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**DIXON, Thomas Nelson, 1939-
A STUDY ON STABILITY AND INCIPIENT
TURBULENCE IN POISEUILLE AND PLANE
POISEUILLE FLOW BY NUMERICAL FINITE
DIFFERENCE SIMULATION.**

**Rice University, Ph.D., 1966
Engineering, chemical**

University Microfilms, Inc., Ann Arbor, Michigan

RICE UNIVERSITY

A STUDY ON STABILITY AND INCIPIENT
TURBULENCE IN POISEUILLE AND PLANE POISEUILLE
FLOW BY NUMERICAL FINITE DIFFERENCE SIMULATION

by

Thomas N. Dixon

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
Doctor of Philosophy

Thesis Director's signature:

Houston, Texas

April, 1966

ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation to the following persons and organizations:

Dr. J. D. Hellums for his supervision and aid in this study and in the preparation of this thesis.

Drs. H. A. Deans and F. A. Wierum, Jr. for their interest in this work and for serving on the oral examination committee.

The Rice University, Humble Oil, and Celanese Corporation for financial support.

The National Science Foundation for financial support of this investigation.

Staff of the Rice University Computer Project for their cooperation and assistance in the use of the Rice University Computer.

To my friends, who took nourishment and refreshments with me and whose companionship made graduate school very enjoyable.

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I. INTRODUCTION

"The theoretical method involved the integration of the equations for unsteady motion in a way that had not been accomplished and which considering the general intracability of the equations was not promising."

Reynolds (1883)

The prediction of the instability of Poiseuille flow has been of interest since the experiments of Reynolds (1883) which demonstrated that laminar flow becomes unstable when the Reynolds number exceeds a critical value of about 2,000. Now, with the use of high speed calculating machines, it is possible to do what Reynolds called "not promising"--the direct integration of the Navier-Stokes equations. Thus with the use of finite differences to directly integrate the equation of motion, the effect of non-linearities on the flow can be studied.

Since techniques for direct solution of the Navier-Stokes equations became available only recently, early attempts at prediction of instability in Poiseuille flow were made by linear analysis which assumes that the amplitude of the disturbance is infinitesimal. The results of Rayleigh (1892) led to the conclusion that Poiseuille flow is stable to infinitesimal disturbances; a result that has been confirmed recently by Corcos and Sellars (1959), Schensted (1962), and Gill (1965). The experimental evidence of Ekman (1910), who

maintained laminar flow up to Reynolds numbers of 40,000, and of Leite (1956), who showed that Poiseuille flow was stable to small axisymmetric disturbances up to Reynolds numbers of 13,000, also tends to confirm this theoretical result.

As a companion problem to the stability of Poiseuille flow, the stability of plane Poiseuille flow had been of great interest to researchers in theoretical fluid mechanics. The result of Lin (1955), who showed that plane Poiseuille flow is stable or unstable to infinitesimal disturbances depending upon the Reynolds number and the frequency of the disturbances has led to a paradox between Poiseuille flow and plane Poiseuille flow. The result that with respect to infinitesimal disturbances Poiseuille flow is stable for all Reynolds numbers and plane Poiseuille flow may be unstable for Reynolds numbers greater than 5,300 has led some researchers as late as 1960 to question the validity of the previous work in the study of stability of Poiseuille flow. To quote Birkhoff (1960)--"In spite of valiant attempts, the observed instability of Poiseuille flow had defied mathematical analysis. It has even been conjectured that Poiseuille flow is stable with respect to infinitesimal perturbations in perfectly smooth circular pipes. However, the instability of plane Poiseuille flow between parallel plates, even for two-dimensional perturbations if $R > 5,300$, is now fairly

reliably established. Hence this conjecture seems unlikely."

The experimental work of Leite (1956) in Poiseuille flow and the theoretical work of Meksyn and Stuart (1951) in plane Poiseuille flow have shown that the critical Reynolds number decreases as the amplitude of the disturbance increases.

It is the purpose of this work to demonstrate the feasibility of investigating the stability of the two before mentioned flows by direct solution of the equation of motion by the method of finite differences. With finite differences, disturbances of finite amplitude can be studied thus enabling direct comparison of the results with those obtained from experiment and non-linear theoretical study. While if very small amplitudes are used, the results may be compared with linear theory.

The use of finite differences is not without limitation. Because of limited speed and/or storage space of the computer, only two dimensional flow could be studied. While this reduction in space dimensions from three to two is major, the experimental results of Leite (1959) and the theoretical investigation of Squire (1933) have shown that two dimensional disturbances are less stable than three dimensional ones. Hence, two dimensional studies in stability yield a lower limit for the critical Reynolds number. Nearly all of the prior work has of course been in two dimensions. An additional

limitation of the difference approach lies in the fact that it is impossible to carry the calculations out to a true steady state. One can only determine the character of the flow from a relatively short initial period of the disturbance.

II. NAVIER-STOKES EQUATIONS

The equation of motion for an incompressible, isothermal fluid in vector form is given by equation 2.1,

$$(2.1) \quad (V_t + (V \cdot \nabla)V) = -\nabla P + \nabla^2 V$$

and the equation of continuity is given by equation 2.2,

$$(2.2) \quad \nabla \cdot V = 0$$

where the operators are given by Bird, Stewart, and Lightfoot (1960).

Plane Poiseuille Flow:

Consider a co-ordinate system situated halfway between two parallel infinite flat plates which are a distance $2h$ apart as shown in Figure 1a. Assuming symmetry in the z direction, we can write equation 2.1 in the following form after making it dimensionless based on h , half the channel width, and U^0 , the centerline velocity of the undisturbed laminar flow.

$$(2.3a) \quad u_t + uu_x + vv_y = -(P/\rho)_x + R^{-1}(u_{xx} + u_{yy})$$

$$(2.3b) \quad v_t + uv_x + vv_y = -(P/\rho)_y + R^{-1}(v_{xx} + v_{yy})$$

where the subscripts denote partial derivatives. The equation of continuity is

$$(2.4) \quad u_x + v_y = 0$$

PLANE POISSEUILLE FLOW

POISEUILLE FLOW

FIGURE 1- CO-ORDINATES FOR PLANE
POISEUILLE AND POISEUILLE
FLOW

It has been found by Aziz (1965) that equations 2.3 and 2.4 are very difficult to solve in this form. Their solution can be greatly facilitated if we eliminate pressure between equations 2.3a and 2.3b and define a new variable, vorticity, as

$$(2.5) \quad \omega = u_y - v_x$$

and introduce a stream function, ψ , defined as

$$(2.6a) \quad u = \psi_y$$

$$(2.6b) \quad v = -\psi_x$$

We obtain equations 2.7 and 2.8

$$(2.7) \quad \omega_t + u\omega_x + v\omega_y = R^{-1}(\omega_{xx} + \omega_{yy})$$

$$(2.8) \quad \omega = \psi_{xx} + \psi_{yy}$$

Poiseuille Flow:

Consider a co-ordinate system situated in the center of a tube as shown in Figure 1b. After assuming symmetry in the angular direction, we can write equation 2.1 in the following form after making it dimensionless based upon h , the radius, and U^0 , the centerline velocity of the undisturbed laminar flow.

$$(2.9a) \quad u_t + uu_x + vv_y = -(P/\rho)_x + R^{-1}(r^{-1}(ru_r)_r + u_{xx})$$

$$(2.9b) \quad v_t + uv_x + vv_y = -(P/\rho)_r + (((rv)_r/r)_r + v_{yy})$$

The equation of continuity is

$$(2.10) \quad u_x + (rv)_r/r = 0$$

Again the equations are not solved in this form. If we eliminate pressure between equations 2.9a and 2.9b and define a new variable, vorticity, as

$$(2.11) \quad \omega = r(u_r - v_x)$$

and introduce a stream function, ψ , defined as

$$(2.12) \quad u = \frac{1}{r} \psi_r$$

$$(2.13) \quad -v = \frac{1}{r} \psi_r$$

we obtain the following relations.

$$(2.14) \quad \omega_t + u\omega_x + v\omega_r - 2\omega v/r = R^{-1}(\omega_{xx} + \omega_{rr} - \frac{\omega_r}{r})$$

$$(2.15) \quad \omega = \psi_{rr} - \frac{\psi_r}{r} + \psi_{xx}$$

To increase accuracy in the computation, a change of variables was made in equations 2.7, 2.8, 2.14, and 2.15 and the definitions of the stream functions. If we let $\omega = \omega_0 + g$, $u = U_0 + u'$, $v = v'$ and $\psi = \psi_0 + f$ where ω_0 , ψ_0 , U_0 are the steady state laminar solutions of the appropriate equations, and then substitute these definitions into equations 2.7, 2.8, 2.12, 2.13, 2.14, 2.6a, 2.6b, and 2.15, we obtain, while remembering that the steady state solutions satisfy the differential equations exactly, for

plane Poiseuille flow the following relations.

$$(2.16) \quad g_t + ug_x + vg_y = R^{-1}(g_{xx} + g_{yy}) - v(\omega_o)_y$$

$$(2.17) \quad g = f_{xx} + f_{yy}$$

$$(2.18) \quad u' = f_y$$

$$(2.19) \quad v = -f_x$$

We obtain for Poiseuille flow these following relations.

$$(2.20) \quad g_t + ug_x + vg_r - 2vg/r = R^{-1}(g_{xx} + g_{rr} - g_r/r)$$

$$(2.21) \quad g = f_{xx} + f_{rr} - f_r/r$$

$$(2.22) \quad u' = f_r/r$$

$$(2.23) \quad v = -f_x/r$$

The steady state profiles are

$$(2.24) \quad U_o = 1 - y^2; \quad \omega_o = -2y; \quad \psi_o = y - y^3/3.0$$

for plane Poiseuille flow, and

$$(2.25) \quad U_o = 1 - y^2; \quad \omega_o = -2r^2; \quad \psi_o = r^2/2.0 - r^4/4.0$$

for Poiseuille flow. It is to be noted that the equations were not linearized in the foregoing change of variables.

Although the vorticity, g , can be eliminated from these equations, it has been found by Pearson (1965) that this increases the computational time.

III. LINEAR THEORY

The method of linear analysis is essentially the same for both plane Poiseuille and Poiseuille flow. A good analysis of the method may be found in a monograph by Lin (1955). The method consists of imposing a disturbance of the form $f = \sum \phi_m(y) \exp(m i(\alpha x + \beta t))$ where β is real and α is complex. Linear theory assumes that a real disturbance can be expressed as an infinite set of small disturbances which have no interaction between themselves or the mean flow. The linear analysis is sometimes called the "normal mode" analysis. The disturbance f is substituted into equation 2.17 or 2.21 and then this result is substituted into equation 2.16 or 2.20 respectively, then terms of order two or greater in ϕ_m are neglected. Thus similar, fourth order complex non-linear ordinary differential equations for each mode, m , of the disturbance can be written. These equations are commonly known as the Orr-Sommerfeld equations. The equation for the first mode for plane Poiseuille flow is

$$(3.1) \quad (D^2 - \alpha^2)^2 \phi_1 = i \alpha R ((U_0 - \beta)(D^2 - \alpha^2) \phi_1 - (D^2 U_0) \phi_1)$$

where
$$D = \frac{d}{dy}$$

with boundary conditions of

$$\phi = \phi_y = 0 \quad y = \pm 1$$

The Orr-Sommerfeld equation for Poiseuille flow is

$$(3.2) \quad ((L - \alpha^2) - (\beta + i\alpha R(1-r^2))(L - \alpha^2))\phi_1 = 0$$

where $L = D^2 + D/r - 1/r^2$; $D = \frac{\partial}{\partial r}$ with boundary conditions

$\phi = \phi_r = 0 \quad r = 1 \quad \phi$ regular at $r = 0$. To determine if the flow is stable or unstable, we need to know whether the imaginary part of α is positive (stable) or negative (instability), since

$$u' = f_y = \text{Re}(\phi_y \exp(i(\alpha x + \beta t))) = \phi_y \exp(-\alpha_i x) \cos(\alpha_r x + \beta t)$$

for plane Poiseuille flow or

$$u' = f_r/r = \text{Re}\left(\frac{\phi_r}{r} \exp(i(\alpha x + \beta t))\right) = \frac{\phi_r}{r} \exp(-\alpha_i x) \cos(\alpha_r x + \beta t)$$

for Poiseuille flow. Thus for the disturbance to grow in distance α_i must be negative.

It has been found for Poiseuille flow that only positive values of α_i exist. This has been the conclusion of Schensted (1960), Corcos and Sellars (1959), and Gill (1965). This indicates that laminar flow in a pipe is stable at all Reynolds numbers for infinitesimal amplitude disturbances. Gill (1965) calculated the set of eigenvalues, α_i , α_r , and β , for equation 3.2 and compared his results with the experimental data of Leite (1956). His results will be discussed with Leite's results in the experimental section. The paper by

Gill (1965) contains an excellent review of the development of the linear theory for Poiseuille flow.

The results for plane Poiseuille flow are somewhat different than for Poiseuille flow. An unstable solution of equation 3.1 may be calculated. If we let $\alpha_1 = 0$, we can determine a curve of neutral stability, i.e. the disturbance neither grows nor decays with distance. Lin (1955) calculated this curve which is displayed in Figure 2. Thomas (1953) made a series of direct numerical integrations of equation 3.1 to confirm this curve. The minimum critical Reynolds number is 5,300.

In summary, linear theory, which does not take into account interactions between the modes of the disturbance and between the disturbance and the mean flow, predicts that, to infinitesimal amplitude disturbances, Poiseuille flow is stable for all Reynolds numbers and frequencies and plane Poiseuille flow exhibits a region of instability for some Reynolds numbers and frequencies.

**FIGURE 2 - CURVE OF NEUTRAL STABILITY
FOR PLANE POISSEUILLE FLOW
(cf. LIN 1955)**

IV. NON-LINEAR THEORY

Since the theory of stability of Poiseuille and plane Poiseuille flow to infinitesimal disturbances has been fairly satisfactorily developed, researchers in recent years have concentrated upon the solution of the problem of what happens to an unstable disturbance as it becomes finite in amplitude. By this type of study, researchers have sought to gain an insight into the mechanism of the flow in the region between the onset of the unstable disturbance and fully developed turbulence. Progress in this area has been somewhat slow. In fact, the only satisfactory analyses of non-linear disturbances have been made in theoretical plane Poiseuille flow. It is to be noted that no theoretical study of non-linear disturbances in Poiseuille flow has been reported in the literature.

The first study carried out to take into account nonlinearities in plane Poiseuille flow was by Meksyn and Stuart (1951). They solved the Orr-Sommerfeld equation for infinitesimal disturbances, equation 3.1, along with the equation of mean motion.

$$(4.1) \quad \rho \overline{u'v'} = P_y + \mu U_y$$

This work allowed for the interaction between the disturbance and the mean flow but did not take into consideration

interactions between disturbances. Their results show a distinct lowering of the minimum critical Reynolds number as the amplitude is increased. The minimum critical Reynolds number which they found was about 2900. The work presents no description of what would be the character of a finite disturbance in an actual physical situation.

A theory of non-linear disturbances in plane Poiseuille flow which takes into account all interactions between the mean flow and the disturbances and the disturbances themselves and is most physically reasonable has been developed by Stuart and/or Watson in a series of papers from 1956 to 1962. The paper which is most closely related to the present work is by Watson (1962). In almost all of the prior work, including most of the papers by Stuart and/or Watson, the disturbances were formulated as being periodic in x while decaying or amplifying with time. In this paper by Watson (1962), the disturbance was formulated as being periodic in time and decaying or amplifying in x . This is the form of the disturbance that one encounters in experimental work and also in the present work. His method consists of letting the stream function be represented by a Fourier Series

$$(4.2) \quad \psi = \psi_0 + \sum_{n=1}^{\infty} (\phi_n \exp(-ni\beta t) + \phi_n \exp(ni\beta t))$$

This series expression is substituted into equations 2.7 and

2.8 and terms of the Fourier components are equated. After some manipulation, there results an infinite set of non-linear ordinary differential equations for the amplitude of the disturbance. These equations have not been solved explicitly but several conclusions about a disturbance were noted by Watson. These conclusions are only applicable near the neutral curve of Figure 2. He found that within the neutral curve, which he called the supercritical region, a disturbance may amplify as it progresses downstream to an equilibrium state. A disturbance of amplitude greater than this equilibrium amplitude will decay to this equilibrium amplitude. Thus states of equilibrium may be stable to infinitesimal disturbances. In the region outside the neutral curve, which he called the subcritical region, if an equilibrium state exists, a disturbance which is smaller than the equilibrium amplitude will decay with distance downstream while a disturbance of amplitude greater than this equilibrium amplitude will amplify with distance downstream. This shows that, although the basic flow is stable to infinitesimal disturbances in this region, it can be unstable to certain finite disturbances. Watson does not present any numerical calculations. The extreme difficulty of solving an infinite set of ordinary differential equations, even by approximate methods, has prevented the author from giving any quantitative

results. Another limitation on this method is that it does not take into account arbitrary disturbances. In summary, the method of perturbations appears to have severe limitations in the study of finite disturbances because of the difficulties involved in the computational procedures used in the problem.

An excellent summary of non-linear stability theory is given in a monograph by Eckhaus (1965). Another summary is given by Shen in a reference book edited by Moore (1964).

V. EXPERIMENTAL RESULTS

In contrast to the prior work of a theoretical nature, the experimental study of the stability of Poiseuille flow and plane Poiseuille flow to small or finite disturbances has been quite limited in scope. Although no work has been done quantitatively to investigate the stability of plane Poiseuille flow to small disturbances by experimental methods, the experimental works of Sherlan (1960) and Whan and Rothfus (1959) in plane Poiseuille flow (made using a rectangular pipe of low depth to width ratio) have shown a transition Reynolds number of about 2,700.

The most important experimental work in the stability of Poiseuille flow has been that of Leite (1959). In this work, he studied the stability of Poiseuille flow to small periodic disturbances by the oscillation of a very thin sleeve inside a pipe and the measurement of the response at various points downstream with a hot wire anemometer and oscilloscope. By comparison of the amplitude of the disturbance at various distances from the disturbance, Leite was able to calculate α_i which is called the decay rate. He was also able to determine α_r which is called the phase lag or the wave number. Decay rates and phase lags for experimental or numerical works can be calculated by assuming that the first mode of the

linear theory dominates and thus the disturbance follows the form of $\Phi_1 \exp(i(\alpha x + \beta t))$. Thus the amplitude is $\Phi_1 \exp(-\alpha_i x)$ and the periodicity with distance is given by $\cos(\alpha_r x)$. If the log of the amplitude of the disturbance from experiment or numerical calculation is plotted versus distance, the decay rate, α_i , can be determined by multiplying the slope by -2.303. The phase lag, α_r , is determined by dividing 2π by the distance between the positive peaks of two waves which are next to each other.

Leite also studied the effect of large amplitude disturbances by using a ring and airfoil suspended in the free stream. He found that for small disturbances of about 0.1 per cent of the mean flow (made using the thin sleeve) the flow was stable up to Reynolds numbers of 13,000, the experimental maximum of his equipment, and that the non-symmetric part of the disturbance decayed more rapidly than the symmetric part. The disturbances of larger amplitude (about 1 per cent of the mean flow made with the ring or airfoil) caused instabilities at much lower Reynolds numbers. Leite's decay rates and phase lags agreed approximately with those calculated by Corcos and Sellar (1959) in a companion theoretical work. But since this theoretical work was based on a disturbance periodic in x , loss of accuracy was incurred in converting Leite's decay rates and phase lags in x to decay rates and

phase lags in time. This problem was alleviated by Gill (1965) when he calculated the eigenvalues for a disturbance periodic in time and compared them to Leite's work. The results that are most pertinent to this work are shown in Figures 3a and 3b. In Figure 3a, the variation of the non-dimensional wave speed, $C = \frac{\beta}{\alpha_r}$, with the non-dimensional frequency, β_R , is shown for several modes of a disturbance near a radius defined by $r_m^2 = 1 - \frac{\beta}{\alpha_r}$. Also given is variation for several modes, q , for a disturbance near the wall. Several experimental points as calculated by Gill from Leite's work are also given. The experimental points are denoted by different symbols for different Reynolds numbers and different radial positions. There is good agreement for the q modes but not the m modes. In Figure 3b, variation of the non-dimensional spatial decay rate, $\alpha_1 R$, with non-dimensional frequency, β_R , is shown for several modes, m , for a disturbance near a radius defined by $r_m^2 = 1 - \frac{\beta}{\alpha_r}$. Several experimental points as determined by Gill from Leite's work are also included. The agreement is good. Gill also calculated theoretical disturbance profiles made up of several modes which agreed with Leite's experimental profiles.

Jenny (1963) using water noted no growth of axisymmetric disturbances up to Reynolds numbers of 5,300 in a 1-inch diameter tube at frequencies from 1/3 to 20,000 cycles per second.

FIGURE 3 - THE CHANGE OF (a) THE NON-DIMENSIONAL WAVE-SPEED $c = \beta/\alpha_r$ AND (b) THE NON-DIMENSIONAL SPATIAL DAMPING-RATE $\alpha_i R$ WITH THE NON-DIMENSIONAL FREQUENCY βR (cf GILL)

VI. PREVIOUS NUMERICAL WORK

The solution of viscous incompressible time independent and time dependent flows in two and three dimensions by finite difference approximation of the appropriate partial differential equations has been of increasing interest in the last few years. In the area of steady viscous flow, several authors have solved the time independent form of the vorticity transport equation and the stream function equation by relaxation methods. In the solution of the vorticity-stream function equations, it is necessary to use stream function values near and on a solid boundary to approximate the vorticity on the boundary. Since the vorticity for plane flow equals $U_y - V_x$ and $V_x = 0$ on a wall, the vorticity on the wall equals U_y or ψ_{yy} in terms of the stream function. Thus the vorticity on the wall can be approximated by a finite difference scheme of form

$$(6.1) \quad \omega_0 = \frac{\psi_{+1} - 2\psi_0 + \psi_{-1}}{(\Delta y)^2} + (\Delta y^2)$$

ψ_{-1} can be eliminated from the approximation since on the wall $u = \psi_y = 0$, i.e. $\psi_{+1} = \psi_{-1}$. Also since ψ_0 is known on the wall, the vorticity on the wall can be calculated from interior stream function points. A similar analysis holds for cylindrical flow.

Thom (1933) solved the problem of steady flow around a cylinder. Thom's work contained an error in that he used the incorrect definition of vorticity on the surface. He used $\omega = \psi_y$ instead $\omega = \psi_{yy}$. Wang and Longwell solved the problem of steady laminar flow in the inlet section of parallel plates. Paris and Whitaker solved the case of flow of a confined wake formed by the merging of two plane Poiseuille flow streams. The authors used the condition, which can be derived from the vorticity transport equation, that $\nabla^2 \omega = 0$. The finite difference scheme for ω_{yy} was centered at a half grid space into the flow field while ω_{xx} was centered on the wall. This boundary condition was used because the authors felt that the stream function on the boundary could only be used once. Since it was used as a boundary condition in the solution of the stream function equation, they concluded that it could not be used again in the calculation of the vorticity on the wall.

In the study of time dependent flow, researchers have used several finite difference schemes to solve the vorticity transport equation but relaxation methods have almost always been used to solve the stream function equation. Fromm (1963) made a study of vortex street development behind a rectangular cylinder using an explicit numerical scheme to solve the vorticity transport equation. The method is closely related

to the well-known DuFort-Frankel (1953) method. Using this same method, Fromm has solved the problem of heat transfer from a rectangular cylinder (1964) and thermal convection in a cell (1965). Wilkes (1963) solved the problem of thermal convection in a two dimensional cell using a method related to the alternating direction method of Peaceman and Rachford (1955). Aziz (1965) solved a two and three dimensional natural convection problem using an alternating direction method adapted from Douglas (1962). Pearson (1965) solved a two dimensional flow problem using an adaptation of the Crank-Nicolson centered scheme for the vorticity transport equation. This scheme was solved by over-relaxation when the flow was almost linear or by a modification of the alternating direction method of Peaceman and Rachford (1955) when the flow was non-linear.

DeSanto and Kellar (1962) used an implicit method based on the alternating direction method of Peaceman and Rachford (1955) to solve the problem of disturbances in transition flow over a flat plate. The method consisted of only using the x direction pass of the Peaceman-Rachford ADI method to advance the vorticity in Equation 2.16. The coefficients were evaluated at the previous time level. The authors imposed a periodic disturbance upon the laminar Blasius flow and followed it in time and space by the solution of Equations

2.16 and 2.17. Disturbances of amplitudes of order 10^{-5} and 10^{-1} of the free stream velocity were considered. The calculations were carried out for 1.75 cycles of the disturbance. The depth of the flow was three boundary layers, δ_0 , and was divided into fifteen equal spaces. The length of the flow field was five natural wave lengths and was spanned by 80 grid spaces for the low amplitudes. For the high amplitude case, 100 spaces were used to span two and one-half wave lengths. The time step was 0.0435, the period of the disturbance for low amplitudes, and 0.0145 for high amplitudes. The authors presented only results for the vorticity deviations of the disturbance. Near the wall at $y = 0.2 \delta_0$ the vorticity showed only decay. At y equal to $0.4 \delta_0$ the disturbance showed amplification and then decay with increasing x . At $0.6 \delta_0$ the flow amplified but the decay which followed was somewhat slower than at $0.4 \delta_0$. At $0.8 \delta_0$, the disturbance was only slightly amplified with correspondingly slight decay. For distances from the wall greater than $0.8 \delta_0$, the vorticity disturbance showed only decay. The flows for the high amplitude disturbance were very highly distorted but did not show indications of breaking up. Stable oscillations were indicated for the high amplitude case.

The amplification and then decay of the disturbance for the low amplitude case was explained on the basis of the

disturbance starting inside the neutral curve and then passing through the neutral curve into a region of stability. Thus the disturbance would amplify and then decay.

VII. FINITE DIFFERENCE FORMULATION AND SOLUTION

A. Vorticity Transport Equation:

Aziz (1965) and Pearson (1965) found that alternating direction implicit methods of solving the parabolic vorticity transport equation was more accurate, converge faster, and was more stable than explicit methods such as DuFort-Frankel. The alternating direction method of Peaceman and Rachford (1955) or a closely related method was used by Wilkes (1963), DeSanto and Kellar (1962) and Pearson (1965). The discretization error for these methods was of $O(\Delta x)^2 + (\Delta y)^2 + (\Delta t)$. Aziz (1965) used an adaptation of the alternating direction scheme developed by Douglas (1962). This method is a perturbation of the Crank-Nicolson equation and has a discretization error of $O(\Delta x)^2 + (\Delta y)^2 + (\Delta t)^2$. An adaptation of this method was used in this work since the method has a lower truncation error in time, greater stability, and faster convergence with only slightly greater computational difficulties than the method of Peaceman and Rachford.

Using the notation given in Appendix A, the finite difference scheme for the vorticity transport equation is given in Equations 7.1 and 7.2 for plane Poiseuille flow and in Equations 7.3 and 7.4 for Poiseuille flow. The first step is to implicitly difference Equation 2.16 for plane Poiseuille

flow or equation 2.20 for Poiseuille flow in the x direction using a Crank-Nicolson centered scheme for the x derivatives and an explicit scheme for the y or r derivatives. The term $-2.0vg/r$ in Equation 2.20 was differenced in the Crank-Nicolson centered scheme for the x-pass. The next step is to difference Equations 2.16 and 2.20 entirely using the Crank-Nicolson centered scheme.

Plane Poiseuille:

x-pass

$$(7.1) \quad g_t^{*n+1} + \frac{1}{2}u^{n+\frac{1}{2}}(g_x^{*n+1} + g_x^n) + v^{n+\frac{1}{2}}g_y^n \\ = R^{-1}(\frac{1}{2}(g_{xx}^{*n+1} + g_{xx}^n) + g_{yy}^n) + 2.0v^{n+\frac{1}{2}}$$

y-pass

$$(7.2) \quad g_t^{n+1} + \frac{1}{2}u^{n+\frac{1}{2}}(g_x^{*n+1} + g_x^n) + \frac{1}{2}v^{n+\frac{1}{2}}(g_y^{n+\frac{1}{2}} + g_y^n) \\ = R^{-1}(\frac{1}{2}(g_{xx}^{*n+1} + g_{xx}^n) + \frac{1}{2}(g_{yy}^{n+1} + g_{yy}^n)) + 2.0v^{n+\frac{1}{2}}$$

Poiseuille Flow:

x-pass

$$(7.3) \quad g_t^{*n+1} + \frac{1}{2}u^{n+\frac{1}{2}}(g_x^{n+1} + g_x^n) + v^{n+\frac{1}{2}}g_r^n - 2.0v^{n+\frac{1}{2}} \cdot \frac{1}{2}(g^{*n+1} + g)/r \\ = R^{-1}(\frac{1}{2}(g_{xx}^{*n+1} + g_{xx}^n) + g_{rr}^n - g_r^n)$$

r-pass

$$\begin{aligned}
 (7.4) \quad g_t^{n+1} + \frac{1}{2}u^{n+\frac{1}{2}}(g_x^{*n+1} + g_x^n) + \frac{1}{2}v^{n+\frac{1}{2}}(g_y^{n+1} + g_y^n) - v^{n+\frac{1}{2}}(g^{*n+1} + g^n)/r \\
 = R^{-1}(\frac{1}{2}(g_{xx}^{*n+1} + g_{xx}^n) + \frac{1}{2}(g_{rr}^{n+1} + g_{rr}^n - g_r^{n+1} - g_r^n))
 \end{aligned}$$

Thus on the x-pass values of vorticity, g^{*n+1} , are predicted and on the y- or r-pass these predicted values and the old level values of vorticity, g^n , are used to calculate "true" values of vorticity, g^{n+1} . The coefficients are evaluated at the half time level. If the definitions given in Appendix A are substituted into Equations 7.1 and 7.2 or Equations 7.3 and 7.4, two algebraic equations can be written. The algebraic equations are similar for plane Poiseuille and Poiseuille flow except for the definitions of the coefficients and forcing terms. The derivation of Equations 7.5 and 7.6 is given in Appendix B.

$$(7.5) \quad -A^{n+\frac{1}{2}} \cdot g_{i,j+1}^{*n+1} + B^{n+\frac{1}{2}} \cdot g_{i,j}^{*n+1} - C^{n+\frac{1}{2}} \cdot g_{i,j-1}^{*n+1} = D^n$$

$$(7.6) \quad -A'^{n+\frac{1}{2}} \cdot g_{i+1,j}^{n+1} + B'^n \cdot g_{i,j}^{n+1} - C'^{n+\frac{1}{2}} \cdot g_{i-1,j}^{n+1} = D'^n + R^{*n+1}$$

B. Stream Function Equation:

In the vorticity transport difference equations, u , u' , and v are evaluated at the half time level. In order to reduce the amount of storage needed in the computer, this

difference scheme of the stream function equation was used.

$$(7.7) \quad \nabla^2 f^{n+\frac{1}{2}} = \frac{1}{2}(g^{n+1} + g^n)$$

This was done in this manner so that values of f^{n+1} would not have to be stored as in the case where $f^{n+\frac{1}{2}}$ would be calculated by Equation 7.8.

$$(7.8) \quad f^{n+\frac{1}{2}} = \frac{1}{2}(f^{n+1} + f^n)$$

The method of successive over-relaxation, SOR, was used to solve Equation 7.7. This method was chosen rather than any of the several other well-known methods for elliptic problems (cf. Varga (1962)). The convergence rate of SOR may be slightly slower than some other methods. However, the efficiency of the computer coding and storage requirements make it the more desirable method. The finite difference scheme for Equation 7.7 is given in Equation 7.9 where ν represents the number of iterations and η is the relaxation parameter.

$$(7.9) \quad f_{i,j}^{\nu+1} = f_{i,j}^{\nu} + \eta(\lambda_1(f_{i,j+1}^{\nu} - 2.0f_{i,j}^{\nu} + f_{i,j-1}^{\nu+1}) \\ + \lambda_2(\lambda_3 f_{i+1,j}^{\nu} - 2.0f_{i,j}^{\nu} + y f_{i-1,j}^{\nu+1}) - \frac{1}{2}\lambda_5(g_{i,j}^{n+1} + g_{i,j}^n))$$

where

$$\lambda_1 = \lambda_3 = \lambda_4 = 1; \quad \lambda_2 \left(\frac{\Delta x}{\Delta y}\right)^2 \quad \lambda_5 = (\Delta x)^2$$

for plane Poiseuille and

$$\lambda_1 = \left(\frac{\Delta r}{\Delta x}\right)^2; \quad \lambda_2 = 1; \quad \lambda_3 = 1 - \frac{\Delta r}{2r}; \quad \lambda_4 = 1 + \frac{\Delta r}{2r}; \quad \lambda_5 = (\Delta r)^2$$

for Poiseuille. The convergence of equation 7.9 was based on the relative error test of equation 7.10. An analysis of the absolute error is given in Appendix C.

$$(7.10) \quad \sum_{i=2}^{m-1} \sum_{j=2}^{n-1} \left| \frac{f_{i,j}^{\nu+1} - f_{i,j}^{\nu}}{f_o(i \Delta y)} \right| \leq \epsilon$$

ϵ was chosen so that the solution did not change significantly with smaller ϵ . The relaxation parameter, η , was chosen by a search technique. First a value of the parameter, η , was found by experiment such that Equation 7.9 converged to within the error given in Equation 7.10 and the number of iterations needed was recorded. Then for the next time step, the value of η was incremented and the number of iterations needed for convergence was noted. If this number of iterations was less than the number needed for the previous time step, the parameter, η , was again incremented in the same direction but if it was greater then η was incremented in the opposite direction. The optimum η was quickly found after a few time steps. Approximately the same η was found when the initial η was both above and below the optimum value of η . The optimum η value of η ranged from 0.35 to 0.63 depending upon the grid

spacing and the parameters of the problem.

C. Boundary Conditions:

The condition that $u = u' = v = 0$ at the wall corresponding to no slip and no mass transfer through the wall was used. The condition at the centerline was symmetry of the flow. In Poiseuille flow, the symmetry is implied by the two-dimensional assumption. The disturbance was formulated to satisfy the boundary conditions at the centerline and wall, the differential equations, and the conservation of mass. The initial condition satisfied the boundary conditions and agreed with the upstream disturbance at x and t equal to zero. It constituted a decaying wave.

The determination of a proper downstream boundary condition in sinuous laminar or incipient turbulent flow for a finite length of flow field is, in principle, impossible. Ideally a flow field of infinite length should be used. In practice, the best condition is one that has minimal influence upon the results upstream. Through experimentation, it was found that forcing the flow to be periodic, $g_{xx} = -c^2 g$ and $f_{xx} = -c^2 f$, at the downstream boundary effected the solution of the flow upstream only slightly. A discussion of the effect of this boundary condition on results upstream is given in Appendix D. DeSanto and Kellar (1962) formed a similar conclusion.

Some runs were made using the condition that the flow at the downstream boundary was Poiseuille flow. In these runs, strong instabilities occurred when the disturbance became large near the boundary. Although these instabilities were reflected far upstream as time progressed, the upstream flow before the instability reached it was very similar to that obtained with the periodic boundary condition.

A third set of runs was made using the condition that $g = -cg_x$ and $f = -cf_x$ at the downstream boundary. This condition has a greater effect upon the upstream results than the periodic condition but it did not cause instability. The periodic condition was used for all results presented in this thesis.

Finite Difference Formulation of Boundary Conditions:

At the wall we have the following conditions.

$$(7.11) \quad u = u' = f_y = v = -f_x = -f_{xx} = 0 \text{ for plane Poiseuille flow}$$

$$(7.12) \quad u = u' = f_r/r = v = -f_x/r = f_{xx} = 0 \text{ for Poiseuille flow.}$$

The stream function, f , equals zero at the wall since the initial condition of a decaying wave must be consistent with the wall conditions. Since $\nabla^2 f = g$, we have at the wall $g = f_{yy}$ for plane Poiseuille flow and $g = f_{rr}$ for Poiseuille

flow. Using the condition that the normal derivative of the stream function is zero, we can obtain this difference form for the vorticity at the wall:

$$(7.13) \quad g_{M,j}^n = (-7.0f_{M,j}^n + 8.0f_{M-1,j}^n - f_{M-2,j}^n)/2(\Delta y)^2 + O(\Delta y)^2$$

Since symmetry was assumed on the centerline, the boundary conditions are $u'_y = f_{yy} = v = -f_x = f_{xx} = 0$ for plane Poiseuille flow and $u'_r = (f_r/r) = v = (-f_x/r) = f_{xx} = 0$ for Poiseuille flow. Thus the vorticity, g , equals zero on the centerline in both flows. The stream function, f , must also be zero so that the initial condition will satisfy the centerline boundary conditions.

The disturbance and the initial condition were chosen such that they satisfied the boundary condition and were consistent with each other. The initial condition was chosen such that it was a decaying wave.

$$(7.14) \quad f_s = A_m \cdot f_D \cdot \exp(-A_o x) \cdot \cos(A_r x)$$

The form of the vorticity initial condition equation was found by substituting Equation 7.14 into Equation 2.17 for plane Poiseuille flow to obtain Equation 7.15 and into Equation 2.21 to obtain Equation 7.16 for Poiseuille flow.

$$(7.15) \quad g_s = A_m \cdot (f_{D_{yy}} + (A_o^2 - A_r^2)f_D) \exp(-A_o x) \cdot \cos(xA_r)$$

$$(7.16) \quad g_s = Am \cdot ((f_{D_{rr}} - f_{D_r}/r) + (A_o^2 - A_r^2)f_D) \exp(-A_o x) \cdot \cos(xA_r)$$

The vorticity disturbance, g_D , is thus equal to

$$f_{D_{yy}} + (A_o^2 - A_r^2)f_D$$

for plane Poiseuille flow and equal to

$$(f_{D_{rr}} - f_{D_r}/r) + (A_o^2 - A_r^2)f_D$$

for Poiseuille flow.

The conservation of mass is given in Equation 7.17 for plane Poiseuille flow and in Equation 7.18 for Poiseuille flow. Thus the disturbance will satisfy the conservation of mass in both flows because the stream function is zero at both the wall and the centerline.

$$(7.17) \quad \int_0^1 u'_D dy = \int_0^1 f_{D_y} dy = 0$$

$$(7.18) \quad \int_0^1 r u'_D dr = \int_0^1 r \frac{f_{D_r} dr}{r} = \int_0^1 f_{D_r} dr = 0$$

The downstream boundary conditions were the same for both flows and are given in Equations 7.19 and 7.20.

$$(7.19a) \quad f_{xx} = -A_r^2 f$$

The constant A_r^2 is obtained by assuming that the spatial frequency is the same as the initial condition. The value of g and f at the downstream boundary were calculated from the first two upstream points by these relations.

$$(7.19b) \quad f_{i,N} = (2 - (\Delta x)^2 A_r^2) f_{i,N-1} - f_{i,N-2}$$

$$(7.20b) \quad g_{i,N} = (2 - (\Delta x)^2 A_r^2) g_{i,N-1} - g_{i,N-2}$$

D. Computational Procedure:

The first step was to select a proper disturbance. A polynomial in y or r was selected as the form of the disturbance. The coefficients were determined from the boundary conditions. The disturbance profile for plane Poiseuille flow is given in Equations 7.21, 7.22, and 7.23. The shapes of these profiles are given in Figure 4.

$$(7.21) \quad f_D = y - 2.0y^3 + y^5 = y(1.0 - y^2)^2$$

$$(7.22) \quad u'_D = f_{D_y} = 1.0 - 6.0y^2 + 5.0y^4 = (1.0 - 5.0y^2)(1.0 - y^2)$$

$$(7.23) \quad g_D = -12.0y + 20.0y^2 + (A_o^2 - A_r^2) f_D$$

The disturbance profiles for Poiseuille flow are given in Equations 7.24, 7.25, and 7.26. The shapes of these profiles

**FIGURE 4 - DISTURBANCE PROFILES FOR
PLANE POISEUILLE FLOW**

are given in Figure 5.

$$(7.24) \quad f_D = \frac{1}{2}r^2 - r^3 + \frac{1}{2}r^4 = \frac{1}{2}r^2(1.0 - r)^2$$

$$(7.25) \quad u'_D = f_D / r = 1.0 - 3.0r + 2.0r^2 = (1.0 - 2.0r)(1.0 - r)$$

$$(7.26) \quad g_D = -3.0r + 4.0r^2 + (A_o^2 - A_r^2)f_D$$

The computational procedure for plane Poiseuille flow is outlined in the following steps. The procedure for pipe flow is exactly the same except a different disturbance is used.

- (1) Determine the disturbance profile to be used and the parameters of the problem $-A_r$, A_o , A_m , Δt , Δx , Δy , etc.
- (2) Specify the initial condition for the vorticity and stream function at each grid point in the flow field by Equations 7.14 and 7.15.
- (3) The disturbance at the upstream boundary is advanced in time by Equations 7.27 and 7.28.

$$(7.27) \quad f_{1,0}^n = f_D(i \Delta y) A_m \cdot \cos(\beta_t(n \Delta t))$$

$$(7.28) \quad g_{1,0}^n = g_D(i \Delta y) A_m \cdot \cos(\beta_t(n \Delta t))$$

- (4) Equation 7.5, the x-pass of the vorticity transport equation, is applied at each grid point $j = 1$ to $N - 1$

FIGURE 5 - DISTURBANCE PROFILES FOR POISEUILLE FLOW

at y equal to $i\Delta y$ to obtain a set of $N - 2$ simultaneous equations in N values of g^{*n+1} . I equals 1 initially.

(5) We have now $N - 2$ equations with N unknowns. One of the unknowns is specified at the upstream boundary by step 3. The value of vorticity at the downstream boundary can be eliminated by using Equation 7.20b. Thus we obtain $N - 2$ unknowns and $N - 2$ tridiagonal equations for g^{*n+1} . These tridiagonal equations can be solved by a simple recursion formula (cf. Richtmyer 1957).

(6) Steps 4 and 5 are repeated at each row, $i = 2$ to $N - 1$. Note that the coefficients are evaluated from the initial stream function data or the previous time level values for the first calculation of g^{*n+1} .

(7) The stream function at the half time level is obtained using g^{*n+1} as g^{n+1} in Equation 7.9. Equation 7.19b was used to obtain values of the stream function at the downstream boundary after each iteration.

(8) A new value of vorticity, g^{*n+1} , is calculated on the wall using Equation 7.13.

(9) Equation 7.6, the y-pass of the vorticity transport equation, is applied at each grid point $i = 1$ to $M - 1$ at

$x = j \Delta x$. The first j is one.

(10) Using the value of g on the wall calculated in step 8 and the centerline vorticity which is zero, the $M - 2$ equations are solved for g^{n+1} .

(11) Steps 9 and 10 are repeated on each column for $J = 2$ to $N - 1$. Note that the coefficients used in Equation 7.6 are not true values at the half time level but only approximations.

(12) g^{n+1} at the downstream boundary is calculated using Equation 7.20b.

(13) New values of the stream function at the half time level are calculated using Equation 7.9.

(14) New values of g^{n+1} are calculated for the wall using Equation 7.13.

(15) If necessary, steps 4 to 14 are repeated. It is to be noted that since the differential equations are non-linear steps 4 to 14 constitute an iteration scheme. The number of iterations needed will depend upon the desired accuracy of the solution, upon the size of the time step, the grid spacing, and upon the parameters of the problem.

(16) Steps 3 to 15 are repeated to advance the solution in time.

The stability of the alternating direction finite difference scheme for linear and mildly non-linear parabolic and elliptic equations has been shown by Douglas (1962). A proof of stability of the difference scheme for the non-linear parabolic equations of the type of this work is not possible. As is customary, it was assumed that since the method was stable for mildly non-linear equations that it was stable for non-linear equations. The results seem to confirm this assumption.

VIII. RESULTS AND DISCUSSION

In the solution of partial differential equations by finite difference methods, one is concerned with these problems: the accuracy of the finite difference method, computational time, and what parameters of the problem should be investigated. All three of these problems are intertwined and cannot be separated.

A. Selection of Parameters:

The parameters of the problem include the Reynolds number, the frequency of the disturbance, the initial disturbance, the shape of the upstream disturbance and its amplitude, and the length of the flow field. The selection of these parameters was based upon theory or experimental results where possible so that comparisons with prior work could be made. Some of the parameters associated with the disturbance and the initial condition had to be chosen through experimentation and are somewhat arbitrary. The decay rate and spatial frequency of the disturbance after a few cycles varied with the spatial dimensions and frequency of the upstream disturbance and were different for the vorticity and stream function. Hence, it was impossible to choose the initial decay rate, A_0 , and the initial space frequency, A_r , such that the transient between the initial condition and a

"steady-state" solution would be small. After some experimental runs, $A_o = 0.1$ and $A_r = 1.0$ were chosen for all runs. These values were chosen as being approximate average values of the decay rate and frequency after a few cycles.

For plane Poiseuille flow the frequency, β_t , was chosen such that Reynolds numbers could be chosen sufficiently far apart and still fall inside and on both sides of the neutral curve in Figure 3. For most runs, a frequency, β_t , of 1.0 was used. Frequencies, β_t , of 1.25 and 0.75 were also used in two runs with $R = 10,000$ so that points above and below the neutral curve could be studied.

For Poiseuille flow the frequency, β_t , was chosen equal to 1.0. The dimensionless frequencies that Leite (1959) used in his experimental work were from about 0.3 to 0.8.

The disturbance profile were required to satisfy the boundary conditions at the wall and centerline and be consistent with the initial conditions. The disturbances used were somewhat arbitrary in shape, but results from the shapes used indicate that the results would hold qualitatively for other shapes.

The length of the flow field was very important. It was found that a length of approximately $37h$ was needed to show the character of the flow in many cases. This was especially true in flow with low amplitude disturbances where

the character of the disturbance depended upon whether the disturbance reamplified downstream or only decayed. From Leite's experimental work with high amplitude disturbances, it was found that most of the transition from laminar to turbulent flow occurred between 20h and 80h from the upstream disturbance. Amplitudes of the order of 10^{-5} and 10^{-1} of the average velocity were used in most runs.

B. Accuracy and Computation Time:

Since the accuracy of the finite difference scheme and the computation time are somewhat proportional to each other, a compromise as to the grid spacing, time step, and accuracy of the iterations was made. The grid spacing and time step used for most of the results was $\Delta x = 0.175$, $\Delta y = 0.1$ and $\Delta t = 0.15$. It was found that when these were changed by up to 50% that the character of the flow changed only slightly. The maxima and minima occurred at approximately the same point in time and space and the absolute value was changed from negligible small amounts up to a maximum of 30%. For example, Δy was reduced to 0.05 from 0.1 and the results were changed by only 1 to 2.0% except near the wall where the change was 10.0 to 20.0%. Δx was changed to 0.125 from 0.175, with a shorter flow field, and after three cycles of the disturbance the results differed by no more than 2.0% and

in most cases by only 0.2 to 0.4% everywhere except near the downstream boundary where they varied by 4.0 to 5.0%. The time step was changed from 0.15 to 0.1 and the results were within 2% of the previous values after three cycles of the disturbance. Thus the use of these grid spacings and time steps seem to give accurate results.

The effect of iteration on the coefficients of the vorticity transport equation was studied. It was found that increasing the number of iterations from one to two changed the results by 0.2 to 1.0% after three cycles. Thus one iteration was used throughout.

The test value for the convergence of the stream function equation was chosen small enough not to influence the results. It was found that $\epsilon = 0.002$ was small enough. In Appendix C an analysis of the absolute error is given. This analysis shows an absolute error of from 2 to 5.0% which is within the range of accuracy as the other parts of the calculation.

The Rice University computer was used in this study. It has a speed roughly equal to an IBM 7040 although of course direct comparison is difficult. The computer time for one step with $\Delta x = 0.175$, $\Delta y = 0.1$ and $\Delta t = 0.15$ with a flow field of length $38h$ was about 50 seconds of which about 24 seconds were for the solution of the vorticity equations and 24 seconds were for the solution of the stream function

equation. It was found that about eight cycles or about 320 time steps were needed to establish the character of the flow. Thus approximately 5 hours per run were needed when the time of output was added to the computation time.

C. Discussion of Results

In Figures R-1 to R-27, to be described, the variables, vorticity, stream function, and u' are presented as functions of x for certain fixed y or r and fixed t by a standard subroutine which normalizes the variable by dividing by the maximum absolute value and plotting it versus the index number, j . Thus to obtain the value of the variable, the ordinate is multiplied by the absolute value of the maxima which is printed by the ordinate. The distance downstream from the upstream disturbance is found by multiplying the index number j , the number of Δx spaces, by Δx . The runs were carried out for almost eight cycles for most runs. The various cases are discussed in detail below. Before reading the detailed results the reader might find it helpful to review Table 1 in the following "Summary and Conclusions" section. This Table lists most of the cases studied and gives a very brief overall view of the principal results.

1. Poiseuille Flow

The stability of Poiseuille flow to disturbances of

infinitesimal amplitudes was studied by using an amplitude, A_m , equal to 10^{-5} with the disturbance profile shown in Figure 5. Since the results at this low amplitude after three cycles of the disturbance differed from those obtained by the solution of the linearized form of the vorticity equation by less than 0.1%, the disturbance was assumed to be close to being linear and thus the results could be compared with linear theory. Three runs with Reynolds number of 10,000; 30,000; and 100,000 and dimensionless frequency of 1.0 were made. In Figures R-1 to R-6, the vorticity, stream function, and velocity for these three Reynolds numbers are shown for two radii, 0.1 and 0.5. The vorticity at all three Reynolds numbers is monotonically decaying with distance and is undistorted at both radii. The decay at Reynolds number of 100,000 is very slight. At r equal to 0.5, the decay is more rapid than at r equal to 0.1. The stream function, f , exhibits no distortion and only decay at $r = 0.1$, but at $r = 0.5$ the stream function shows slight reamplification downstream and slight distortion at the point of the start of the reamplification. The reamplifications are greater with Reynolds numbers. While the velocity, u' , at $r = 0.1$ is undistorted and monotonically decaying for all Reynolds numbers, the velocity at $r = 0.5$ shows rapid amplification, then decay to a small value, and then reamplification. The

FIGURE R-1 POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-5}$) RADIUS = 0.1

FIGURE R-2 POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-3}$) RADIUS = 0.5

FIGURE R-5 POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-5}$) RADIUS = 0.1

FIGURE R-4 POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-2}$) RADIUS = 0.5

FIGURE R-6 POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-3}$) RADIUS = 0.1

FIGURE R-6 POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-5}$) RADIUS = 0.6

reamplifications are very strong at the higher Reynolds numbers. There is some distortion at the point of the start of the reamplification. Although reamplification does occur, this does not necessarily indicate instability since the disturbance may reamplify, decay, reamplify, decay, etc. with each reamplification being less than the previous one until steady flow is obtained at some point downstream.

The spatial decay rate constants, α_x , for the amplitude of the vorticity, stream function, and velocity were calculated and correlated by the method, explained in section V, that Gill (1965) used to compute decay rates from Leite's experimental data and from linear theory. The results are shown in Figure 6 along with the first theoretical mode by Gill from Figure 3b and with Leite's experimental points. The decay rates for the vorticity are much less than for the stream function and velocity and the decay rates for $r = 0.1$ are less than for $r = 0.5$. This constitutes agreement in some sense with linear theory since the first mode from linear theory falls between the correlation for the stream function and vorticity. The space frequencies, α_x , could not be correlated according to the method that Gill (1965) presents in Figure 3a. The points fell in the general region of between 1.0 and 0.8 but were scattered. The experimental points from Leite's work fall well below these points.

FIGURE 6 - VARIATION OF NON-DIMENSIONAL SPATIAL DAMPINGRATE, $\alpha_i R$, WITH NON-DIMENSIONAL FREQUENCY, βR , FROM NUMERICAL SIMULATION

The effect of non-linearities in Poiseuille flow was studied by using an amplitude, A_m , equal to 0.1 with the profile given in Figure 5. This amplitude was chosen so that the average amplitude of the disturbance in u' was close to the magnitude found in fully developed turbulent flow at $R = 10,000$. Runs of about eight cycles each were carried out for Reynolds numbers of 500; 3,000; 10,000; and 100,000 with a dimensionless frequency of 1.0. The results are presented in Figures R-7 to R-13. The greatest distortions occur near the centerline at r equal to 0.1 with the vorticity exhibiting greater distortions than the stream function and velocity. At the Reynolds number of 500, the vorticity, stream function, and velocity monotonically decay very rapidly with no distortion or reamplification of the disturbance at all radii. At $R = 3,000$ and $r = 0.1$, the vorticity decays monotonically with distance with slight distortions. While at $r = 0.5$ the vorticity decays more rapidly and is not distorted. The stream function and velocity at $R = 3,000$ show only decay and slight distortion at $r = 0.1$ and at $r = 0.5$ they show very little distortion. At $R = 10,000$, the vorticity at $r = 0.1$ is distorted with reamplification. While at $r = 0.5$, it is monotonically decaying with distance with no distortion. The stream function and velocity monotonically decay with distance at $r = 0.1$ with some distortion at 10,000. But at 0.5, slight

FIGURE R-7. POISEUILLE FLOW HIGH AMPLITUDE ($A_m = 10^{-1}$); $\nu = 40$ RADIUS = 0.1

FIGURE R-6 POISEUILLE FLOW HIGH AMPLITUDE ($A_m = 10^{-1}$) ; $\epsilon = 48$ RADIUS = 0.1

NUMBER OF Δx INCREMENTS

FIGURE R-9 POISEUILLE FLOW HIGH AMPLITUDE ($A_m = 10^{-1}$); $\nu = 48$ RADIUS = 0.1

FIGURE R-10 POISEUILLE FLOW HIGH AMPLITUDE ($A_m = 10^{-1}$); $\nu = 40$ RADIUS = 0.5

FIGURE R-11 POISEUILLE FLOW HIGH AMPLITUDE ($A_m = 10^{-1}$) $\Gamma = 48$ RADIUS = 0.5

NUMBER OF Δx INCREMENTS

FIGURE R-12 POISEUILLE FLOW HIGH AMPLITUDE ($A_m = 10^{-1}$) ; $\alpha = 40$ RADUS $= 0.5$

FIGURE R-13 POISEUILLE FLOW HIGH AMPLITUDE ($A_0 = 10^{-1}$) ; $\nu = 0.1$ RADIUS = 0.1 R = 100,000

reamplification with slight distortion occurs at the Reynolds number of 10,000. At 100,000, the vorticity at $r = 0.1$ is very distorted with strong reamplification. At 0.5, the vorticity is still undistorted with only slight decay. The stream function at $r = 0.1$ shows strong distortions with reamplification. While at 0.5, the distortions are slight with strong reamplification. At $r = 0.1$, the velocity is not quite as distorted as the stream function. While at 0.5, the velocity shows very strong reamplifications with slight distortions.

An early study on the effect of amplitude upon the stability of Poiseuille flow was made using a different disturbance and a flow field of length 8.0h. The initial condition was not of the same form as that given by Equation 7.14 and 7.16 but was similar. Although this disturbance was made under somewhat different conditions than the disturbance considered in the foregoing paragraph, the results are consistent with other results obtained in the course of this work. After two or three cycles of the disturbance, the character, in the nature of distortions, of the flow could be discerned. As shown in Figure 7, the effect of amplitude was very pronounced between amplitudes of order 10^{-3} and 10^{-2} .

2. Plane Poiseuille Flow

The stability of plane Poiseuille flow to disturbances

FIGURE 7 - VARIATION OF CRITICAL REYNOLDS NUMBER WITH AMPLITUDE FOR PIOSEUILLE FLOW

of infinitesimal amplitudes was studied by using an amplitude, A_m , equal to 10^{-5} and the disturbance profile in Figure 4. The results for a frequency of 1.0 and Reynolds number of 10,000 are given in Figures R-14 to R-17. This frequency and Reynolds number were chosen so that the disturbance would lie within the neutral curve of Figure 2. Also these constants were used by Thomas (1953) when he directly integrated the Orr-Sommerfeld equations to determine that the flow was unstable. The vorticity near the wall at $y = 0.9$ shows rapid decay. At $y = 0.7$, the vorticity shows amplification, decay, and then reamplification with distance and with no distortions. At 0.6, the vorticity first exhibits rapid amplification with distance then oscillates with a constant amplitude for about 14h, and then suddenly decays. There is some slight distortion after the decay. At 0.5, the vorticity shows a slight initial amplification, then decays, then reamplifies, and then decays with distance. The vorticity is not as constant as at y equal to 0.6. There is no distortion. For y equal to 0.3 and less, the vorticity decays with very slight reamplification far downstream and shows no distortion. The stream function at $y = 0.7$ shows rapid decay, then reamplification, and then decay with some distortions. At $y = 0.5$, the stream function shows rapid decay, amplification, decay, reamplification, and then decay

NUMBER OF Δx INCREMENTS

FIGURE R-14 PLANE POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-5}$) $I = 48$ $R = 10,000$ $\beta = 1.0$

FIGURE 10 PLANE POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-3}$) $\epsilon = 48$ $R = 10,000$ $A = 1.0$

FIGURE R-16 PLANE POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-5}$) $\epsilon = 48$ $R = 10,000$ $\beta_1 = 1.0$

NUMBER OF Δx INCREMENTS

FIGURE R-17 PLANE POISEUILLE FLOW - LOW AMPLITUDE ($A_m = 10^{-3}$) $i = 48$ $R = 10,000$ $A_1 = 1.0$

with distance. The stream function is undistorted at $y = 0.5$. While at $y = 0.3$, the stream function decays to a fairly stable oscillation which does show some amplification far downstream with slight distortion. The velocity at $y = 0.7$ shows rapid decay, amplification, decay, reamplification, and then decay with distance and no distortion. At $y = 0.5$, u' shows rapid amplification and then decays to a fairly stable oscillation with slight distortions. At $y = 0.3$, the velocity shows decay to a very small value, amplification, decay to a very small value, and then reamplification to a value equal to or greater than the previous amplification. The flow is slightly distorted where the reamplification starts to occur.

The shape of the vorticity curves described in the previous paragraph as one progresses away from the wall is similar to those shown by DeSanto and Kellar (1962) in their study on transition flow over a flat plate. Their computations were carried out with a shorter flow field and for fewer cycles of the disturbance than in this work. They did not find that the disturbance reamplified downstream. They attributed the initial amplification then decay on the basis of linear theory. In flow over a flat plate, the Reynolds number, which is based upon the length parameter of the distance from the upstream edge of the plate, increases with distance. Thus a disturbance which is started out inside the

neutral curve for flow over a flat plate, a curve very similar to Figure 2, will progress from a region of instability to a region of stability as it progresses downstream. Hence, the disturbance will amplify and then decay with distance. This explanation does not appear to be valid in plane Poiseuille flow since the Reynolds number is constant in this case. The flow follows a mechanism formulated by Landau (1944) and described by Shen (1964) more than that of DeSanto and Kellar. To quote Shen: "Given a basic flow, it may become unstable with respect to certain disturbances as the Reynolds number is increased. The nonlinear effects, however, may keep the disturbance to finite amplitude and thus a distorted laminar flow prevails at slightly higher Reynolds numbers. This distorted flow naturally may fluctuate with time, but should repeat itself periodically."

The existence of a stable oscillation as predicted by Stuart (1962) is not easily evident. The vorticity at y equal to 0.5 and 0.6 does show that the amplitude of the oscillation remains constant up to about $10h$ but then it decays rapidly. The stream function and velocity show definite amplification, decay, reamplification, and decay with distance. The reamplification may be due to the same effects that give rise to the intermittent regions of turbulent and laminar flow as one progresses downstream in transition flow over a flat

plate which was measured by Schubauer and Skramstad (1948) or due to the mechanism postulated by Landau (1944).

A run with dimensionless frequency of 1.0 and a Reynolds number of 100,000 was made to see if the stable region to the right of the neutral curve could be discerned. As shown in Figures R-18 to R-20, the disturbance only differs from the run made at 10,000 and $\beta_t = 1.0$ in that the reamplifications are much greater. The vorticity at $y = 0.7$ with distance amplifies, decays, and then reamplifies to a value equal to the previous value with no distortions. At $y = 0.5$, the vorticity remains constant for 14h and then monotonically decays with no distortion. At 0.1, the vorticity decays with slight distortion. The stream function and velocity are also similar to the run at 10,000 except that the reamplifications are of order of the initial amplifications. The strong reamplification near the downstream boundary is the initial condition which has not had time to be swept away and has not decayed.

It was possible to discern the left boundary of the curve of neutral stability in Figure 2. Two runs were made with a dimensionless frequency of 1.0 using a shorter flow field of 19h. At R equal to 6,000, the vorticity amplified decayed some and reamplified with distance for some y . While at 4,000, the vorticity while amplifying at some y 's initial

FIGURE R-16 PLANE POISEUILLE FLOW LOW AMPLITUDE ($A_m=10^{-3}$) $i=48$ $R=100,000$ $A_1=1.0$

FIGURE R-19 PLANE POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-3}$) $t = 48$ $R = 100,000$ $A_1 = 1.0$

FIGURE R-80 PLANE POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-5}$) $\epsilon = 48$ $R = 100,000$ $A = 1.0$

did not reamplify downstream. The character of the flow at 4,000 was not greatly different than at 6,000 but it was still possible to discern a difference. A run was made at 1,000 and the vorticity decayed for all y . The theoretical neutral Reynolds number for this frequency is about 5,300.

Two runs were ran with dimensionless frequencies, β_t , of 0.75 and 1.25 and with a Reynolds number of 10,000. These points were chosen as being above and below the neutral curve of Figure 2 for $R = 10,000$. The results are shown in Figures R-21 to R-24. The results were very similar to the run made at $R = 10,000$ and $\beta_t = 1.0$ except the frequencies are different. The runs were carried out for eight cycles. The motion is sinuous and does not show any indication of being stable in both cases.

The effect of a high amplitude disturbance is shown in Figures R-25 to R-27. Three runs with Reynolds numbers of 2,000; 4,000; and 10,000 were made with $A_m = 10^{-1}$ and $\beta_t = 1.0$ and the disturbance profile shown in Figure 4. The flow at $y = 0.5$ is presented as being representative of the character of the flow at these Reynolds numbers. At 2,000, the vorticity shows monotone decay with slight distortions. While at 4,000, the decay rate is less and the disturbance is still oscillating far downstream where it was not at 2,000. There were slight distortions at 4,000. At 10,000 the distortions are severe

FIGURE R-81 PLANE POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-5}$) $R = 10,000$ ($\beta_1 = 0.75$) 1-64

FIGURE R-22 PLANE POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-3}$) $R = 10,000$ $\beta_1 = 0.75$ $1 = 64$

FIGURE R-23 PLANE POISEUILLE FLOW LOW AMPLITUDE ($A_m=10^{-4}$) $R=10,000$ $R_1=1.25$ $t=40$

FIGURE R-24 PLANE POISEUILLE FLOW LOW AMPLITUDE ($A_m = 10^{-5}$) $R=10,000$ $A=1.25$ $t=40$

FIGURE R-25 PLANE POISEUILLE FLOW HIGH AMPLITUDE ($A_m = 10^{-1}$) $t = 48$ $y = 0.5$ $\sqrt{\nu} = 1.0$

FIGURE R-26 PLANE POISEUILLE FLOW HIGH AMPLITUDE ($A_m = 10^{-1}$) $i = 48$ $y = 0.5$ $\beta = 1.0$

FIGURE R-27 PLANE POISEUILLE FLOW HIGH AMPLITUDE ($A_m = 10^{-1}$) $1-40$ $\gamma = 0.5$ $\beta_1 = 1.0$

with no decay. The stream function and velocity at 2,000 show only decay with no distortion. While at 4,000, they decay to an oscillation of fairly constant amplitude with some distortion. At 10,000, the stream function and velocity are distorted and decay to a constant amplitude oscillation with some reamplification far downstream. The experimental critical Reynolds number is 2,700.

D. Summary and Conclusions:

It was found that the instability or incipient turbulence of the flow could be discerned after a few cycles of the disturbance. In the case of low amplitude disturbances the instability manifested itself in the nature of a sinuous motion while for high amplitude disturbances the incipient turbulence was evident in the nature of strong distortions and reamplification of the disturbance. It was found that the character of the vorticity disturbance best indicated the nature of the flow in both plane Poiseuille and Poiseuille flow. The character of the stream function and velocity generally did not differ from that of the vorticity except in that they decayed more rapidly and showed the sinuous motion sometimes where the vorticity did not. Also the distortions are much greater in vorticity than in the stream function and velocity. Near the point of transition, the

differences in the character of the flow in the nature of distortions were very slight. In Table 1, a short description of the vorticity is given for each run.

The following statements can be made concerning the flow.

1. Poiseuille flow is stable to low amplitude disturbances for Reynolds numbers up to 100,000. It is to be noted that the stream function and velocity do show some reamplifications at high Reynolds numbers but are not distorted. This result confirms linear theory.

2. The effect of amplitude in Poiseuille flow is very pronounced. It was found that the critical Reynolds number decreased very rapidly with amplitude of the disturbance. Stable undistorted regions below the critical curve could be discerned while regions of distorted flow above the critical curve could be found. A minimum critical Reynolds number of about 2,000 was found.

3. Plane Poiseuille flow for low amplitude disturbances was found to be sinuous in nature when the Reynolds number exceeded a certain critical Reynolds number between 4,000 and 6,000. The minimum critical Reynolds number from linear theory is 5,300. The flow did not become undistorted but the disturbance amplified, decayed, reamplified, decayed, reamplified, etc. with distance. Runs made inside, above,

below, and to the right of the neutral curve of Figure 2 were all similar and sinuous in nature. Thus the effect of frequency was found to be minimal. Below a critical Reynolds number of about 5,000 the flow was not sinuous in nature but was stable. A stable oscillation in the disturbance as predicted by Watson (1962) was also strongly indicated. Although a completely stable oscillation was not found, the amplitude of the disturbance did remain constant for over 14h. These results show that although plane Poiseuille flow may be unstable to infinitesimal disturbances as predicted by linear theory, these instabilities do not lead to incipient turbulence but to a stable sinuous laminar flow.

4. The effect of amplitude in plane Poiseuille flow was also strong. The character in the nature of distortions of the flow was significantly different at $R = 4,000$ from the character at $R = 2,000$. The experimental transition Reynolds number is 2,700.

In conclusion, the method of studying stability and incipient trubulence by direct solution of the Navier-Stokes equations by finite difference techniques, gives results which are in good agreement with linear and non-linear theory and experimental results. This method has permitted the study of non-linear disturbances in Poiseuille flow which has not been accomplished by other theoretical methods. The

finite difference scheme used, an adaptation of the alternating direction method of Douglas (1962), is accurate and stable.

It appears that the turbulence paradox (so called by Birkhoff (1960)) is not really a paradox since both plane Poiseuille and Poiseuille flow appear to be stable to very small disturbances. The difference being that a disturbance leads to a stable sinuous laminar motion in plane Poiseuille flow, while in Poiseuille flow the disturbed profile returns to the stable parabolic profile.

NOMENCLATURE

<u>Roman Letter</u>	<u>Definition</u>
A_m	Amplitude constant
A_o	Initial decay rate constant
A_r	Initial spatial frequency constant
c	Constant of proportionality
f	Deviation of stream function from steady laminar flow
g	Deviation of vorticity from steady laminar flow
h	One-half the plate spacing or the radius of the pipe
m, q	Various modes of the linear disturbance for Poiseuille flow, Gill (1965)
P	Pressure
r	Dimensionless distance along the radius of the tube
R	Reynolds Number = $U^o h / \nu$
t	Dimensionless time
u, v	Dimensionless components of velocity
U^o	Maximum dimension centerline velocity of the steady laminar flow
u'	Deviation of the u velocity from steady laminar flow
V	Velocity vector u, v
x, y, z	Dimensionless distance along the x, y, z axis

Greek LettersDefinition

α	Complex wave number in x-space
α_r	Spatial wave number
α_i	Spatial decay constant
β	Temporal wave number for linear disturbance in theoretical studies
β_t	Temporal wave number for disturbance in present numerical study
$\Delta r, \Delta x, \Delta y$	Grid spacing in the r, x, and y directions
Δt	Time step
∇	The "del" operator
∇^2	The Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ for plane Poiseuille flow and $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} - \frac{\partial}{r \partial r}$ for Poiseuille flow
δ_o	Boundary layer
ϵ	Test parameter on the convergence of the stream function equation iteration
η	Relaxation parameter
θ	Angle
μ	Viscosity
ν	Kinematic viscosity
ρ	Density
$\theta_m(y)$	Dimensionless amplitude function
ψ	Dimensionless stream function
ω	Dimensionless vorticity

OverlinesDefinition

~

Complex Conjugate

-

Time averaged

SubscriptsDefinition

i,j

Denotes grid points

o

Refers to condition of steady laminar flow

D

Refers to upstream disturbance profile

s

Refers to initial condition of the disturbance

SuperscriptsDefinition

*

Refers to value of vorticity at time level, $n + 1$, calculated in the x-pass of the alternating direction scheme

v

Number of iteration

AbbreviationsDefinition

A.D.I.

Alternating direction implicit

S.O.R.

Successive overrelaxation

Note: All of the symbols used are not defined here. Some of the symbols which have localized meaning are defined in text at appropriate places.

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APPENDIX A

Definitions of Finite Difference Approximations

Consider the rectangular flow field of Figure 1a or 1b as being divided by a number of equally spaced horizontal lines and a number of equally spaced vertical lines where the two spacings may be different. If the horizontal lines are indexed by i with $i = 0$ at the centerline and $i = M$ at the wall and the vertical lines are indexed by j with $j = 0$ at the left boundary and $j = N$ at the right boundary, we can use the notation that $\theta_{i,j}^n$ represents the value of $\theta(x,y,t)$ at $x = j\Delta x$, $y = i\Delta y$ and $t = n\Delta t$. The time scale is divided by equal spaced intervals indexed by n .

Using this notation, some symbols for the various difference schemes for the dependent variables and their derivatives are defined. Only centered difference schemes were used.

$$(A.1) \quad \theta_x^n \equiv \left(\frac{\partial \theta}{\partial x} \right)_{t,y} = \frac{\theta_{i,j+1}^n - \theta_{i,j-1}^n}{2\Delta x} + O(\Delta x)^2$$

$$(A.2) \quad \theta_y^n \equiv \left(\frac{\partial \theta}{\partial y} \right)_{t,x} = \frac{\theta_{i+1,j}^n - \theta_{i-1,j}^n}{2\Delta y} + O(\Delta y)^2$$

$$(A.3) \quad \theta_{xx}^n \equiv \left(\frac{\partial^2 \theta}{\partial x^2} \right)_{t,y} = \frac{\theta_{i,j+1}^n - 2\theta_{i,j}^n + \theta_{i,j-1}^n}{(\Delta x)^2} + O(\Delta x)^2$$

$$(A.4) \quad \theta_{yy}^n \equiv \left(\frac{\partial^2 \theta}{\partial y^2} \right)_{t,x} = \frac{\theta_{i+1,j}^n - 2\theta_{i,j}^n + \theta_{i-1,j}^n}{(\Delta y)^2} + \mathcal{O}(\Delta y)^2$$

$$(A.5) \quad \theta_t^n \equiv \left(\frac{\partial \theta}{\partial t} \right)_{x,y} = \frac{\theta_{i,j}^{n+1} - \theta_{i,j}^n}{\Delta t} + \mathcal{O}(\Delta t)$$

APPENDIX B

Derivation of the Vorticity Transport Finite
Difference Equations for Poiseuille and Plane Poiseuille Flow

Plane Poiseuille Flow:

If we substitute into Equations 7.1 and 7.2 the finite difference formulation given in Appendix A and then multiply by Δt , we obtain relations B.1 and B.2.

$$\begin{aligned}
 (B.1) \quad & g_{i,j}^{*n+1} - g_{i,j}^n + \frac{1}{2} \beta_x^{u,n+\frac{1}{2}} (g_{i,j+1}^{*n+1} - g_{i,j-1}^{*n+1} + g_{i,j+1}^n - g_{i,j-1}^n) \\
 & + \beta_y^{v,n+\frac{1}{2}} (g_{i+1,j}^n - g_{i-1,j}^n) \\
 & = \frac{1}{2} \alpha_x (g_{i,j+1}^{*n+1} - 2g_{i,j}^{*n+1} + g_{i,j-1}^{*n+1} + g_{i,j+1}^n - 2g_{i,j}^n + g_{i,j-1}^n) \\
 & + \alpha_y (g_{i+1,j}^n - 2g_{i,j}^n + g_{i-1,j}^n) + 2 \Delta t v_{i,j}^{n+\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 (B.2) \quad & g_{i,j}^{n+1} - g_{i,j}^n + \frac{1}{2} \beta_x^{u,n+\frac{1}{2}} (g_{i,j+1}^{*n+1} - g_{i,j-1}^{*n+1} + g_{i,j+1}^n - g_{i,j-1}^n) \\
 & + \frac{1}{2} \beta_y^{v,n+\frac{1}{2}} (g_{i+1,j}^{n+1} - g_{i-1,j}^{n+1} + g_{i+1,j}^n - g_{i-1,j}^n) \\
 & = \frac{1}{2} \alpha_x (g_{i,j+1}^{*n+1} - 2g_{i,j}^{*n+1} + g_{i,j-1}^{*n+1} + g_{i,j+1}^n - 2g_{i,j}^n + g_{i,j-1}^n) \\
 & + \frac{1}{2} \alpha_y (g_{i+1,j}^{n+1} - 2g_{i,j}^{n+1} + g_{i-1,j}^{n+1} + g_{i+1,j}^n - 2g_{i,j}^n + g_{i-1,j}^n) \\
 & + 2 \Delta t v_{i,j}^{n+\frac{1}{2}}
 \end{aligned}$$

where $\beta_x \equiv \frac{\Delta t}{2\Delta x}$; $\beta_y \equiv \frac{\Delta t}{2\Delta y}$; $\alpha_x \equiv \frac{\Delta t}{R(\Delta x)^2}$; $\alpha_y \equiv \frac{\Delta t}{R(\Delta y)^2}$

Collecting terms, we have equations B.3 and B.4.

$$\begin{aligned}
 (B.3) \quad & - (\frac{1}{2}\alpha_x - \frac{1}{2}\beta_x^{u^{n+\frac{1}{2}}})g_{i,j+1}^{*n+1} + (1+\alpha_x)g_{i,j}^{*n+1} \\
 & - (\frac{1}{2}\alpha_x + \frac{1}{2}\beta_x^{u^{n+\frac{1}{2}}})g_{i,j-1}^{*n+1} \\
 & = (\frac{1}{2}\alpha_x - \frac{1}{2}\beta_x^{u^{n+\frac{1}{2}}})g_{i,j+1}^n + (1-\alpha_x)g_{i,j}^n \\
 & + (\frac{1}{2}\alpha_x + \frac{1}{2}\beta_x^{u^{n+\frac{1}{2}}})g_{i,j-1}^n + (\alpha_y - \beta_y^{v^{n+\frac{1}{2}}})g_{i+1,j}^n \\
 & - 2\alpha_y g_{i,j}^n + (\alpha_y + \beta_y^{v^{n+\frac{1}{2}}})g_{i-1,j}^n + 2\Delta t v_{i,j}^{n+\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 (B.4) \quad & - (\frac{1}{2}\alpha_y - \frac{1}{2}\beta_y^{v^{n+\frac{1}{2}}})g_{i+1,j}^{n+1} + (1+\alpha_y)g_{i,j}^{n+1} \\
 & - (\frac{1}{2}\alpha_y + \frac{1}{2}\beta_y^{v^{n+\frac{1}{2}}})g_{i-1,j}^{n+1} \\
 & = (\frac{1}{2}\alpha_y - \frac{1}{2}\beta_y^{v^{n+\frac{1}{2}}})g_{i+1,j}^n + (1-\alpha_y)g_{i,j}^n \\
 & + (\frac{1}{2}\alpha_y + \frac{1}{2}\beta_y^{v^{n+\frac{1}{2}}})g_{i-1,j}^n + (\frac{1}{2}\alpha_x - \frac{1}{2}\beta_x^{u^{n+\frac{1}{2}}}) \\
 & + (g_{i,j+1}^{*n+1} + g_{i,j+1}^n) - \alpha_x (g_{i,j}^{*n+1} + g_{i,j}^n) \\
 & + (\frac{1}{2}\alpha_x + \frac{1}{2}\beta_x^{u^{n+\frac{1}{2}}})(g_{i,j-1}^{*n+1} + g_{i,j-1}^n) + 2\Delta t v_{i,j}^{n+\frac{1}{2}}
 \end{aligned}$$

Thus the coefficients in Equations 7.5 and 7.6 are given by these definitions.

$$A \equiv \frac{1}{2}(a_x - \beta_x u_{i,j}^{n+\frac{1}{2}}); \quad B \equiv (1 + a_x); \quad C \equiv \frac{1}{2}(a_x + \beta_x u_{i,j}^{n+\frac{1}{2}})$$

$$A' \equiv \frac{1}{2}(a_y - \beta_y v_{i,j}^{n+\frac{1}{2}}); \quad B' \equiv (1 + a_y); \quad C' \equiv \frac{1}{2}(a_y + \beta_y v_{i,j}^{n+\frac{1}{2}})$$

$$D^n \equiv Ag_{i,j+1}^n + (2-B)g_{i,j}^n + Cg_{i,j-1}^n + 2A'g_{i+1,j}^n + (2-2B')g_{i,j}^n \\ + 2C'g_{i-1,j}^n + 2\Delta t v_{i,j}^{n+\frac{1}{2}}$$

$$D'^n \equiv A'g_{i+1,j}^n + (2-B')g_{i,j}^n + C'g_{i-1,j}^n + Ag_{i,j+1}^n + (1-B)g_{i,j}^n \\ + Cg_{i,j-1}^n + 2\Delta t v_{i,j}^{n+\frac{1}{2}}$$

$$R^{*n+1} \equiv Ag_{i,j+1}^{*n+1} + (1-B)g_{i,j}^{*n+1} + Cg_{i,j-1}^{*n+1}$$

Poiseuille Flow:

The procedure for Poiseuille flow is exactly the same as for plane Poiseuille flow. The definitions of the coefficients in Equation 7.5 and 7.6 for Poiseuille flow are given by the following relations.

$$A \equiv \frac{1}{2}(a_x - \beta_x u_{i,j}^{n+\frac{1}{2}}); \quad B \equiv (1 + a_x - 2v \quad t/r); \quad C \equiv \frac{1}{2}(a_x + \beta_x u_{i,j}^{n+\frac{1}{2}})$$

$$A' \equiv \frac{1}{2}(a_r - \beta_n/r - \beta_y v_{i,j}^{n+\frac{1}{2}}); \quad B' \equiv (1 + a_r);$$

$$C' \equiv \frac{1}{2}(a_r + \beta_n/r + \beta_y v_{i,j}^{n+\frac{1}{2}})$$

$$D^n \equiv Ag_{i,j+1}^n + (2-B)g_{i,j}^n + Cg_{i,j-1}^n + 2A'g_{i+1,j}^n \\ + (2-B')g_{i,j}^n + 2C'g_{i-1,j}^n$$

$$D'^n \equiv A'g_{i+1,j}^n + (2-B')g_{i,j}^n + C'g_{i-1,j}^n + Ag_{i,j+1}^n \\ + (1-B)g_{i,j}^n + Cg_{i,j-1}^n$$

$$R^{*n+1} \equiv Ag_{i,j+1}^{*n+1} + (1-B)g_{i,j}^{*n+1} + Cg_{i,j-1}^{*n+1}$$

where

$$\beta_n \equiv \frac{\Delta t}{R \cdot \Delta r}; \quad a_r \equiv \frac{\Delta t}{R(\Delta r)^2}; \quad a_x \equiv \frac{\Delta t}{R(\Delta x)^2}; \quad \beta_r \equiv \frac{\Delta t}{2\Delta r};$$

$$\beta_x \equiv \frac{\Delta t}{2\Delta x}$$

APPENDIX C

Absolute Error in the Solution
of the Stream Function Equation

The problem of the absolute error in the stream function equation iteration, Equation 7.9, may be approached as follows. Consider the linear iteration scheme of the difference equation $Af' = B$, which is the matrix form of Equation 7.9, as given by Equation C.1.

$$(C.1) \quad f_k = H(\eta)f_{k-1} + M_{k-1}$$

Now consider an error, Y_k , defined as $Y_{k-1} = f_k - f_{k-1}$. Thus

$$(C.2) \quad Y_k = [H(\eta)]^k Y_0$$

Now if we take the norm of Equation C.2, we can obtain

$$(C.3) \quad \left\| \frac{Y_k}{Y_{k-1}} \right\| < \|H(\eta)\|$$

Now if η_1 is the dominate eigenvalue of $H(\eta)$, we obtain from relation C.3 as $k \rightarrow \infty$

$$(C.4) \quad \left\| \frac{Y_k}{Y_{k-1}} \right\| \rightarrow \eta_1$$

If we define another error, E_k , equal to $f_k - A^{-1}B$, we obtain

$$(C.5) \quad E_{k+1} \approx \eta_1 E_k \quad \text{as } k \rightarrow \infty$$

For, if $A^{-1}B = f_k + E_k$ and if

$$(C.6) \quad A^{-1}B = f_{k+1} + \eta_1 E_k + \epsilon_k \quad \|\epsilon_k\| \text{ small},$$

then by elimination of E_k we can obtain this relationship for the asymptotic answer, f' , if the norm of ϵ_k is small in comparison to $1 - \eta_1$.

$$(C.7) \quad f' = (f_{k+1} - \eta_1 f_k) / (1 - \eta_1)$$

See Forsythe and Wasow (1960) for the details of the foregoing analysis.

The norm used to calculate η_1 is given by Equation 7.10. It was found that the error for the k th iteration divided by the $k-1$ iteration was fairly constant after a few iterations. This constant value, η_1 , varied from between 0.5 and 0.8 depending upon the grid size and the parameters of the problem. If f' is called the asymptotic answer, then $100 (f' - f_k) / f_k$ equals the per cent error after k iterations. This error will depend upon the error test. The test used for most of the runs in this study was $\epsilon = 0.002$. Thus the average error $(f_{k+1} - f_k) / f_D$ was less than 0.2%. Now as shown in the result section, the upstream disturbance stream function, f_D , is larger than f_k . An average value of $f_k = 0.2 f_D$ is approximately correct for the average value of f_k at each y . Thus we can obtain the following per cent errors.

$$(C.8) \quad f_{k+1} - f_k \leq .002 f_D \leq .01 f_k$$

$$(C.9) \quad f_{k+1} \leq 1.01 f_k$$

$$(C.10) \quad \frac{f_{k+1} - f_k}{f_{k+1}} = \frac{\frac{f_{k+1} - \eta_1 f_k}{1 - \eta_1}}{f_k} = f_k = \frac{101 - \eta_1}{1 - \eta_1} - 1$$

$$\% \text{ Error} = \left(\frac{.01}{1 - \eta_1} \right) \times 100$$

if $\eta_1 = 0.5$, then $\% \text{ Error} = 2\%$

if $\eta_1 = 0.8$, then $\% \text{ Error} = 5\%$.

APPENDIX D

Downstream Boundary Condition

The assumption of periodic flow, $g_{xx} = -cg$ and $f_{xx} = -cf$, in the x direction at the downstream boundary was chosen after some experimentation which showed that this boundary condition had the minimum effect upon the upstream results. This effect was studied by making a run with flow field of 8h and then the run was duplicated except a flow field of 9 or 10h was used. The results were compared after the same number of cycles. The results were very close for the first 6 or 7h. Near the downstream boundary of the shorter flow field, that is from 7 to 8h, the results differed by only 1 or 2 per cent from those obtained in the same region by the longer flow field.

The condition that $g_x = -cg$ and $f_x = -cf$ influenced the results by only 2 or 3 per cent in the same region.

The constant, $c = A_r^2$, was obtained from the initial condition of the disturbance. The periodic condition was applied to all y at the downstream boundary. The periodic condition in vorticity can be derived from the periodic condition on the stream function as follows.

$$(D.1) \quad g = f_{xx} + f_{yy}$$

$$(D.2) \quad f_{xx} = -A_r^2 f$$

We assume that the periodic condition on vorticity holds.

$$(D.3) \quad g_{xx} = -A_r^2 g$$

Then we differentiate Equation D.1 twice with respect to x and substitute the result and Equation D.1 into Equation D.3 to obtain Equation D.4.

$$(D.4) \quad f_{xxxx} + f_{yyxx} = -A_r^2 (f_{xx} + f_{yy})$$

Using Equation D.2, we obtain Equation D.5, an identity.

$$(D.5) \quad -A_r^2 f_{xx} - A_r^2 f_{yy} = -A_r^2 (f_{xx} + f_{yy})$$

Q.E.D.