad A.5$$

This series converges for $0 \leq U < 1$, which is the desired range. Multiplying Equation A.5 by U^{N-1} and integrating term by term one obtains:

$$q(V) = - \left[\frac{V^N}{N} - \frac{(M-1)}{(N+1)} V^{N+1} + \frac{(M-1)(M-2)}{2!(N+2)} V^{N+2} - \dots \right]$$

$$q(V) = - \frac{V^N}{N} + \sum_{I=1}^{\infty} \left\{ (-1)^{I+1} \frac{V^{N+I}}{(N+I)I!} \left[\prod_{J=1}^I (M-J) \right] \right\} \quad A.6$$

Equations A.3, and A.6 allow the generation of the Beta function to any desired degree of accuracy. For $Z \leq .5$, Equation A.3 alone is used to generate the Incomplete Beta function. If $Z > .5$, the following yields the Incomplete Beta function:

$$\beta_z(M, N) = H(.5) + q(.5) - q(1-z) \quad . \quad A.7$$

Equations A.3, A.6, and A.7 were programmed, as a subroutine, in Fortran IV for the IBM 7040. This subroutine was checked by generating a table of $\beta_z(.5, .5)$ which was then compared against values available

in the literature (Reference 18). The results of the subroutine agreed exactly.

A listing of this subroutine, BETAF, is presented in Appendix C.

APPENDIX B

Table of $\beta_x(.3, .5)$

Table of $\beta_x(.375, .5)$

Tables of φ

TABLE OF THE INCOMPLETE BETA FUNCTION --- M=.3, N=.5

X	$B_x(.3, .5)$	X	$B_x(.3, .5)$
.01	.838265	.51	2.934536
.02	1.033230	.52	2.957385
.03	1.168248	.53	2.980165
.04	1.275061	.54	3.002886
.05	1.364967	.55	3.025560
.06	1.443440	.56	3.048198
.07	1.513594	.57	3.070809
.08	1.577384	.58	3.093406
.09	1.636130	.59	3.115999
.10	1.690768	.60	3.138599
.11	1.741988	.61	3.161219
.12	1.790315	.62	3.183869
.13	1.836160	.63	3.206561
.14	1.879848	.64	3.229308
.15	1.921646	.65	3.252122
.16	1.961771	.66	3.275017
.17	2.000407	.67	3.298005
.18	2.037707	.68	3.321102
.19	2.073803	.69	3.344323
.20	2.108808	.70	3.367683
.21	2.142820	.71	3.391199
.22	2.175925	.72	3.414890
.23	2.208200	.73	3.438774
.24	2.239711	.74	3.462873
.25	2.270517	.75	3.487208
.26	2.300673	.76	3.511805
.27	2.330227	.77	3.536689
.28	2.359222	.78	3.561890
.29	2.387697	.79	3.587440
.30	2.415691	.80	3.613376
.31	2.443234	.81	3.639736
.32	2.470358	.82	3.666568
.33	2.497091	.83	3.693920
.34	2.523459	.84	3.721854
.35	2.549486	.85	3.750435
.36	2.575194	.86	3.779744
.37	2.600605	.87	3.809873
.38	2.625739	.88	3.840934
.39	2.650613	.89	3.873062
.40	2.675246	.90	3.906423
.41	2.699654	.91	3.941227
.42	2.723853	.92	3.977742
.43	2.747858	.93	4.016325
.44	2.771684	.94	4.057466
.45	2.795344	.95	4.101872
.46	2.818851	.96	4.150629
.47	2.842219	.97	4.205568
.48	2.865459	.98	4.270265
.49	2.888584	.99	4.353972
.50	2.911606	1.00	4.554441

TABLE OF THE INCOMPLETE BETA FUNCTION --- M-.375, N=.5

X	$B_x(.375, .5)$	X	$B_x(.375, .5)$
.01	.474857	.51	2.262464
.02	.616663	.52	2.284204
.03	.718928	.53	2.305909
.04	.801946	.54	2.327589
.05	.873170	.55	2.349254
.06	.936288	.56	2.370913
.07	.993428	.57	2.392577
.08	1.045948	.58	2.414255
.09	1.094773	.59	2.435958
.10	1.140565	.60	2.457695
.11	1.183817	.61	2.479478
.12	1.224906	.62	2.501317
.13	1.264128	.63	2.523223
.14	1.301723	.64	2.545208
.15	1.337884	.65	2.567284
.16	1.372773	.66	2.589463
.17	1.406524	.67	2.611760
.18	1.439253	.68	2.634186
.19	1.471057	.69	2.656757
.20	1.502022	.70	2.679488
.21	1.532222	.71	2.702396
.22	1.561722	.72	2.725498
.23	1.590581	.73	2.748813
.24	1.618848	.74	2.772361
.25	1.646570	.75	2.796165
.26	1.673788	.76	2.820249
.27	1.700539	.77	2.844638
.28	1.726858	.78	2.869362
.29	1.752776	.79	2.894452
.30	1.778319	.80	2.919945
.31	1.803516	.81	2.945881
.32	1.828389	.82	2.972303
.33	1.852960	.83	2.999264
.34	1.877252	.84	3.026822
.35	1.901282	.85	3.055045
.36	1.925069	.86	3.084012
.37	1.948630	.87	3.113815
.38	1.971980	.88	3.144567
.39	1.995136	.89	3.176402
.40	2.018112	.90	3.209486
.41	2.040920	.91	3.244030
.42	2.063574	.92	3.280303
.43	2.086087	.93	3.318661
.44	2.108471	.94	3.359596
.45	2.130736	.95	3.403814
.46	2.152895	.96	3.452404
.47	2.174959	.97	3.507196
.49	2.218841	.99	3.655385
.48	2.196937	.98	3.571771
.50	2.240680	1.00	3.855804

TABLE OF ϕ $\gamma = 40.0$ $\tau_s = 0.00$

$\tilde{W}$	τ_F						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98903	.98965	.99024	.99082	.99137	.99191	.99242
0.7500	.97666	.97791	.97912	.98030	.98144	.98255	.98362
1.0000	.96113	.96310	.96502	.96690	.96873	.97051	.97224
1.2500	.94347	.94617	.94882	.95143	.95397	.95647	.95890
1.5000	.92454	.92794	.93129	.93459	.93784	.94104	.94417
1.7500	.90498	.90901	.91301	.91696	.92087	.92473	.92854
2.0000	.88526	.88986	.89443	.89898	.90349	.90796	.91238

 $\tau_s = 0.25$

$\tilde{W}$	τ_F						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98907	.98969	.99029	.99087	.99143	.99196	.99248
0.7500	.97676	.97801	.97923	.98041	.98155	.98266	.98374
1.0000	.96130	.96327	.96520	.96708	.96892	.97070	.97244
1.2500	.94372	.94643	.94909	.95170	.95426	.95676	.95920
1.5000	.92488	.92829	.93165	.93497	.93823	.94144	.94459
1.7500	.90542	.90947	.91348	.91745	.92138	.92525	.92908
2.0000	.88581	.89043	.89502	.89958	.90412	.90861	.91306

 $\tau_s = 0.50$

$\tilde{W}$	τ_F						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98978	.99042	.99103	.99163	.99220	.99276	.99330
0.7500	.97827	.97956	.98082	.98205	.98324	.98439	.98551
1.0000	.96385	.96590	.96790	.96986	.97178	.97365	.97547
1.2500	.94749	.95031	.95309	.95582	.95851	.96114	.96373
1.5000	.92999	.93357	.93710	.94059	.94404	.94745	.95080
1.7500	.91199	.91626	.92050	.92471	.92889	.93303	.93713
2.0000	.89392	.89882	.90370	.90858	.91343	.91826	.92307

 $\tau_s = 0.75$

$\tilde{W}$	τ_F						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99282	.99353	.99422	.99490	.99556	.99620	.99682
0.7500	.98478	.98625	.98768	.98909	.99047	.99182	.99315
1.0000	.97477	.97713	.97945	.98175	.98401	.98624	.98845
1.2500	.96352	.96683	.97011	.97337	.97660	.97981	.98299
1.5000	.95163	.95589	.96014	.96439	.96862	.97285	.97706
1.7500	.93958	.94476	.94995	.95516	.96039	.96563	.97089
2.0000	.92770	.93374	.93983	.94597	.95215	.95838	.96465

TABLE OF φ $\gamma = 30.0$ $\tau_s = 0.00$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98914	.98974	.99033	.99090	.99144	.99197	.99248
0.7500	.97697	.97819	.97938	.98053	.98165	.98273	.98378
1.0000	.96175	.96366	.96553	.96736	.96914	.97088	.97257
1.2500	.94446	.94708	.94966	.95219	.95466	.95709	.95946
1.5000	.92594	.92923	.93248	.93568	.93884	.94195	.94500
1.7500	.90679	.91070	.91457	.91840	.92220	.92595	.92965
2.0000	.88748	.89193	.89636	.90077	.90514	.90949	.91379

 $\tau_s = 0.25$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98918	.98979	.99038	.99095	.99149	.99202	.99253
0.7500	.97707	.97829	.97948	.98064	.98176	.98285	.98390
1.0000	.96192	.96384	.96571	.96755	.96933	.97108	.97277
1.2500	.94471	.94734	.94992	.95246	.95495	.95738	.95976
1.5000	.92628	.92958	.93284	.93606	.93923	.94235	.94541
1.7500	.90723	.91115	.91504	.91889	.92270	.92647	.93019
2.0000	.88802	.89249	.89694	.90137	.90577	.91013	.91446

 $\tau_s = 0.50$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98989	.99051	.99112	.99170	.99227	.99282	.99335
0.7500	.97857	.97984	.98107	.98227	.98343	.98456	.98566
1.0000	.96445	.96644	.96839	.97030	.97217	.97400	.97578
1.2500	.94845	.95119	.95389	.95655	.95917	.96174	.96426
1.5000	.93136	.93483	.93826	.94165	.94500	.94832	.95158
1.7500	.91377	.91791	.92202	.92611	.93017	.93420	.93819
2.0000	.89611	.90085	.90559	.91032	.91504	.91974	.92442

 $\tau_s = 0.75$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99291	.99360	.99429	.99495	.99560	.99623	.99685
0.7500	.98503	.98646	.98786	.98924	.99060	.99193	.99324
1.0000	.97526	.97755	.97982	.98207	.98428	.98647	.98863
1.2500	.96431	.96753	.97072	.97390	.97706	.98019	.98330
1.5000	.95277	.95691	.96104	.96517	.96930	.97341	.97753
1.7500	.94108	.94610	.95115	.95622	.96130	.96641	.97153
2.0000	.92956	.93542	.94133	.94730	.95331	.95937	.96548

TABLE OF ω $\gamma = 20.0$ $\tau_s = 0.000$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98935	.98993	.99050	.99105	.99158	.99209	.99258
0.7500	.97756	.97873	.97986	.98096	.98204	.98308	.98409
1.0000	.96290	.96471	.96649	.96823	.96993	.97158	.97320
1.2500	.94630	.94877	.95120	.95359	.95595	.95825	.96051
1.5000	.92851	.93161	.93468	.93770	.94069	.94364	.94654
1.7500	.91011	.91379	.91744	.92106	.92465	.92820	.93171
2.0000	.89152	.89571	.89989	.90406	.90820	.91231	.91639

 $\tau_s = 0.25$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98939	.98998	.99055	.99110	.99163	.99214	.99264
0.7500	.97766	.97883	.97996	.98107	.98215	.98319	.98420
1.0000	.96307	.96488	.96667	.96841	.97011	.97178	.97340
1.2500	.94654	.94902	.95146	.95387	.95622	.95854	.96081
1.5000	.92885	.93196	.93503	.93808	.94108	.94403	.94695
1.7500	.91055	.91424	.91791	.92154	.92515	.92872	.93225
2.0000	.89206	.89627	.90047	.90466	.90882	.91295	.91705

 $\tau_s = 0.50$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99009	.99069	.99128	.99184	.99239	.99293	.99344
0.7500	.97914	.98035	.98153	.98268	.98380	.98489	.98595
1.0000	.96556	.96745	.96931	.97113	.97292	.97466	.97637
1.2500	.95024	.95283	.95538	.95791	.96039	.96284	.96525
1.5000	.93388	.93715	.94039	.94360	.94678	.94993	.95304
1.7500	.91703	.92094	.92482	.92869	.93254	.93637	.94016
2.0000	.90010	.90458	.90906	.91354	.91801	.92247	.92692

 $\tau_s = 0.75$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99307	.99374	.99440	.99505	.99568	.99630	.99690
0.7500	.98548	.98685	.98821	.98954	.99085	.99213	.99340
1.0000	.97616	.97835	.98051	.98266	.98479	.98689	.98897
1.2500	.96577	.96882	.97186	.97489	.97790	.98090	.98388
1.5000	.95485	.95877	.96269	.96662	.97054	.97447	.97840
1.7500	.94382	.94857	.95335	.95815	.96298	.96784	.97272
2.0000	.93295	.93849	.94409	.94974	.95544	.96120	.96702

TABLE OF φ

$\gamma = 15.0$

$\tau_s = 0.00$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98955	.99011	.99066	.99119	.99170	.99220	.99268
0.7500	.97811	.97922	.98031	.98137	.98240	.98340	.98438
1.0000	.96395	.96567	.96736	.96902	.97065	.97224	.97379
1.2500	.94795	.95029	.95260	.95488	.95712	.95932	.96148
1.5000	.93082	.93375	.93666	.93953	.94238	.94518	.94794
1.7500	.91308	.91656	.92003	.92346	.92687	.93025	.93359
2.0000	.89513	.89911	.90307	.90702	.91096	.91487	.91875

$\tau_s = 0.25$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98959	.99016	.99071	.99124	.99175	.99225	.99273
0.7500	.97821	.97932	.98041	.98147	.98251	.98352	.98449
1.0000	.96411	.96584	.96754	.96920	.97083	.97243	.97398
1.2500	.94819	.95055	.95286	.95515	.95740	.95960	.96177
1.5000	.93115	.93410	.93702	.93990	.94276	.94557	.94835
1.7500	.91352	.91701	.92049	.92394	.92737	.93076	.93412
2.0000	.89567	.89967	.90365	.90762	.91157	.91551	.91942

$\tau_s = 0.50$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99028	.99086	.99143	.99198	.99251	.99303	.99353
0.7500	.97967	.98083	.98196	.98306	.98414	.98520	.98622
1.0000	.96658	.96838	.97015	.97189	.97360	.97528	.97692
1.2500	.95185	.95431	.95674	.95914	.96152	.96386	.96616
1.5000	.93614	.93924	.94232	.94537	.94841	.95141	.95438
1.7500	.91996	.92366	.92735	.93103	.93469	.93834	.94196
2.0000	.90367	.90793	.91218	.91644	.92070	.92495	.92920

$\tau_s = 0.75$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99322	.99387	.99451	.99514	.99576	.99636	.99696
0.7500	.98590	.98722	.98852	.98981	.99108	.99232	.99355
1.0000	.97698	.97907	.98115	.98321	.98525	.98728	.98929
1.2500	.96709	.97000	.97290	.97579	.97868	.98156	.98442
1.5000	.95673	.96045	.96419	.96793	.97168	.97543	.97920
1.7500	.94627	.95078	.95533	.95990	.96451	.96914	.97381
2.0000	.93598	.94124	.94656	.95194	.95737	.96286	.96841

TABLE OF φ $\gamma = 10.0$ $\tau_s = 0.00$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98991	.99044	.99095	.99145	.99194	.99241	.99287
0.7500	.97908	.98011	.98112	.98210	.98306	.98400	.98491
1.0000	.96579	.96737	.96892	.97044	.97194	.97341	.97485
1.2500	.95083	.95296	.95506	.95715	.95920	.96122	.96321
1.5000	.93481	.93748	.94012	.94274	.94534	.94791	.95044
1.7500	.91821	.92137	.92452	.92765	.93076	.93385	.93692
2.0000	.90136	.90497	.90858	.91218	.91578	.91935	.92291

 $\tau_s = 0.25$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.98995	.99048	.99100	.99150	.99199	.99246	.99292
0.7500	.97918	.98021	.98122	.98220	.98317	.98411	.98502
1.0000	.96595	.96753	.96909	.97062	.97212	.97360	.97504
1.2500	.95107	.95321	.95532	.95741	.95947	.96150	.96350
1.5000	.93514	.93782	.94047	.94310	.94571	.94829	.95084
1.7500	.91864	.92181	.92498	.92812	.93125	.93436	.93744
2.0000	.90190	.90553	.90916	.91278	.91639	.91999	.92357

 $\tau_s = 0.50$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99063	.99117	.99171	.99223	.99274	.99323	.99371
0.7500	.98061	.98168	.98273	.98376	.98477	.98576	.98672
1.0000	.96837	.97002	.97165	.97325	.97483	.97639	.97792
1.2500	.95465	.95690	.95912	.96133	.96351	.96567	.96780
1.5000	.94005	.94287	.94568	.94848	.95126	.95402	.95676
1.7500	.92500	.92837	.93174	.93511	.93846	.94181	.94515
2.0000	.90983	.91371	.91760	.92150	.92540	.92931	.93321

 $\tau_s = 0.75$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99349	.99411	.99472	.99532	.99591	.99649	.99705
0.7500	.98665	.98788	.98910	.99031	.99150	.99268	.99384
1.0000	.97843	.98036	.98228	.98419	.98609	.98799	.98987
1.2500	.96939	.97206	.97472	.97739	.98006	.98272	.98539
1.5000	.95997	.96337	.96679	.97022	.97367	.97714	.98062
1.7500	.95049	.95460	.95876	.96295	.96717	.97143	.97572
2.0000	.94117	.94596	.95082	.95574	.96071	.96574	.97084

TABLE OF ϕ

$\gamma = 5.0$

$\tau_s = 0.00$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99079	.99124	.99169	.99212	.99254	.99295	.99335
0.7500	.98136	.98220	.98303	.98385	.98466	.98545	.98622
1.0000	.96995	.97123	.97249	.97373	.97496	.97617	.97737
1.2500	.95721	.95891	.96061	.96229	.96395	.96560	.96723
1.5000	.94357	.94570	.94782	.94992	.95202	.95411	.95617
1.7500	.92938	.93191	.93444	.93695	.93947	.94197	.94446
2.0000	.91491	.91781	.92071	.92361	.92652	.92942	.93232

$\tau_s = 0.25$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99084	.99129	.99173	.99216	.99259	.99300	.99340
0.7500	.98145	.98230	.98313	.98395	.98476	.98555	.98633
1.0000	.97011	.97139	.97265	.97390	.97514	.97635	.97755
1.2500	.95744	.95915	.96085	.96254	.96422	.96587	.96751
1.5000	.94389	.94603	.94816	.95028	.95239	.95448	.95656
1.7500	.92981	.93235	.93489	.93742	.93995	.94247	.94497
2.0000	.91544	.91836	.92128	.92420	.92713	.93005	.93296

$\tau_s = 0.50$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99148	.99195	.99241	.99286	.99330	.99373	.99416
0.7500	.98282	.98370	.98457	.98544	.98629	.98713	.98795
1.0000	.97242	.97376	.97510	.97642	.97772	.97902	.98030
1.2500	.96089	.96270	.96450	.96630	.96808	.96986	.97162
1.5000	.94865	.95092	.95319	.95546	.95772	.95998	.96224
1.7500	.93603	.93874	.94146	.94419	.94693	.94966	.95240
2.0000	.92326	.92639	.92955	.93272	.93590	.93909	.94230

$\tau_s = 0.75$

$\tilde{W}$	τ_f						
	.98	.96	.94	.92	.90	.88	.86
0.5000	.99417	.99471	.99524	.99577	.99629	.99680	.99731
0.7500	.98843	.98946	.99049	.99151	.99253	.99354	.99455
1.0000	.98172	.98331	.98489	.98648	.98807	.98966	.99125
1.2500	.97449	.97665	.97883	.98101	.98321	.98541	.98763
1.5000	.96703	.96978	.97254	.97533	.97814	.98098	.98384
1.7500	.95958	.96289	.96624	.96962	.97304	.97650	.98000
2.0000	.95228	.95613	.96004	.96400	.96801	.97208	.97620

APPENDIX C

Program Listings

PROGRAM BETAF

Program BETAF is a subroutine written in Fortran IV for the IBM 7040. It uses Equation 23 to generate the Incomplete Beta Function. The Incomplete Beta Function is defined by Equation C.1.

$$\beta_x(P, Q) = \int_0^x x^{(P-1)}(1-x)^{Q-1} dx \quad . \quad C.1$$

In the notation of Equation 70, the calling sequence is:

```
CALL BETAF(P, Q, X, BETAX)
```

where BETAX is the value of the Incomplete Beta Function. BETAX is a dummy variable and is included in the call statement in order to return its value to the calling program. P, Q, and X must be less than or equal to one.

The value of the Incomplete Beta Function is accurate to six significant figures.


```

$IBFTC BETAF
      SUBROUTINE BETAF(P,Q,X,BETAX)
      DIMENSION BCONS(41),ACONS(41)
      ITRAN = 0
      IF(X-.5) 29,29,28
28  XSTO = X
      X = .5
      ITRAN = 1
C      THE SERIES FOR X LESS THAN OR EQUAL TO 0.5 STARTS HERE AND ENDS
C      WITH STATEMENT 101.
29  BCONS(1) = (Q-1.)/(P+1.0)
30  N = 1
      ASIGN = -1.0
      BETAX = (X**P)/P
      FLON = N
      GO TO 41
40  FLON = N
      BCONS(N)=(BCONS(N-1)/FLON)*((Q-FLON)*(P+(FLON-1.))/(P+FLON))
41  SUBS = BCONS(N)*(X**(P+FLON))
45  N = N+1
      BETAX = BETAX + ASIGN*SUBS
      ASIGN = -ASIGN
      SUBS=ABS (SUBS)
      IF(SUBS-1.E-8) 120,120,80
80  IF(N-40)40,40,90
90  WRITE(6,101)
101 FORMAT(30H INSUFFICIENT TERMS IN SERIES )
120 IF(ITRAN) 180,122,180
122 RETURN
C      THE SERIES FOR X GREATER THAN 0.5 STARTS HERE AND ENDS WITH
C      STATEMENT 201.
180 ACONS(1) = (P-1.0)/(Q+1.0)
      U = .5
      ICOUN = 0
      GO TO 239
339 U= 1. - XSTO
239 J = 1
      BETAU = (U**Q)/Q
      ASIGN =-1.0
      FLOJ = J
      GO TO 241
240 FLOJ = J
      ACONS(J)=(ACONS(J-1)/FLOJ)*((P-FLOJ)*(Q+FLOJ-1.)/(Q+FLOJ))
241 SUBS2 = ACONS(J)*(U**(Q+FLOJ))
      J = J+1
      BETAU = BETAU + ASIGN*SUBS2
      ASIGN = -ASIGN
      SUBS2 = ABS(SUBS2)
      IF(SUBS2 - 1.E-8) 279,279,220
220 IF(J-40)240,240,290
290 WRITE(6,201)
201 FORMAT (37H INSUFFICIENT TERMS IN SERIES 2 )
279 ICOUN= ICOUN + 1
      IF(ICOUN - 1) 281,281,381
281 HALF = BETAU
      GO TO 339
381 BETAX = BETAX + HALF - BETAU
      X = XSTO
      RETURN
      END

```

PROGRAMS FTUBE0 AND TABF

Program FTUBE0 is the design program for a zero sink temperature. The operation of this program is described on pages 22 and 23. Program TABF is a function subprogram which is called by FTUBE0.

FTUBE0

The input variables are α , γ , $2\tilde{Q}$, number of strips, $\tilde{W}$, and IFAC. IFAC is an output control character. If IFAC has the value k, the program will print τ_f , τ_b , τ_{mp} , $\Delta\tilde{L}$, and the running total of the $\Delta\tilde{L}$'s for the first strip and for every k'th strip.

The data cards have the following form:

	Analysis Variable	Program Variable	Format	Columns
First Card		IFAC	I4	1-4
Second Card	α	ALPHA	F14.8	1-14
	γ	GAMMA	F14.8	15-28
Third Card	$2\tilde{Q}$	AQU	F14.8	1-14
	No. of Strips	STRIP	F14.8	15-28
	$\tilde{W}$	W	F14.8	29-32

TABF

TABF is a function subprogram used by FTUBE0. It is a special form of subroutine BETAF. It returns the difference between the Complete and Incomplete Beta Function for the case where $P=0.3$ and $Q=0.5$, i.e., the call $Y = \text{TABF}(X)$ returns:

$$Y = \beta(.3, .5) - \beta_x(.3, .5)$$

```

$IBFTC FTUBEO
C      SSWTCH 4 CAUSES WRITING OF DATA FOR EACH LOOP OF ITERATION ON TAUB
C      FIRST LINE TAUB, SECOND LINE TAUMP, XI, AND THIRD LINE DELTA AND
C      DIFF. SSWTCH 5 WILL CAUSE ABORT FROM DO LOOP AND READING OF
C      NEW DATA AFTER PAUSE TO ALLOW RESETING OF SSWTCHES
C      THE SYMBOL 0 UNDER 'TYPE OF TRANSFER' MEANS THE DO LOOP WAS
C      COMPLETED WITH OUT CONVERGENCE. THE SYMBOL 2 MEANS THE CORRECTION
C      ON THE TEMPERATURE WAS LESS THAN 1.E-8 TIMES THE TEMPERATURE
C      BEING CORRECTED. THESE SYMBOLS ARE ALSO PRINTED AS THE FIRST
C      COLUMN OF OUTPUT. NO SYMBOL MEANS THE CRITERION FOR CONVERGENCE
C      OF THE NEWTON-RAPHSON ITERATION WAS SATISFIED
      DIMENSION DIFF(35),DERDIF(35),D(500),A(500),ICTYPE(500)
      4 READ(5,20) IFAC
      20 FORMAT(I4)
      READ(5,5) ALPHA, GAMMA
      5 FORMAT(2F14.8)
      READ(5,6) AQU, STRIP, W
      6 FORMAT(3F14.8)
      WRITE(6,7) ALPHA, GAMMA
      7 FORMAT(1H10X6HALPHA=F12.6,10X6HGAMMA=F12.6 )
      WRITE(6,8) AQU, STRIP, W
      8 FORMAT(1H 10X5HHEAT=F12.6,10X7HSTRIPS=F7.2,10X6HWIDTH=F12.6)
      WRITE(6,21) IFAC
      21 FORMAT(1H 5HEVERYI4,32HTH STRIP CALCULATED IS PRINTED )
      WRITE(6,10)
      10 FORMAT(1H09X,9HTAU FLUID10X,8HTAU BASE8X,13HTAU MID POINT12X,2HX11
      14X,7HDELTA L15X,1HL )
      INDICT = 0
      L = 0
      IC = 1
      ICC = 1
      AL = 0.0
      QU = AQU/(STRIP*2.)
      BQU = 2.*QU
      COUNT = 0.0
      J11 = 1
      C = GAMMA*GAMMA
      TAUF = 1. -(ALPHA*BQU)/10.
      TAUB = TAUF -.025
      GO TO 60
      50 TAUB = TAUF - GUESS
      60 SUM = 0.0
      SUM1 = C.0
C      THE NEWTON-RAPHSON ITERATION STARTS HERE AND ENDS WITH STATEMENT
C      450
      DO 450 I=1,35
      61 F = TAUF - TAUB
      B = F*F
      CALL SSWTCH(4,J1)
      IF(J1.EQ.1) WRITE(6,65) TAUB
      65 FORMAT(1H E18.8 )
      XI = 1. -((C*B)/(TAUB**5.))*0.2500
      IF(XI.GE.0.) GO TO 89
      SIG = .5*(TAUF-TAUB)
      TAUB = TAUF - SIG
      GO TO 61
      89 IF(1.-XI) 90,91,91
      90 XI = .99191919
      91 TA = TABF(XI)
      TAUMP = (TAUB**5. - .25*C*B)**.200
      DIFF(I) = W - TA/(TAUMP**1.5)
      IF(J1.EQ.1) WRITE(6,66)TAUMP,XI
      66 FORMAT(1H 2E18.8 )
      IF(ABS(DIFF(I)) - 0.000001)500,500,100

```

```

100 DERDIF(I)=(.3*TA*(5.*TAUB**4+.5*C*F)/(TAUMP**6.5))+.25*C*((5.*R/TA
    IUB**6)+(2.*F/(TAUB**5)))/((XI**7)*SQRT(1.-XI)*(TAUMP**1.5))
    DELTA = - DIFF(I)/DERDIF(I)
    IF(J1.EQ.1) WRITE(6,67)DELTA,DIFF(I)
67  FORMAT(1H 2E18.8 )
    IF(ABS(DELTA).LT.1.E-8) GO TO 453
    CALL SSWTCH(5,J5)
    IF(J5.EQ.1) GO TO 1769
    GO TO 1770
1769 PAUSE 7777
    GO TO 4
1770 SUM = (DELTA/TAUB)**2
    IF(I.EQ.1) GO TO 1790
    IF(SUM1 -SUM)1780,1780,1790
1780 RED= .5*RED
    GO TO 1791
1790 RED = 1.0
1791 TAUB = TAUB + RED*DELTA
450 SUM1 = SUM
C   THE NEXT 7 CARDS ARE ENCOUNTERED IF THE DC LOOP ABOVE IS COMPLETED
C   WITHOUT SATISFYING THE CONVERGENCE CRITERION. THEY ARE USED TO
C   STORE INFORMATION REGARDING CONVERGENCE
    L = L+1
    XI = 1. -((C*B)/(TAUB**5.))*0.2500
    IF(1.-XI)451,452,452
451 XI = .99191919
452 TAUMP = (TAUB**5. - .25*C*B)**.200
    A(L) = COUNT + 1.
    D(L) = W- TABF(XI)/(TAUMP**1.5)
    GO TO 454
C   THE NEXT 4 STATEMENTS ARE ENCOUNTERED IF THE CORRECTION GENERATED
C   BY THE NEWTON-RAPHSON ITERATION IS LESS THAN 1.E-8 TIMES THE
C   TEMPERATURE BEING CORRECTED. THEY ARE USED TO STORE INFORMATION
C   REGARDING CONVERGENCE.
453 L= L+ 1
    D(L) = DIFF(I)
    A(L) = COUNT + 1.
    INDICT = 2
454 GUESS = TAUF - TAUB
    DELTAL = QU/SQRT(TAUB**5 - TAUMP**5)
    AL = AL + DELTAL
    ICTYPE(L) = INDICT
    IF(ICC.EQ.1) GO TO 1900
    IF(IC.EQ.1FAC) GO TO 1900
    IC= IC + 1
    GO TO 2100
1900 WRITE(6,2000) INDICT,J11,TAUF,TAUB,TAUMP,XI,DELTAL, AL
2000 FORMAT(1H I1,I4,F13.6,5F19.6 )
    IF(ICC.EQ.1) GO TO 2010
    IC = 1
    GO TO 2100
2010 ICC = ICC + 1
    IC = IC +1
2100 TAUF = TAUF -(ALPHA*BQU)/5.
    INDICT = 0
    COUNT = COUNT + 1.
    J11 = COUNT + 1.
    IF(IFAC.EQ.1) IC = 1
    IF(STRIP - COUNT) 10000,2999,50
10000 WRITE(6,10001)
10001 FORMAT(1H 25H COUNT GREATER THAN STRIP )
    GO TO 4
C   THE FOLLOWING SECTION UP TO STATEMENT 3520 IS ENCOUNTERED AT THE
C   END OF THE RADIATOR DESIGN. IT PRINTS INFORMATION REGARDING

```

```

C      CONVERGENCE OF THE ITERATION.
2999 WRITE(6,3000)
3000 FORMAT(1H128H ITERATION CHECKING VALUES          )
3001 WRITE(6,3002)
3002 FORMAT(1H0,5HSTRIP,5X,16HTYPE OF TRANSFER,15X,4HDIFF  )
      M= 1
3100 E=D(M)
      IJIJ = A(M)
      IJIJ = ICTYPE(M)
      WRITE(6,3500) IJIJ,IJIJ,E
3500 FORMAT(1H I4,13X,I1,11X,F19.8)
      IF(L-M) 3520,3520,3510
3510 M= M+1
      GO TO 3100
3520 D(1) = C.0
      GO TO 4
C      THE FOLLOWING SECTION IS ENCOUNTERED WHEN THE CONVERGENCE
C      CRITERION IN THE NEWTON-RAPHSON ITERATION IS SATISFIED.
500  GUESS = TAUF - TAUB
      DELTA = QU/SQRT(TAUB**5 - TAUMP**5)
      AL = AL + DELTA
      IF(ICC.EQ.1) GO TO 700
      IF(IC.EQ.IFAC) GO TO 700
      IC = IC + 1
      GO TO 1100
700  WRITE(6,1000) J11,TAUF,TAUB,TAUMP,XI,DELTA,AL
1000 FORMAT(1H ,1X,I4,F13.6,5F19.6 )
      IF(ICC.EQ.1) GO TO 1010
      IC = 1
      GO TO 1100
1010 ICC = ICC + 1
      IC = IC + 1
1100 TAUF = TAUF -(ALPHA*BQU)/5.
      COUNT = COUNT + 1.
      J11 = COUNT + 1.
      IF(IFAC.EQ.1) IC = 1
      IF(STRIP - COUNT) 1C000,2999,50
      END

```

```

$IBFTC TABF
      FUNCTION TABF(X)
      DIMENSION ACONS(30),BCONS(30)
C      THE SECTION FROM THIS POINT UP TO BUT NOT INCLUDING STATEMENT 8079
C      IS THE SERIES USED FOR X GREATER THAN .5
      7999 IF(X- .5) 9000,9000,8000
      8000 U= 1. - X
           ACONS(1) = - .7/1.5
           BETAU = 2.*SQRT(U)
           J = 1
           ASIGN= -1.0
      8010 N= J+1
           FLOJ = J
           FLON = N
           ACONS(N) = (ACONS(N-1)/FLOJ)*((.3 - FLON)*(.5+FLON-1.)/(.5+FLON))
           SUBS2 = ACONS(J)*U**(.5 + FLOJ)
           J= J+ 1
           BETAU = ASIGN*SUBS2 + BETAU
           ASIGN = -ASIGN
           IF(ABS(SUBS2) - .0000001) 8079,8079,8010
      8079 TABF = BETAU
           GO TO 9999
C      THE SECTION FROM THIS POINT UP TO BUT NOT INCLUDING STATEMENT 9079
C      IS THE SERIES USED FOR X LESS THAN OR EQUAL TO .5.
      9000 BCONS(1) = -.5/1.3
           BETAX = (X**.3)/.3
           J= 1
           ASIGN = -1.0
      9020 N=J+1
           FLOJ = J
           FLON=N
           BCONS(N) = (BCONS(N-1)/FLOJ)*((.5-FLON)*(.3+FLON-1.)/(.3+FLON))
           SUBS = BCONS(J)*(X**(.3+FLOJ)
           J=J+1
           BETAX = BETAX + ASIGN*SUBS
           ASIGN = -ASIGN
           IF(ABS(SUBS)- .0000001)9079,9079,9020
      9079 TABF = 4.5544413 - BETAX
      9999 RETURN
      END

```

PROGRAMS FTBSNC AND INTGRA

Program FTBSNC is the design program for sink temperatures not equal to zero. Its operation is described on pages 28 and 29. Program INTGRA is a subroutine which is called by FTBSNC.

FTBSNC

The inputs to FTBSNC are α , γ , $2\tilde{Q}$, $\tilde{W}$, τ_s , IFAC, ICHECO, CSTART, and CHSTOP. The meaning of α , γ , $2\tilde{Q}$, $\tilde{W}$, and τ_s are described in the main body of the thesis. IFAC is an output control character. If IFAC has the value k, the program will print τ_f , τ_b , τ_{mp} , $\Delta\tilde{L}$, and the running total of the $\Delta\tilde{L}$'s for the first and every k'th strip.

ICHECO, CSTART, and CHSTOP are used to allow printing of information used in debugging the program. If ICHECO is equal to one the following information is printed:

- 1 the difference between the left and right side of Equations 41 and 43
- 2 τ_s/τ_{mp} , τ_{mp}/τ_b , τ_{mp} and τ_b
- 3 the corrections on τ_b and τ_{mp} which are generated by the Newton-Raphson iteration, and the number of the iteration

If CSTART is equal to n and CHSTOP is equal to k, the intermediate output starts at the n'th strip and ends with the k'th strip.

The data cards have the following form:

	Analysis Variable	Program Variable	Format	Columns
First Card		IFAC	I4	1-4
		ICHECO	I2	37-38
		CSTART	F14.8	39-52
		CHSTOP	F14.8	53-66

Second Card	α	ALPHA	F14.8	1-14
	γ	GAMMA	F14.8	15-28
Third Card	$2\tilde{Q}$	AQU	F14.8	1-14
	No. of Strips	STRIP	F14.8	15-28
	$\tilde{W}$	W	F14.8	29-42
	τ_s	TASINK	F14.8	43-56

INTGRA

INTGRA is a subroutine which evaluates the integral in Equation 31. It returns values of the function written $\lambda(\zeta, \theta_s)$ in the main body of the thesis.

The calling sequence is:

CALL INTGRA(TSINK,TAU,VALUE)

Where $TASINK = \tau_s / \tau_{mp} = \theta_s$
 $TAU = \tau_{mp} / \tau_b = \zeta$
 $VALUE = \lambda(\zeta, \theta_s)$

The number returned for $\lambda(\zeta, \theta_s)$ is accurate to the sixth decimal place.


```

$IBFTC FTBSNC
      DIMENSION ICTYPE(500),A(500),D(500),F(500),TH(500)
C     SENSE SWITCH 5 IF ON WILL CAUSE A PAUSE TO ALLOW RESETTNG OF
C     SWITCH. PRESSING START WILL CAUSE AN ABORT AND NEW DATA TO BE READ
C     THE SYMBOL 0 UNDER 'TYPE OF TRANSFER' MEANS THE DO LOOP WAS
C     COMPLETED WITHOUT SATISFYING THE CONVERGENCE CRITERION. THE
C     SYMBOL 2 MEANS THE CORRECTIONS ON THE TEMPERATURES WERE BOTH LESS
C     THAN 10E-8 TIMES THE TEMPERATURE BEING CORRECTED. THESE SYMBOLS
C     ARE ALSO PRINTED AS THE FIRST COLUMN OF OUTPUT. NO SYMBOL MEANS
C     THE ITERATION SATISFIED THE CONVERGENCE CRITERION.
      1 READ(5,250) IFAC,ICHECC,CSTART,CHSTOP
250  FORMAT(14,32X,12,2F14.8)
      READ(5,260) ALPHA,GAMMA
260  FORMAT(2F14.8)
      READ(5,270) AQU,STRIP,W,TASINK
270  FORMAT(4F14.8)
      WRITE(6,750) ALPHA,GAMMA,TASINK
750  FORMAT(1H110X6HALPHA=F12.6,10X6HGAMMA=F12.6,10X10HSINK TEMP=F9.6)
      WRITE(6,760) AQU,STRIP,W
760  FORMAT(1H 10X5HHEAT=F12.6,10X7HSTRIPS=F7.2,10X6HWIDTH=F12.6)
      WRITE(6,770) IFAC
770  FORMAT(6H EVERY14,32HTH STRIP CALCULATED IS PRINTED      )
      IF(ICHECC.NE.1) GO TO 779
      WRITE(6,820)
820  FORMAT(119H0BECAUSE 'ICHECC' HAS BEEN SET TO 1, INTERMEDIATE OUTPU
      IT WILL BE PRINTED FOR STRIPS STARTING WITH NO.GIVEN BY 'CSTART' )
      WRITE(6,828)
828  FORMAT(77H AND ENDING WITH NO. GIVEN BY 'CHSTOP'. THE OUTPUT IS A
      IRRANGED AS FOLLOWS      )
      WRITE(6,832)
832  FORMAT(49H **A** THETA SINK, THETA BASE TEMP, TAUMP, TAUH      )
      WRITE(6,834)
834  FORMAT(83H **B** DIFF.RAD. EQ., DIFF. TEMP. EQ.,DDIFRL, DDIFRB,DU
      IFFL, DDIFFB, DETERM      )
      WRITE(6,836)
836  FORMAT(70H **C** CORRECTION ON TAUB(DTAUB), CORRECTION ON TAUMP (
      IDTAUL), INFO      )
779  WRITE(6,780)
780  FORMAT(1H016X9HTAU FLUID,12X8HTAU BASE,6X13HTAU MID POINT,10X7HDEL
      ITA L,14X1HL      )
      IF(TASINK.LT.(1.-ALPHA*AQU/5.)) GO TO 888
      WRITE(6,788)
788  FORMAT(66H THE SINK TEMP IS HIGHER THAN THE DESIRED RADIATOR OUTLE
      IT TEMP      )
      GO TO 1
888  INDICT = 0
      L=0
      IC=1
      ICC=1
      AL=0.0
      QU=AQU/(STRIP*2.)
      BQU=2.*QU
      COUNT = 0.0
      J11= 1
      C= GAMMA/GAMMA
      TAUF = 1. -(ALPHA*BQU)/10.
C     THE NEXT 2 CARDS ARE GUESSES AT TAUB AND TAUMP FOR THE FIRST
C     STRIP.
      TAUB = TAUF - .04
      TAUMP = TAUB*.98
      GO TO 60
50  TAUB = TAUF- GUESS
      TAUMP=TAUB-GUESMP
C     THE NEWTON-RAPHSON ITERATION STARTS HERE AND ENDS WITH STATEMENT

```

```

C      450.
60 DO 450 I=1,35
    CONVER = 1.
    BACTB = 1.
    BACTL = 1.
    THSTAR = TASINK/TAUMP
    IF(THSTAR.GT.1.) THSTAR = .98765432
    THETAB = TAUMP/TAUB
    IF(ICHECO.NE.1) GO TO 61
    IF(COUNT.GE.CSTART.AND.COUNT.LE.CHSTOP) WRITE(6,27)THSTAR,THETAB,T
    1AUMP,TAUB
27 FORMAT(6H **A** 4E16.8)
61 CALL INTGRA(THSTAR,THETAB,FLAM)
    DIFRAD = W - FLAM*5.*(TAUMP**-1.5)
    DIFFIN=-(5.*(THSTAR**4)*(TAUMP**5 -TAUB*(TAUMP**4))+TAUB**5 - TAUM
    1P**5 -.25*C*((TAUF -TAUB)**2) )
    IF(ABS(DIFRAD).LT.1.E-5.AND.ABS(DIFFIN).LT.1.E-5) GO TO 500
    CHTL = TAUMP - .001
    CHTHS= TASINK/CHTL
    CHTHB = CHTL/TAUB
    CALL INTGRA(CHTHS,CHTHB,CHFLA)
    DPARTL =-(CHFLA-FLAM)/.001
    DDIFRL = -7.5*FLAM*(TAUMP**-2.5) + DPARTL*5.*(TAUMP**-1.5)
    DDIFRB= 5.*SQRT((THETAB**3)/((TAUMP**3)*(1.+(5.*(THSTAR**4)-1.)*(T
    1HETAB**5) - 5.*(THSTAR**4)*(THETAB**4))))/TAUB
    DDIFFL= 5.*(TAUMP**4)*(THSTAR**4 -1.)
    DDIFFB= 5.*(TAUB**4-((THSTAR*TAUMP)**4)) +.5*C*(TAUF-TAUB)
    DETERM = DDIFFL*DDIFRB -DDIFRL*DDIFFB
    DTAUB = (DIFRAD*DDIFFL - DIFFIN*DDIFRL)/DETERM
    DTAUL = (DIFFIN*DDIFRB - DIFRAD*DDIFFB)/DETERM
    INFO = 1
C      THE NEXT 5 STATEMENTS ARE INCLUDED TO SUPPRESS THE CONVERGENCE AND
C      AVOID THE POSSIBILITY OF THE TEMPERATURES EXCEEDING 1.
38 IF(ABS(DTAUL).LT.(.25*TAUMP).AND.ABS(DTAUB).LT.(.25*TAUMP)) GO TO
    1477
    INFO = INFO + 1
    DTAUL = .5*DTAUL
    DTAUB = .5*DTAUB
    GO TO 38
477 IF(ICHECO.NE.1) GO TO 449
    IF(COUNT.GE.CSTART.AND.COUNT.LE.CHSTOP) GO TO 448
    GO TO 449
448 WRITE(6,37) DIFRAD,DIFFIN,DDIFRL,DDIFRB,DDIFFL,DDIFFB,DETERM
37 FORMAT(6H **B**2X7E15.8 )
    WRITE(6,47) DTAUB,DTAUL,INFC
47 FORMAT(6H **C**20X,2E16.8,10X,14 )
449 TAUMP = TAUMP + DTAUL
    TAUB = TAUB + DTAUB
    CALL SSWTCH(5,IQUIT)
    IF(IQUIT.NE.1) GO TO 446
    PAUSE
    GO TO 1
C      THE NEXT 8 STATEMENTS ARE INCLUDED TO INSURE THAT IN THE ITERATION
C      THE TEMPERATURES ALWAYS REMAIN IN THE PROPER RELATIONSHIP TO EACH
C      OTHER AS IS INDICATED IN STATEMENT 446.
446 IF(TAUMP.LT.TAUB.AND.TAUB.LT.1.) GO TO 447
    IF(ABS(BACTB).LT.1.E-8.AND.ABS(BACTL).LT.1.E-8) GO TO 444
    CONVER = CONVER*.5
    BACTB = CONVER*DTAUB
    BACTL = CONVER*DTAUL
    TAUMP = TAUMP-BACTL
    TAUB = TAUB - BACTB
    GO TO 446
444 WRITE(6,443)

```

```

443 FORMAT(49H NO CONVERGE. SEE STATEMENT NO.446 AND 446 + 1      )
      GO TO 1
447 IF(ABS(DTAUL).LT.(TAUMP*(1.E-8)).AND.ABS(DTAUB).LT.(TAUB*(1.E-8)))
      IGO TO 453
450                                CONTINUE
C   THE NEXT 6 STATEMENTS ARE USED TO STORE INFORMATION REGARDING
C   CONVERGENCE. THEY ARE ENCOUNTERED IF THE DO LOOP ABOVE IS
C   COMPLETED WITHOUT SATISFYING THE CONVERGENCE CRITERION.
      L = L+1
      A(L) = COUNT + 1.
      D(L) = DIFRAD
      F(L) = DIFFIN
      TH(L) = THSTAR
      GO TO 454
C   THE NEXT 6 STATEMENTS ARE USED TO STORE INFORMATION REGARDING
C   CONVERGENCE. THEY ARE ENCOUNTERED IF BOTH OF THE CORRECTIONS
C   GENERATED BY THE NEWTON-PARHSCN ITERATION ARE LESS THAN 1.E-8
C   TIMES THE VALUE OF THE TEMPERATURE BEING CORRECTED.
453 L = L+1
      A(L) = COUNT + 1.
      D(L) = DIFRAD
      F(L) = DIFFIN
      TH(L) = THSTAR
      INDICT = 2
454 GUESS = TAUF - TAUB
      GUESMP = TAUB - TAUMP
      DELTAL = BQU/(GAMMA*(TAUF - TAUB))
      AL = AL + DELTAL
      ICTYPE(L) = INDICT
      IF(ICC.EQ.1) GO TO 1900
      IF(IC.EQ.1FAC) GO TO 1900
      IC = IC + 1
      GO TO 2100
1900 WRITE(6,2000) INDICT,J11,TAUF,TAUB,TAUMP,DELTAL,AL
2000 FORMAT(1H 8X11,I4,F13.6,4F19.6)
      IF(ICC.EQ.1) GO TO 2010
      IC = 1
      GO TO 2100
2010 ICC = ICC + 1
      IC = IC + 1
2100 TAUF = TAUF - (ALPHA*BQU)/5.
      INDICT = 0
      COUNT = COUNT + 1.
      J11 = COUNT + 1.
      IF(1FAC.EQ.1) IC = 1
      IF(STRIP - COUNT) 10000,2999,50
10000 WRITE(6,10001)
10001 FORMAT(1H 25H COUNT GREATER THAN STRIP      )
      GO TO 1
C   THE FOLLOWING SECTION UP TO BUT NOT INCLUDING STATEMENT 500 IS
C   ENCOUNTERED AT THE END OF THE RADIATOR DESIGN. IT PRINTS A TABLE
C   OF INFORMATION REGARDING CONVERGENCE OF THE ITERATION.
2999 WRITE(6,3000)
3000 FORMAT(1H128H ITERATION CHECKING VALUES      )
3001 WRITE(6,3002)
3002 FORMAT(1H0,5HSTRIP,5X16H TYPE OF TRANSFER,15X8H DIFF RAD,22X8H DIF TE
      1MP,22X6H THSINK )
      M = 1
3100 E = D(M)
      THS = TH(M)
      G = F(M)
      III = A(M)
      IJIJ = ICTYPE(M)
      WRITE(6,3500) III,IJIJ,E,G,THS

```

```

3500 FORMAT(1H I4,13X,I1,3(11X,F19.8))
      IF(L-M) 3520,3520,3510
3510 M= M+1
      GO TO 3100
3520 D(1) = 0.0
      F(1) = 0.0
      TH(1) = 0.0
      A(1) = 0.0
      ICTYPE(1) = 0
      GO TO 1
C     THE FOLLOWING SECTION IS ENCOUNTERED WHEN THE CONVERGENCE
C     CRITERION IN THE NEWTON-RAPHSON ITERATION IS SATISFIED
500  GUESS = TAUF - TAUB
      GUESMP = TAUB-TAUMP
      DELTAL = BQU/(GAMMA*(TAUF - TAUB))
      AL = AL + DELTAL
      IF(ICC.EQ.1) GO TO 700
      IF(IC.EQ.IFAC) GO TO 700
      IC = IC +1
      GO TO 1100
700  WRITE(6,1000) J11,TAUF,TAUB,TAUMP,DELTAL, AL
1000 FORMAT(1H 9X I4,F13.6,4F19.6 )
      IF(ICC.EQ.1) GO TO 1010
      IC = 1
      GO TO 1100
1010 ICC = ICC + 1
      IC = IC +1
1100 TAUF = TAUF -(ALPHA*BQU)/5.
      COUNT = COUNT + 1.
      J11 = COUNT + 1.
      IF(IFAC.EQ.1) IC = 1
      IF(STRIP - COUNT) 10000,2999,50
      END

```

```

$IBFTC INTGRA
      SUBROUTINE INTGRA(TSINK,TAU,VALUE)
C      THE CRITERION FOR THE CONVERGENCE OF THE SERIES IS THAT THE
C      ABSOLUTE VALUE OF THE TERM BE LESS THAN .000001.
C      NAME COMMON 'CHECK' IS INCLUDED FOR DEBUGGING AND FOR MAIN
C      PROGRAM CHECKING PURPOSES. THE SUBROUTINE WILL PRINT A MESSAGE
C      IF THE SERIES DOES NOT CONVERGE. NAME COMMON 'CHEPI' IS
C      INCLUDED TO ALLOW A METHOD FOR THE CALLING PROGRAM TO TAKE
C      PROPER ACTION IF THE SERIES DOES NOT CONVERGE TO THE REQUIRED
C      ACCURACY. 'ITEST' IS 0 IF CONVERGENCE HAS OCCURED AND GREATER THAN
C      0 IF CONVERGENCE HAS NOT OCCURED.
C      'SERPAR' IS THE INTEGRAL FROM 0 TO TAU. 'PARLOW' IS THE VALUE OF
C      THE INTEGRAL FROM 0 TO .8. IF TAU IS GREATER THAN .8 'SERIES' IS
C      THE INTEGRAL FROM TAU TO 1. IF TAU IS LESS THAN .8 'SERIES' IS
C      THE VALUE OF THE INTEGRAL FROM .8 TO 1.0.
      COMMON/CHECK/B,C,SERPAR,PARLOW,SERIES
      COMMON/CHEPI/ITEST
      ITEST = 0
      B=0.0
      C=0.0
      SERPAR = 0.0
      PARLOW= 0.0
      SERIES = 0.0
      STOR1 = TAU
      LOW = 0
      IF(TAU.GE..8) GO TO 4999
      LOW = 1
      TAU = .8
      NNN= 1
1050 C=-5.*(TSINK**4)
      IF(TSINK.EQ.0.) GO TO 1010
      B = -(C+1.)
      IF(ABS(B).EQ.0.) GO TO 2020
C      LOW END SERIES
      STNONE = 0.0
      ATERM = 500.
      DO 1000 J=1,30
      CONM1 = 1.
      DO 100 N=1,J
      FN=N
100 CONM1 =(1.5-FN)*CONM1
      CONMU = CONM1*2.
      PART = 0.0
      DO 500 K=1,J
      FK = K
      FJ = J
      FACK1 = 1.
      DO 200 I=1,K
      FI = I
      FAC = FI - 1.
      IF(FAC.NE.0.0) GO TO 102
      FAC = 1.
102 FACK1 = FACK1*FAC
200 CONTINUE
      FACJK = 1.0
      DO 300 I=K,J
      FI = I
      FAC = FJ-FI
      IF(FAC.NE.0.0) GO TO 202
      FAC = 1.
202 FACJK = FACJK*FAC
300 CONTINUE
      PART1 = TAU**((10.*FJ-2.*FK-5.)*.5)
      PART2 =((2.*CONMU*(B**(J-K))*(C**(K-1))*PART1)/((10.*FJ-2.*FK-5.)*

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      IFACK1*FACJK))
      PART = PART2 + PART
500 CONTINUE
      TERM = PART
      STNONE = STNONE + TERM
      IF(ABS(TERM).LT..CCC001) GC TO 3000
1000 CONTINUE
      WRITE(6,10)
      10 FORMAT(90H LOW END SERIES HAS NOT CONVERGED TO .00C001. COMPUTATI
        ON IS CONTINUING. THE VALUES ARE )
      WRITE(6,11) LOW,NNN,TERM,TAU,STNONE,TSINK
      11 FORMAT(5H LOW=I2,5X4HNNN=I2,5X5HTERM=E16.8,5X4HTAU=E16.8,5X7HSTNUN
        IE=E16.8,5X6HTSINK=E16.8)
      ITEST = ITEST + 1
      GO TO 3000
C      SPECIAL CASE OF LOW END SERIES FOR TSINK=0
C      THIS SECTION FROM STATEMENT 1010 TO THE NEXT SET OF COMMENT CARDS
C      IS ADDED TO MAKE THE SUBROUTINE VALID FOR A 0 TEMP. SINK. THIS IS
C      NECESSARY BECAUSE THE OPTION TAKEN BY THE IBM 7040 SYSTEM IN
C      RAISING 0 TO A 0 POWER IS NOT THE OPTION REQUIRED TO MAKE THE
C      GENERAL SERIES VALID.
      1010 STNONE = (2./3.)*(TAU**1.5)
      TERM = (TAU**6.5)/13.
      STNONE = STNONE + TERM
      DO 600 N=3,30
      FN = N
      TERM=((TAU**5)*(FN-1.5)*(10.*FN-17.))/((FN-1.)*(10.*FN-7.))*TERM
      STNONE = STNONE + TERM
      IF(ABS(TERM).LT..COC001) GO TO 3000
600 CONTINUE
      WRITE(6,12)
      12 FORMAT(106H SPECIAL CASE OF LOW END SERIES FOR TSINK=0 HAS NOT CON
        VERGED TO .00C001. COMPUTATION IS CONTINUING WITH )
      WRITE(6,11) LOW,NNN,TERM,TAU,STNONE,TSINK
      ITEST = ITEST + 1
      GO TO 3000
C      SPECIAL CASE OF LOW END SERIES FOR B=0
C      THIS SECTION FROM STATEMENT 2020 TO STATEMENT 3000 IS ADDED TO
C      MAKE THE SUBROUTINE VALID FOR B=0. THIS IS NECESSARY BECAUSE THE
C      OPTION TAKEN BY THE IBM 7040 SYSTEM IN RAISING 0 TO A 0 POWER IS
C      NOT THE OPTION REQUIRED TO MAKE THE GENERAL SERIES VALID.
      2020 STNONE = (2.*(TAU**1.5))/3.
      TERM = (TAU**5.5)/11.
      STNONE = STNONE + TERM
      DO 660 N=3,30
      FN = N
      TERM=((TAU**4)*(FN-1.5)*(8.*FN-13.))/((FN-1.)*(8.*FN-5.))*TERM
      STNONE = STNONE + TERM
      IF(ABS(TERM).LT..0CC001) GC TO 3000
660 CONTINUE
      WRITE(6,13)
      13 FORMAT(102H SPECIAL CASE OF LOW END SERIES FOR B=0 HAS NOT CONVERG
        ED TO .000001. COMPUTATION IS CONTINUING WITH )
      WRITE(6,11) LOW,NNN,TERM,TAU,STNONE,TSINK
      ITEST = ITEST + 1
3000 IF(NNN.EQ.2) GO TO 1020
      PARLOW = STNONE
      TAU = STOR1
      NNN= 2
      GO TO 1050
4999 U=1.-TAU
      GO TO 1021
1020 SERPAR = STNONE
      U = .2

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C      HIGH END SERIES
C      THIS SECTION, FROM STATEMENT 1021 TO STATEMENT 9898, IS THE HIGH
C      END SERIES AND IS USED FOR TAU GREATER THAN .8. THIS SERIES IS
C      VALID FOR ALL VALUES OF SINK TEMPERATURE LESS THAN 1.
1021  SERIES = 0.0
      A = TSINK**4
      B = -(5.*A-5.)
      C = 20.*A-10.
      D = -(30.*A-10.)
      E = 20.*A-5.
      F = -(5.*A-1.)
      AP = B**-.5
      TERM = 2.*AP*(U**-.5)
      SERIES = SERIES + TERM
      IF(ABS(TERM).LT..00001) GO TO 9001
      AP1 = -.5*(B**-.1.5)*C
      F1 = -.5*AP+AP1
      TERM = (2./3.)*F1*(U**1.5)
      SERIES = SERIES + TERM
      IF(ABS(TERM).LT..00001) GO TO 9001
      AP2 = .75*(B**-.2.5)*(C**2) -.5*(B**-.1.5)*(2.*D)
      F2 = AP2 -(AP1 +.25*AP)
      TERM = .2*F2*(U**2.5)
      SERIES = SERIES + TERM
      IF(ABS(TERM).LT..00001) GO TO 9001
      B3 = B**-.1.5
      B5 = B**-.2.5
      B7 = B**-.3.5
      G2 = C*C
      G3 = G2*C
      C3 = .75
      C5 = 2.5*C3
      H = 2.*D
      AP3 = 2.25*B5*C*H -(C5*B7*G3+.5*B3*6.*E)
      F3 = AP3-(1.5*AP2 +C3*AP1 +.375*AP)
      TERM = (1./21.)*F3*(U**3.5)
      SERIES = SERIES + TERM
      IF(ABS(TERM).LT..00001) GO TO 9001
      B9 = B**-.4.5
      H2 = H*H
      G4 = G3*C
      C7 = 3.5*C5
      AP4 = (C7*B9*G4+B5*(3.*C*6.*E +2.25*H2))-(6.*C5*B7*G2*H+.5*B3*24.*F)
      F4 = AP4-(2.*AP3 +1.5*(AP1 +AP2) +.5*C5*AP)
      TERM = (1./108.)*F4*(U**4.5)
      SERIES = SERIES + TERM
      IF(ABS(TERM).LT..00001) GO TO 9001
      C9 = 4.5*C7
      B11 = B**-.5.5
      G5 = G4*C
      AP5 = (H*(10.*C7*B9*G3+7.5*B5*6.*E)+3.75*B5*C*24.*F) -(C5*B11*G5 +10
1. *C5*B7*G2*6.*E +15.*C5*B7*C*H2)
      F5 = AP5-(2.5*(AP3 +AP4)+3.75*AP2+2.5*C5*AP1 +.5*C7*AP)
      TERM = (1./660.)*F5*(U**5.5)
      SERIES = SERIES + TERM
      IF(ABS(TERM).LT..00001) GO TO 9001
      C11 = 5.5*C9
      B13 = B**-.6.5
      G6 = G5*C
      AI = 6.*E
      AI2 = 6.*E*AI
      H3 = H*H2
      AP6 = (C11*B13*G6 +45.*C7*B9*G2*H2 +20.*C7*B9*G3*6.*E +7.5*B5*AI2 +
111.25*B5*H*24.*F) - (15.*C9*B11*G4*H + 60.*C5*B7*C*H*6.*E + 15.*C5

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2*B7*G2*24.*F + 15.*C5*B7*H3)
F6 = AP6 - (3.*AP5 + 3.75*AP4 + 4.*C5*AP3 + 7.5*C5*AP2 + 3.*C7*AP1 + .5*C9*AP)
TERM = (1./4680.)*F6*(U**6.5)
SERIES = SERIES + TERM
IF(ABS(TERM).LT..CCC001) GC TC 9001
C13 = 6.5*C11
B15 = B** - 7.5
G7 = G6*C
AP7 = (21.*C11*B13*G5*H + 105.*C7*B9*C*H3 + 210.*C7*B9*G2*H*AI + 35.*
1C7*B9*G3*24.*F + 26.25*B5*AI*24.*F) - (C13*B15*G7 + 105.*C9*B11*G3*H
22 + 35.*C9*B11*G4*AI + 105.*C5*B7*H2*AI + 70.*C5*B7*C*AI2 + 105.*C5*B7
3*C*H*24.*F)
F7 = AP7 - (3.5*AP6 + 5.25*AP5 + 7.*C5*AP4 + 17.5*C5*AP3 + 10.5*C7*AP2
1 + 3.5*C9*AP1 + .5*C11*AP)
TERM = (1./37800.)*F7*(U**7.5)
SERIES = SERIES + TERM
IF(ABS(TERM).LT..CCC001) GC TO 9001
C15 = 7.5*C13
B17 = B** - 8.5
G8 = G7*C
H4 = H*H3
AP8 = (C15*B17*G8 + 210.*C11*B13*G4*H2 + 56.*C11*B13*G5*AI + 105.*(C7*B
19*H4 + 8.*C7*B9*C*H2*AI + 4.*C7*B9*G2*H*24.*F) + 280.*C7*B9*G2*AI2
2 + 35.*C3*B5*((24.*F)**2)) - (28.*C13*B15*G6*H + 420.*C9*B11*G2*H3 + 5
360.*C9*B11*G3*H*AI + 70.*C9*B11*G4*24.*F + 280.*C5*B7*(H*AI2 + C*AI*24.*
4F) + 210.*C5*B7*H2*24.*F)
F8 = AP8 - (4.*AP7 + 7.*AP6 + 21.*AP5 + 35.*C5*AP4 + 28.*C7*AP3 + 14.*C9*AP
12 + 4.*C11*AP1 + .5*C13*AP)
TERM = (1./342720.)*F8*(U**8.5)
SERIES = SERIES + TERM
IF(ABS(TERM).LT..COC001) GC TO 9001
AJ = 24.*F
C17 = 8.5*C15
B19 = B** - 9.5
G9 = G8*C
AI3 = AI*AI2
AJ2 = AJ*AJ
AP9 = (36.*C15*B17*G7*H + 1260.*C11*B13*(G3*H3 + G4*H*AI) + 126.*C11*B
113*G5*AJ + 1260.*C7*B9*(H3*AI + G2*AI*AJ) + 2520.*C7*B9*C*H*AI2 + 189
20.*C7*B9*C*H2*AJ) - (C17*B19*G9 + 378.*C13*B15*G5*H2 + 84.*C13*B15*G6
3*AI + 945.*C9*B11*C*H4 + 3780.*C9*B11*G2*H2*AI + 840.*C9*B11*G3*AI2 +
41260.*H*(C9*B11*G3*AJ + C5*B7*AI*AJ) + 280.*C5*B7*AI3 + 315.*C5*B7*C*
5AJ2)
F9 = AP9 - (4.5*AP8 + 9.*AP7 + 42.*C3*AP6 + 63.*C5*AP5 + 63.*C7*AP4 + 42.
1*C9*AP3 + 18.*C11*AP2 + 4.5*C13*AP1 + .5*C15*AP)
TERM = (1./3447360.)*F9*(U**9.5)
SERIES = SERIES + TERM
IF(ABS(TERM).LT..CCC001) GC TC 9001
B21 = B** - 10.5
C19 = 9.5*C17
G10 = G9*C
H5 = H*H4
AP10 = (C19*B21*G10 + C15*B17*(630.*G6*H2 + 120.*AI*G7) + C11*B13*(4725
1.*G2*H4 + 12600.*G3*H2*AI + 2100.*G4*AI2 + 3150.*G4*H*AJ) + C7*B9*(630
20.*H2*AI2 + 3150.*H3*AJ + 2800.*C*AI3 + 12600.*C*H*AI*AJ + 1575.*G2*AJ
32)) - (45.*C17*B19*G8*H + C13*B15*(3150.*G4*H3 + 2520.*G5*H*AI +
4210.*G6*AJ) + C9*B11*(945.*H5 + 12600.*C*H3*AI + 12600.*G2*H*AI2 +
59450.*G2*H2*AJ + 4200.*G3*AI*AJ) + C5*B7*(2100.*AI2*AJ + 1575.*H*AJ2))
F10 = AP10 - (5.*AP9 + 11.25*AP8 + 45.*AP7 + 105.*C5*AP6 + 126.*C7*AP5
1 + 105.*C9*AP4 + 60.*C11*AP3 + 22.5*C13*AP2 + 5.*C15*AP1 + .5*C17*AP)
TERM = (1./38102400.)*F10*(U**10.5)
SERIES = SERIES + TERM
IF(ABS(TERM).LT..COC001) GC TO 9001
WRITE(6,1)

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1 FORMAT(105H THE TERM IN THE SERIES HAS NOT REDUCED TO LESS THAN .0
100001 CONTROL WAS RETURNED TO MAIN PROGRAM WITH )
WRITE(6,2) LOW,TERM,TAU,SERIES,TSINK
2 FORMAT(5H LOW=I2,5X5HTERM=E16.8,5X4HTAU=E16.8,5X7HSERIES=E16.8,5X6
1HTSINK=E16.8)
ITEST = ITEST + 1
9001 IF(LOW.EQ.0) GO TO 9898
VALUE = PARLOW + SERIES - SERPAR
RETURN
9898 VALUE = SERIES
9999 RETURN
END

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REFERENCES

- ¹ D. B. MacKay and G. P. Bacha, "Space radiator analysis and Design," ASD Technical Report 61-30, Part I, October, 1961.
- ² J. G. Bartas and W. H. Sellers, "Radiation Fin Effectiveness," Trans. ASME, Series C, Journal of Heat Transfer, 82, 1960.
- ³ S. Lieblein, "Analysis of Temperature Distribution and Radiant Heat Transfer along a Rectangular Fin of Constant Thickness, NASA TN D-196, November 1959.
- ⁴ N. O. Stockman and J. L. Kramer, "Effect of Variable Thermal Properties on One-Dimensional Heat Transfer in Radiating Fins," NASA TN D-1878, October, 1963.
- ⁵ E. M. Sparrow, V. K. Jonsson, and W. J. Minkowycz, "Heat Transfer from Fin-Tube Radiators Including Longitudinal Conduction and Radiant Interchange between Longitudinally-Non-Isothermal Finite Surfaces," NASA TN D-2077, December, 1963.
- ⁶ R. J. French and J. M. Shield, "Computer Program for Space Radiator Analysis and Design," LTV Report No. 00.391, February, 1964.
- ⁷ A. V. Saule, R. P. Krebs, and B. M. Auer, "Design Analysis and General Characteristics of Flat-Plate Central-Fin-Tube Sensible-Heat Space Radiators," NASA TN D-2839, June 1965.
- ⁸ Chen-Ya Liu, "On Minimum-Weight Rectangular Radiating Fins," Journal of the Aerospace Sciences, November, 1960.
- ⁹ Chen-Ya Liu, "On Optimum Rectangular Cooling Fins," Quarterly of Applied Mathematics, Vol. XIX, No. 1, April 1961.
- ¹⁰ J. E. Wilkins, Jr., "Minimum-Mass Thin Fins Which Transfer Heat Only by Radiation to Surroundings at Absolute Zero," Journal of the Society of Industrial and Applied Mathematics, Vol. 8, No. 4, December 1960.
- ¹¹ J. E. Wilkins, Jr., "Minimum Mass Thin Fins for Space Radiators", Proceedings of the 1960 Heat Transfer and Fluid Mechanics Institute, Stanford, California.
- ¹² J. E. Wilkins, Jr., "Minimum Mass Thin Fins with Specified Minimum Thickness," Journal of the Society of Industrial and Applied Mathematics, Vol. 9, No. 2, June, 1961.
- ¹³ Chen-Ya Liu, "A Variational Problem with Applications to Cooling Fins," Journal of the Society of Industrial and Applied Mathematics, Vol. 10, No. 1, March 1962.

- ¹⁴ B. V. Karlekar, and B. T. Chao, "Mass Minimization of Radiating Trapezoidal Fins with Negligible Base Cylinder Interaction," International Journal of Heat and Mass Transfer, Vol. 6, 1963.
- ¹⁵ C. L. Walker, C. R. Smith, and D. G. Gritton, "Weight Optimization of Heat Rejections Systems for Space Applications," Proceedings of the 1960 Heat Transfer and Fluid Mechanics Institute, Stanford, California.
- ¹⁶ W. C. Reynolds, "A Design-Oriented Optimization of Simple Tapered Radiating Fins," Transactions of the ASME, Series C, Journal of Heat Transfer, 85, August, 1963.
- ¹⁷ A. F. Anderson, R. J. Jameson, A. L. Johnson, R. T. Stake, and R. M. Wright, "Radiator Design for Space Vehicles," Aircsearch Publication MS-AP 0069, 1963.
- ¹⁸ K. Pearson (Editor), "Tables of the Incomplete Beta-Function", Cambridge University Press (Proprietors of Biometrika), 1934.
- ¹⁹ L. D. Russell, "Thermal Analysis of Space Radiators", Unpublished Thesis at Rice University, 1966.
- ²⁰ R. J. French, "Redesign of the Apollo Environmental Control System Radiator (Contract NAS 9-2772)" LTV Report No. 00.465, June, 1964.