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ASYMPTOTIC BEHAVIOR
OF FUNCTIONS HOLOMORPHIC
IN THE UNIT DISC

by

Robert Lester Hall

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Gerald R. MacLane

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TO MY WIFE,

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1. Introduction. This paper is concerned primarily with the asymptotic behavior of holomorphic functions in the unit disc. The work of G. R. MacLane [7] is taken as the foundation. In section 2, we will summarize those definitions and results of MacLane which will be useful to us.

If f is holomorphic in $\{|z| < 1\}$, we say that the point $\zeta \in \{|z| = 1\}$ is an asymptotic point of f for the complex number a ($a = \infty$ included) if there is an arc $\Gamma \subset \{|z| < 1\}$ tending to ζ such that $f \rightarrow a$ on Γ . The limit a is called an asymptotic value of f . MacLane has shown that under the growth condition

$$(III) \quad \int_0^1 (1-r) \log M(r) dr < \infty$$

where $M(r) = \max_{|z|=r} |f(z)|$, the asymptotic points are dense

on $\{|z| = 1\}$. Now the modular function (section 8, example 1) satisfies (III) and has a denumerable set of asymptotic points. In section 3, we will show that if $f \neq 0$ and

$$\log M(r) = o\left(\frac{1}{1-r}\right),$$

then on any arc $\gamma \subset \{|z| = 1\}$, either the measure of the set of asymptotic points for finite values of a is positive or the set of asymptotic points for ∞ is uncountable.

Various implications are considered in the remainder of the section.

In section 4, we consider some aspects of the behavior of a function which has a finite asymptotic value on a curve $\Gamma \subset \{|z| < 1\}$ which tends to a subarc (or all) of $\{|z| = 1\}$. In particular we consider the question of when such functions have asymptotic tracts for ∞ . We show that under certain conditions, the function will either have an arc tract for ∞ or infinitely many point tracts for ∞ . Then adding the hypothesis that $f \in \mathcal{L}^*$, we see that f can not have an arc tract for ∞ ; here $f \in \mathcal{L}^*$ if and only if every level curve of f "ends at points" of $\{|z| = 1\}$. The precise definition is given in section 2.

Our principal objective in section 5 is to apply a technique of Collingwood and Cartwright [4] to the study of level sets of holomorphic functions. The first result of the section is a geometric proof of a generalization of a Lemma of MacLane; MacLane's proof was essentially analytic. We then consider some sufficient conditions for $f \in \mathcal{L}$, where $\mathcal{L}$ is the class of non-constant holomorphic functions in $\{|z| < 1\}$ whose level sets "end at points" of $\{|z| = 1\}$. Also, some consideration is given to level sets of real harmonic functions.

Section 6 is a collection of results for real harmonic functions, analogous to those of MacLane for holomorphic functions. It is pointed out, by using a simple transformation, that the results of MacLane have, in effect, already

established these results for real harmonic functions.

In section 7 we show that homeomorphisms of the interior of the disc do not satisfy the class relation $\mathcal{A} = \mathcal{B} = \mathcal{C}$ given by MacLane, where we now define $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ appropriately for continuous functions.

In section 8, several examples are given to illustrate and clarify some of the results of the preceding sections.

2. Definitions, Notation, and Useful Results.

The definitions and results in this section can be found in MacLane [7]. We list them here for convenience.

Let $\{\gamma_n\}$ be a sequence of continuous compact curves in $\{|z| < 1\}$ and let γ be an arc $\{z \mid |z|=1, \alpha \leq \arg z \leq \beta\}$.

Definition. $\gamma_n \rightarrow \gamma$ if and only if for each $\epsilon > 0$ there exists n_0 such that

$$\left. \begin{aligned} \gamma_n &\subset \{1-\epsilon < |z| < 1\} \\ |\inf_{\gamma_n} \arg z - \alpha| &< \epsilon, \quad |\sup_{\gamma_n} \arg z - \beta| < \epsilon \end{aligned} \right\} \quad (n > n_0).$$

The following concept of asymptotic tract is very useful.

A tract $\{D(\epsilon), a\}$ associated with the finite value a is a set of non-void domains $D(\epsilon)$, one for each $\epsilon > 0$, such that

$$(2.1) \quad D(\epsilon) \text{ is a component of } \{z \mid |f(z) - a| < \epsilon\},$$

$$(2.2) \quad 0 < \epsilon_1 < \epsilon_2 \Rightarrow D(\epsilon_1) \subset D(\epsilon_2),$$

$$(2.3) \quad \bigcap_{\epsilon > 0} D(\epsilon) = \emptyset.$$

If a is replaced by ∞ the only change is to replace $|f(z) - a| < \epsilon$ in (2.1) by $|f(z)| > \frac{1}{\epsilon}$.

$$\text{Clearly} \quad K = \bigcap_{\epsilon > 0} \overline{D(\epsilon)}$$

is a non-void, connected, closed subset of $\{|z|=1\}$. We shall call K the end of the tract. If K is a point

(arc) then the tract will be called a point - (arc -) tract.

A tract is uniquely determined by $D(\epsilon_n)$ satisfying (2.1), (2.2), and (2.3) for some sequence $\epsilon_n \downarrow 0$. The two tracts $\{D_1(\epsilon), a_1\}$ and $\{D_2(\epsilon), a_2\}$ are distinct if and only if $D_1(\epsilon) \cap D_2(\epsilon) = \emptyset$ for ϵ sufficiently small; if $a_1 \neq a_2$ they are automatically distinct.

Let $\{D(\epsilon), a\}$ be a tract and let $\Gamma : z = \phi(t), 0 \leq t < 1$ be a continuous curve in $\{|z| < 1\}$ such that $\phi(t) \in D(\epsilon)$ for $1 - \delta(\epsilon) < t < 1$. In this case we shall say that Γ belongs to $\{D(\epsilon), a\}$. Because of (2.3), $|\phi(t)| \rightarrow 1$ as $t \uparrow 1$. Clearly, $f(z) \rightarrow a$ as $|z| \rightarrow 1$ on Γ ; that is, f has the asymptotic value a on Γ . Given the tract, there exist curves belonging to it. Conversely, if f has the asymptotic value a on some curve Γ on which $|z| \rightarrow 1$, then there is precisely one tract $\{D(\epsilon), a\}$ to which Γ belongs. A curve Γ belonging to $\{D(\epsilon), a\}$ will tend either to a single point of K or to an arc of K . There always will be some Γ which tends to the whole set K ; there may or may not be curves Γ belonging to the tract which tend to proper subarcs of K .

Let Γ_1 and Γ_2 be two arcs in $\{|z| < 1\}$ ending at the same point $\zeta, |\zeta| = 1$, such that $f \rightarrow a_i$ on Γ_i, a_i finite. If Γ_1 and Γ_2 belong to different tracts (which is certainly the case if $a_1 \neq a_2$) then there is a curve Γ , between Γ_1 and Γ_2 and ending at ζ , on which $f \rightarrow \infty$. "Between" means that Γ_1 and Γ_2 may be extended so that

$\Gamma_1 \cup \Gamma_2$ is a Jordan curve and Γ lies in the corresponding Jordan domain; that is, the interior of $\Gamma_1 \cup \Gamma_2$.

(2.4) We shall say that f has the asymptotic value a at ζ , $|\zeta|=1$, if and only if there is an arc $\Gamma \subset \{|z|<1\}$ tending to ζ such that $f \rightarrow a$ on Γ .

Let f be holomorphic and non-constant in $\{|z|<1\}$. For any complex number a ($a = \infty$ permitted) we define the set A_a as follows. The point ζ belongs to A_a if and only if $|\zeta|=1$ and f has the asymptotic value a at ζ .

If S is any set on the sphere we set

$$(2.5) \quad A(S) = \bigcup_{a \in S} A_a, \quad A(S) = \emptyset \quad \text{if } S = \emptyset.$$

In particular we set

$$(2.6) \quad A^* = \bigcup_{a \neq \infty} A_a, \quad A = A^* \cup A_\infty.$$

Definition. f belongs to A if and only if f is holomorphic and non-constant in $\{|z|<1\}$ and A is dense on $\{|z|=1\}$.

We define the set B^* as follows. The point ζ , $|\zeta|=1$, belongs to B^* if and only if there is an arc Γ in $\{|z|<1\}$, ending at ζ , such that $|f|$ is bounded by some finite constant M on Γ . In general the bound M will vary as ζ and Γ vary. Also we set

$$(2.7) \quad B = B^* \cup A_\infty.$$

Definition. f belongs to B if and only if f is holomorphic and non-constant in $\{|z|<1\}$ and B is dense

on $\{|z|=1\}$.

For the following two definitions we distinguish between level set and level curve. Given f defined in $\{|z|<1\}$, then the level set $L(\lambda)$ is $\{z \mid |f(z)| = \lambda\}$; here λ is any constant ≥ 0 . A level curve $C(\lambda)$ is any component of $L(\lambda)$.

Now let S be any subset of $\{|z|<1\}$. For each r , $0 < r < 1$, let the components of

$$S \cap \{r < |z| < 1\}$$

be $S_i(r)$, $i \in I$. Let $\delta_i(r) = \text{diam } S_i(r)$ and set

$$\delta(r) = \sup_{i \in I} \delta_i(r)$$

with $\delta(r) \equiv 0$ if I is void. Clearly $\delta(r) \downarrow$ as $r \uparrow$.

We shall say that S "ends at points" of $\{|z|=1\}$ if and only if $\delta(r) \downarrow 0$ as $r \uparrow 1$.

Definition. f belongs to the class $\mathcal{L}(\mathcal{L}^*)$ if and only if f is holomorphic and non-constant in $\{|z|<1\}$ and every level set $L(\lambda)$ (level curve $C(\lambda))$ ends at points of $\{|z|=1\}$.

MacLane proves the following theorem about the classes defined above.

Theorem [7, Theorem 1]. $\mathcal{A} = \mathcal{B} = \mathcal{L} \subset \mathcal{L}^*$ and the inclusion is proper.

3. The Size of A Under Restricted Growth.

We will show that restricting the growth of f more than MacLane's condition (III) [7, p.37] will guarantee a larger set of asymptotic points than condition (III) alone.

Theorem 1. Let f be holomorphic and non-constant in $\{|z| < 1\}$. Suppose further that $f \neq 0$ in $\{|z| < 1\}$ and

$$(3.1) \quad \log M(r) = o\left(\frac{1}{1-r}\right).$$

Then for any arc $\gamma \subset \{|z|=1\}$, either

$$(3.2) \quad m(A^* \cap \gamma) > 0, \text{ or}$$

$$(3.3) \quad A_\infty \cap \gamma \text{ is } \underline{\text{uncountable}}.$$

Remarks. We can't hope to do essentially better than Theorem 1 since the modular function ω (Example 1, section 8) is a non-constant holomorphic function in $\{|z| < 1\}$ having the following properties: ω omits 0 and 1, $\log M(r) = O((1-r)^{-1})$ and A is countable.

Clearly, the condition that f be non-constant is not needed, but its addition to the hypotheses expedites the proof slightly. Also, the condition $f \neq 0$ can be replaced by the condition that f omit one finite value.

We shall say that f is locally unbounded near the arc $\gamma \subset \{|z|=1\}$ if $\limsup |f(z)| = \infty$ as $z \rightarrow \zeta$, $z \in \{|z| < 1\}$, for every point $\zeta \in \gamma$.

In the interesting case where f is locally unbounded near the arc $\gamma \subset \{|z|=1\}$, we shall actually prove (Theorem 2) that either

$$(3.2') \quad m(A(\{|w| > \delta\}) \cap \gamma) > 0 \text{ for arbitrary } \delta,$$

$$0 < \delta < \infty, \text{ or}$$

$$(3.3) \quad A_\infty \cap \gamma \text{ is uncountable.}$$

We do not include ∞ in the set $\{|w| > \delta\}$. Hence, it follows by a simple measure theoretic argument that under the same condition we have either

$$(3.2'') \quad m(A(\{|w| < \eta\}) \cap \gamma) > 0 \text{ for some } \eta > 0, \text{ or}$$

$$(3.3) \quad A_\infty \cap \gamma \text{ is uncountable.}$$

The measurability of $A(S)$ for $f \in \mathcal{A}$ and S a Borel set on the sphere was proved by MacLane [7, Theorem 10] and extended by McMillan [8, Theorem 1] to include arbitrary holomorphic functions.

Proof: Suppose, contrary to the conclusion, that there exists an arc $\gamma \subset \{|z|=1\}$ such that

$$(3.4) \quad m(A^* \cap \gamma) = 0, \text{ and}$$

$$(3.5) \quad A_\infty \cap \gamma \text{ is countable.}$$

We first observe that (3.1) implies $f \in \mathcal{A}$ by MacLane's condition (III). Also, since $f \neq 0$, it follows from MacLane [7, Theorem 4 and Theorem 7] that f has no arc tracts.

By still another theorem of MacLane [7, Theorem 11], (3.4) implies that A_∞ is dense on γ . Let $\overset{\circ}{\gamma}$ denote the interior of the arc γ . Let $\zeta \in A_\infty \cap \overset{\circ}{\gamma}$ and choose a domain $D(\epsilon)$ belonging to the tract for f at ζ such that

$$(3.6) \quad \overline{D(\epsilon)} \cap \{|z|=1\} \subset \overset{\circ}{Y}, \text{ and}$$

$$(3.7) \quad L\left(\frac{1}{\epsilon}\right) = \{z \mid |f(z)| = \frac{1}{\epsilon}\} \text{ has no multiple points.}$$

Obviously, $D(\epsilon)$ is simply connected since $f \neq 0$. In fact, since $f \in \mathfrak{L}$ and $f \neq 0$, it follows that $D(\epsilon)$ is a Jordan Domain.

We assume without loss of generality that the origin belongs to $D(\epsilon)$ since we can transform D onto itself by a linear transformation while preserving (3.1).

Let ϕ be a one-one conformal map of $D(\epsilon)$ onto $\{|w|<1\}$ such that $\phi(0) = 0$.

Consider the function $g(w) = f(\phi^{-1}(w))$. We shall establish the following properties of g :

$$(3.8) \quad |g(w)| > \frac{1}{\epsilon} \text{ for } |w|<1, \text{ and}$$

$$(3.9) \quad |g(e^{i\theta})| = \frac{1}{\epsilon} \text{ except possibly for a set of } \theta$$

of measure zero. Here, $g(e^{i\theta})$ denotes the finite radial limit of g at $e^{i\theta}$ ($\lim g(\rho e^{i\theta})$ as $\rho \uparrow 1$) if it exists.

Note that it follows from the Theorems of Fatou and F. and

M. Riesz that g has finite radial limits almost everywhere.

Property (3.8) is immediate since if $w_0 \in \{|w| < 1\}$ then $\phi^{-1}(w_0) \in D(\epsilon)$ and $D(\epsilon)$ is a component of $\{z \mid |f(z)| > \frac{1}{\epsilon}\}$. To obtain (3.9), we apply a technique of MacLane.

If (3.9) fails to hold, then $|g(e^{i\theta})| > \frac{1}{\epsilon}$ for $\theta \in E^*$ with $m(E^*) > 0$. The fact that $|g(e^{i\theta})| > \frac{1}{\epsilon}$ follows from (3.8). Since $D(\epsilon)$ is a Jordan domain, each radial limit $g(e^{i\theta})$ corresponds to a point asymptotic value of f in $D(\epsilon)$. The set E^* is mapped onto a subset of γ since $|f| = \frac{1}{\epsilon}$ on the remaining boundary of $D(\epsilon)$. Letting $F = \overline{D(\epsilon)} \cap \gamma$, we have E^* mapped onto E , $E \subset F$.

Let m_e denote exterior measure. We will demonstrate that

$$(3.10) \quad m_e(E) \geq m(E^*).$$

This will contradict (3.4), for F is closed; hence $E_0 = F \cap A^*$ is measurable and $E_0 \supset E$, and therefore

$$(3.11) \quad m(E_0) \geq m_e(E).$$

Now to prove (3.10), let $E_1^* \subset E^*$ be closed and satisfy

$$(3.12) \quad m(E_1^*) > m(E^*) - \epsilon.$$

Let $\Omega(w)$ be the harmonic measure in $\{|w| < 1\}$ of E_1^* . Thus $1 - \Omega(w)$ is the harmonic measure of the open set $E_1^{*'}$, where the complement is of course taken with respect to $\{|w| = 1\}$.

Then $2\pi[1 - \Omega(0)] = m(E_1^{*'})$ or

$$(3.13) \quad 2\pi \Omega(0) = m(E_1^*).$$

Now transfer the harmonic measure $\Omega(w)$ to $D(\epsilon)$ and call it $\omega(z) = \Omega(\phi(z))$. Since $\phi(0) = 0$, we have from (3.13)

$$(3.14) \quad 2\pi \omega(0) = m(E_1^*).$$

Let G be an open set on $\{|z|=1\}$ covering E such that

$$(3.15) \quad m(G) < m_e(E) + \varepsilon.$$

Let $\omega_1(z)$ be the harmonic measure of G in $\{|z|<1\}$.

Then

$$(3.16) \quad 2\pi \omega_1(0) = m(G).$$

The function $u(z) = \omega_1(z) - \omega(z)$ is harmonic in $D(\varepsilon)$. If ζ is an interior point of one of the cross-cuts bounding $D(\varepsilon)$, then $\omega_1(\zeta) > 0$ and $\omega(\zeta) = 0$ and $u(\zeta) > 0$. If $\zeta \in F \cap G$, then $\omega_1(\zeta) = 1$ and $\limsup_{z \rightarrow \zeta} \omega(z) \leq 1$, and thus

$$(3.17) \quad \liminf_{\substack{z \rightarrow \zeta \\ z \in D(\varepsilon)}} u(z) \geq 0.$$

If $\zeta \in F \cap G'$, then ζ corresponds to a point $\xi \in \{|w|=1\}$ and $\zeta \in G'$ implies $\zeta \notin E$ and hence $\xi \notin E^*$, or $\xi \in E^{*'}.$ Hence $(E_1^{*'} \text{ is open}) \Rightarrow \Omega(w) \rightarrow 0$ as $w \rightarrow \xi$; i.e., $\omega(z) \rightarrow 0$ as $z \rightarrow \zeta, z \in D(\varepsilon)$. Thus $\liminf u(z) \geq 0$ for $z \rightarrow \zeta, z \in D(\varepsilon)$. Therefore at all boundary points of $D(\varepsilon)$, $\liminf u(z) \geq 0$ and hence $u(z) \geq 0$ in $D(\varepsilon)$. In particular $u(0) \geq 0$, or

$$(3.18) \quad \omega_1(0) \geq \omega(0).$$

Combining (3.12), (3.14), (3.15), and (3.16), we get

$$(3.10) \quad m_e(E) \geq m(E^*).$$

Thus we have a contradiction, and (3.9) is established.

It follows from (3.8) that the function $u(w) = \log(\epsilon |g(w)|)$ is positive harmonic in $\{|w| < 1\}$. Also, from (3.9),

$$(3.19) \quad u(e^{i\theta}) = 0 \quad \text{for almost all } \theta \in [0, 2\pi].$$

Here again, $u(e^{i\theta})$ denotes the radial limit of u at the point $e^{i\theta}$ if it exists.

Since u is positive harmonic in $\{|w| < 1\}$, it can be represented as a Poisson-Stieltjes Integral with respect to a non-decreasing function ψ :

$$(3.20) \quad u(w) = \frac{1}{2\pi} \int_0^{2\pi} P(w, \theta) d\psi(\theta).$$

We now distinguish two cases according to whether or not ψ is continuous.

First, we suppose that ψ has a discontinuity at the point θ_0 , and let $K = \psi(\theta_0+) - \psi(\theta_0-)$ where $\psi(\theta_0+)$ and $\psi(\theta_0-)$ denote respectively the right and left limits of ψ at θ_0 . Fix $w = \rho e^{i\theta_0}$; for $\epsilon > 0$ choose a partition of $[0, 2\pi]$ such that

$$(3.21) \quad \left| u(w) - \sum_{k=1}^n P(w, \theta'_k) (\psi(\theta_k) - \psi(\theta_{k-1})) \right| < \epsilon,$$

where $\theta'_k \in [\theta_{k-1}, \theta_k]$, and $\theta'_j = \theta_0$ for some j , $1 \leq j \leq n$.

Therefore we have

$$(3.22) \quad u(w) > \sum_{k=1}^n P(w, \theta'_k) (\psi(\theta_k) - \psi(\theta_{k-1})) - \epsilon.$$

And since P is positive and ψ is non-decreasing,

$$(3.23) \quad u(w) > P(w, \theta_0) (\psi(\theta_j) - \psi(\theta_{j-1})) - \epsilon.$$

Consequently, using the fact that $w = \rho e^{i\theta_0}$, $0 < \rho < 1$,

$$(3.24) \quad u(w) > \frac{1+\rho}{1-\rho} [\psi(\theta_0+) - \psi(\theta_0-)] - \epsilon.$$

The arbitrariness of $\epsilon > 0$ implies

$$(3.25) \quad u(w) \geq \frac{K}{1-|w|}, \quad \arg w = \theta_0.$$

Let $\{\rho_n\}$ be a sequence of positive numbers such that $\rho_n \uparrow 1$ and $w_n = \rho_n e^{i\theta_0}$. Then from (3.25) it follows that

$$(3.26) \quad u(w_n) \geq \frac{K}{1-|w_n|}.$$

Let $z_n = \phi^{-1}(w_n)$, $n = 1, 2, \dots$. From (3.26) it follows that $u(w_n) \rightarrow \infty$ as $n \rightarrow \infty$ since $K > 0$ and $|w_n| = \rho_n \uparrow 1$. By definition, $u(w) = \log(\epsilon |g(w)|)$ and $g(w) = f(\phi^{-1}(w))$; therefore $u(w_n) = \log(\epsilon |f(z_n)|)$ and we have $|f(z_n)| \rightarrow \infty$ as $n \rightarrow \infty$. Hence $|z_n| \rightarrow 1$ as $n \rightarrow \infty$.

Since $\phi^{-1}(0) = 0$, it follows from Schwarz' Lemma that

$$(3.27) \quad |\phi^{-1}(w)| \leq |w|.$$

Using (3.26) and the definition of u we have

$$(3.28) \quad \log(\epsilon |f(z_n)|) \geq \frac{K}{1-|z_n|}.$$

Since $K > 0$ and $|z_n| \rightarrow 1$, (3.28) contradicts (3.1).

If ψ is continuous, then we consider the Fourier-Stieltjes Series of ψ :

$$(3.29) \quad d\psi \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

where
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos n\theta \, d\psi(\theta), \quad n = 0, 1, 2, \dots;$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin n\theta \, d\psi(\theta), \quad n = 1, 2, \dots$$

Then from standard results, we have

$$(3.30) \quad u(\rho e^{i\theta}) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\theta + b_n \sin n\theta).$$

Let $\Sigma A_n(\theta)$ be a trigonometric series. Then define $u^*(\theta)$ and $u_*(\theta)$ as follows:

$$(3.31) \quad u^*(\theta) = \limsup_{\rho \uparrow 1} \Sigma A_n(\theta) \rho^n,$$

$$(3.32) \quad u_*(\theta) = \liminf_{\rho \uparrow 1} \Sigma A_n(\theta) \rho^n.$$

The limits $u^*(\theta)$ and $u_*(\theta)$ are called the upper and lower Abel sums respectively of $\Sigma A_n(\theta)$ (at the point θ). If $u^*(\theta) = u_*(\theta)$, we say that $\Sigma A_n(\theta)$ is Abel summable to the common value at θ .

We are now in a position to apply the following theorem [13, p.356].

Theorem (Verblunsky). Let $\Sigma A_n(\theta)$ be a trigonometric series such that:

$$(3.33) \quad \Sigma A_n(\theta) \text{ has coefficients } o(n),$$

$$(3.34) \quad u^*(\theta) \cong \chi(\theta) \text{ where } \chi(\theta) \text{ is integrable,$$

$$(3.35) \quad u_*(\theta) \text{ is finite except possibly in a denumerable set } E,$$

$$(3.36) \quad \lim_{\rho \uparrow 1} (1-\rho)u(\rho e^{i\theta}) = 0 \text{ for } \theta \in E, \text{ where } \\ u(\rho e^{i\theta}) = \Sigma A_n(\theta) \rho^n.$$

Then $\Sigma A_n(\theta)$ is a Fourier Series.

It follows from Zygmund [13, p.108] that the continuity of ψ implies

$$(3.37) \quad \sum_{k=1}^n (|a_k| + |b_k|) = o(n)$$

and thus (3.33) is satisfied. The fact that $u > 0$ shows that (3.34) is satisfied.

Since $D(\epsilon)$ is a Jordan Domain, $\phi^{-1}(w)$ may be extended one-one and continuously to $\{|w|=1\}$. Recalling that $u(\rho e^{i\theta}) = \sum A_n(\theta) \rho^n$ and $u(\rho e^{i\theta}) = \log(\epsilon |f(\phi^{-1}(\rho e^{i\theta}))|)$ we see that $u_*(\theta) = \infty$ implies that f has the asymptotic value ∞ at the point $\phi^{-1}(e^{i\theta})$. Also $\phi^{-1}(e^{i\theta}) \in \gamma$ since $\overline{D(\epsilon)} \cap \{|z|=1\} \subset \gamma^\circ$ by (3.6). Therefore, if the set E where $u_*(\theta)$ is infinite were uncountable then $A_\infty \cap \gamma$ would be uncountable since ϕ^{-1} is one-one on $\{|w|=1\}$. But $A_\infty \cap \gamma$ is countable by (3.5). Therefore E is countable and u satisfies (3.35).

Finally, (3.36) is satisfied by the same argument as that given between (3.25) and (3.28).

Therefore the series (3.29) is a Fourier Series for some function F . Then (3.29) is Abel summable to F almost everywhere by the Theorem of Fejér. But by (3.19) and (3.30) it follows that (3.29) is Abel summable to 0 almost everywhere. Thus $F = 0$ a.e. and (3.29) is a constant. But this contradicts the hypothesis that f be non-constant since $u(w) = \log(\epsilon |f(\phi^{-1}(w))|)$. This concludes the proof of Theorem 1.

We now proceed to state some immediate corollaries to Theorem 1. The first corollary is merely the contrapositive of Theorem 1.

Corollary 1.1. Let f be holomorphic in $\{|z|<1\}$,

and suppose $f \neq 0$. Suppose further that there exists an arc $\gamma \subset \{|z|=1\}$ such that

$$(3.38) \quad m(A^* \cap \gamma) = 0 \text{ and}$$

$$(3.39) \quad A_\infty \cap \gamma \text{ is countable.}$$

Then $\log M(r) > o\left(\frac{1}{1-r}\right)$.

The next corollary is a particular case of Theorem 1.

Corollary 1.2. Let f be holomorphic in $\{|z|<1\}$,
and suppose $f \neq 0$ in $\{|z|<1\}$. Suppose further that

$$(3.40) \quad \log M(r) = o\left(\frac{1}{1-r}\right)$$

and $A_\infty \cap \gamma$ is countable for some arc $\gamma \subset \{|z|=1\}$.

Then $m(A^* \cap \gamma) > 0$.

The following theorem shows that if f is locally unbounded near the arc γ , then we can sharpen the conclusion of Theorem 1.

Theorem 2. Let f be holomorphic in $\{|z|<1\}$, and
suppose $f \neq 0$ in $\{|z|<1\}$. Suppose further that

$$(3.41) \quad \log M(r) = o\left(\frac{1}{1-r}\right)$$

Let γ be an arc of $\{|z|=1\}$ such that f is locally
unbounded near γ . Then either

$$(3.42) \quad m(A(\{|w|>\delta\}) \cap \gamma) > 0 \text{ for arbitrary } \delta,$$

$0 < \delta < \infty$, or

$$(3.43) \quad A_\infty \cap \gamma \text{ is uncountable.}$$

Proof. Suppose that there exists a $\delta > 0$ such that

$$(3.44) \quad m(A(\{|w|>\delta\}) \cap \gamma) = 0 \text{ and}$$

$$(3.45) \quad A_\infty \cap \gamma \text{ is countable.}$$

In the proof of Theorem 1, replace (3.4) and (3.5) by (3.44)

and (3.45) respectively.

The fact that A_∞ is dense on γ now follows from a theorem of McMillan [8, Theorem 3] since f is locally unbounded near the arc γ . The only additional restriction needed in the proof of Theorem 1 is that in selecting $D(\epsilon)$ we choose $\epsilon > 0$ so small that $\frac{1}{\epsilon} > \delta$. Then the proof goes through as before and we obtain a contradiction. Thus Theorem 2 is proved.

Let $A'(\{|w| \leq \lambda\})$ denote the complement of $A(\{|w| \leq \lambda\})$ relative to $\{|z|=1\}$. Then we have the following corollary.

Corollary 2.1. Let f satisfy the hypotheses of Theorem 2. Then for any real λ , $0 < \lambda < \infty$, $A'(\{|w| \leq \lambda\}) \cap \gamma$ is uncountable.

Proof. We need the following theorem [1, Corollary 1].

Theorem of Bagemihl. Let f be a function in $\{|z| < 1\}$. The set $E \subset \{|z|=1\}$ at which f has more than one asymptotic value is countable.

Now, suppose $A'(\{|w| \leq \lambda\}) \cap \gamma$ is countable for some $\lambda > 0$. By Theorem 2, either

$$(3.42) \quad m(A(\{|w| > \lambda\}) \cap \gamma) > 0, \text{ or}$$

$$(3.43) \quad A_\infty \cap \gamma \text{ is uncountable.}$$

But either (3.42) or (3.43) implies that the set of points E at which f has more than one asymptotic value is uncountable since $A'(\{|w| \leq \lambda\}) \cap \gamma$ is countable. This contradicts the above theorem, and therefore $A'(\{|w| \leq \lambda\}) \cap \gamma$ is uncountable for every λ , $0 < \lambda < \infty$. This proves Corollary 2.1.

Corollary 2.2. Let f satisfy the hypotheses of
Theorem 2 and let $a \neq \infty$ be a complex number. Then
 $A'_a \cap \gamma$ is uncountable.

Proof. Recall that $A_a = A(\{a\})$. Then Corollary 2.2 is an immediate consequence of Corollary 2.1 since $A'(\{|w| \leq |a|\}) \cap \gamma$ is uncountable and $A_a \subset A(\{|w| \leq |a|\})$. Thus Corollary 2.2 is proved.

Corollary 2.3. Let f satisfy the hypotheses of
Theorem 2. Suppose also that $A_\infty \cap \gamma$ is countable. Then
for any real λ , $0 < \lambda < \infty$, $m(A'(\{|w| \leq \lambda\}) \cap \gamma) > 0$.

Proof. Suppose $m(A'(\{|w| \leq \lambda\}) \cap \gamma) = 0$. Since $A_\infty \cap \gamma$ is countable, it follows from (3.42) that $m(A(\{|w| > \lambda\}) \cap \gamma) > 0$. But this implies that $m(A(\{|w| > \lambda\}) \cap A(\{|w| \leq \lambda\}) \cap \gamma) > 0$ in contradiction to the theorem of Bagemihl and Corollary 2.3 is proved.

The following theorem shows that we can drop the condition $f \neq 0$ at some expense to the growth condition and still obtain the conclusion of Theorem 1.

Theorem 3. Let f be holomorphic in $\{|z| < 1\}$ and
satisfy

$$(3.46) \quad M(r) = o\left(\frac{1}{1-r}\right).$$

Then for any arc $\gamma \subset \{|z|=1\}$, either

$$(3.47) \quad m(A^* \cap \gamma) > 0, \text{ or}$$

$$(3.48) \quad A_\infty \cap \gamma \text{ is uncountable.}$$

Remark. MacLane has shown [5] that there exist holomorphic functions in $\{|z| < 1\}$ of arbitrarily slow growth which omit 0 and have no radial limits. However, Theorem 1

and Corollary 2.1 show that under certain growth conditions (depending respectively on whether or not f omits 0) these functions must have asymptotic points in abundance.

Proof. Let $F(z) = e^{f(z)}$ and $M_1(r)$ be the maximum modulus of F . Then we have

$$(3.49) \quad F \neq 0 \text{ in } \{|z| < 1\}, \text{ and}$$

$$(3.50) \quad \log M_1(r) = o\left(\frac{1}{1-r}\right).$$

Now if F has the asymptotic value a , where $a \neq 0$ and $a \neq \infty$, then f has the finite asymptotic value $\log a$ on the same path. If F has the asymptotic value ∞ , then $\operatorname{Ref} \rightarrow \infty$ and thus f has the asymptotic value ∞ on the same path.

Let γ be any arc of $\{|z|=1\}$. If there exists a point $\zeta \in \gamma$ and a neighborhood N of ζ such that F is bounded in $N \cap \{|z| < 1\}$ then it follows from the Theorems of Fatou and F. and M. Riesz that $m(A(\{|w| > 0\}) \cap \gamma) > 0$ for the function F . Thus by the above remarks we have (3.47) for the function f in this case.

If F is unbounded near γ , then by (3.49) and (3.50) F satisfies the hypotheses of Theorem 2 and we have either (3.42) or (3.43) for the function F . Again by the above remarks it follows that f satisfies either (3.47) or (3.48). This concludes the proof of Theorem 3.

Remarks. Note that all we actually need to get (3.46) is the fact that $\mu(r) = o((1-r)^{-1})$ where $\mu(r)$ is the

maximum of $|\operatorname{Re} f(z)|$ on $\{|z|=r\}$. Similarly, we may consider $F(z) = e^{if(z)}$ in the proof of Corollary 2.1 so that it is also sufficient to have $v(r) = o((1-r)^{-1})$ where $v(r)$ is the maximum of $|\operatorname{Im} f(z)|$ on $\{|z|=r\}$.

Example 2 in section 8 shows that under the hypotheses of Corollary 2.1 we may have $A^* = \emptyset$. Therefore the conclusion that $m(A^*) > 0$ or A^* is uncountable is impossible.

The following theorem and its corollaries parallel the development of Theorem 2 and its corollaries where we have again dropped the condition $f \neq 0$.

Theorem 4. Let f be holomorphic in $\{|z|<1\}$ and satisfy

$$(3.51) \quad M(r) = o\left(\frac{1}{1-r}\right).$$

Let γ be an arc of $\{|z|=1\}$ such that f is locally unbounded near γ . Then either

$$(3.52) \quad m(A(\{|w|>\delta\}) \cap \gamma) > 0 \quad \text{for arbitrary } \delta, 0 < \delta < \infty,$$

or

$$(3.53) \quad A_\infty \cap \gamma \text{ is uncountable.}$$

Proof. By definition, f locally unbounded near γ means

$$(3.54) \quad \limsup_{\substack{z \rightarrow \zeta \\ z \in \{|z|<1\}}} |f(z)| = \infty \quad \text{for every point } \zeta \in \gamma.$$

Let $F_1 = e^f$, $F_2 = e^{-f}$, $F_3 = e^{if}$, and $F_4 = e^{-if}$. Then for each point $\zeta \in \gamma$ we have

$$(3.55) \quad \limsup_{\substack{z \rightarrow \zeta \\ z \in \{|z| < 1\}}} |F_j(z)| = \infty \quad \text{for at least one } j, 1 \leq j \leq 4,$$

by considering real and imaginary parts of f . Let

$$E_j = \{\zeta \in \gamma \mid \limsup_{z \rightarrow \zeta, z \in \{|z| < 1\}} |F_j(z)| = \infty\},$$

$j = 1, 2, 3, 4$. Then from (3.55) it follows that $\gamma = \bigcup_{j=1}^4 E_j$.

Thus at least one of the E_j 's, say E_k , is dense on some subarc γ' of γ .

We now restrict our attention to F_k and γ' . Since E_k is dense on γ' , it follows that F_k is locally unbounded near γ' . Also, $F_k \neq 0$ in $\{|z| < 1\}$, and

$$(3.56) \quad \log M_1(r) = o\left(\frac{1}{1-r}\right)$$

where $M_1(r)$ is the maximum modulus of F_k by (3.51).

Applying Theorem 2 to F_k we have either

$$(3.57) \quad m(A(\{|w| > e^\delta\}) \cap \gamma') > 0 \quad \text{for arbitrary } \delta, \\ 0 < \delta < \infty, \text{ or}$$

$$(3.58) \quad A_\infty \cap \gamma' \text{ is uncountable.}$$

Note that if F_j has the asymptotic value ∞ then f has the asymptotic value ∞ on the same path, $j = 1, 2, 3, 4$. Also, if F_j has the asymptotic value a where $|a| > e^\delta$, $\delta > 0$, then f has the asymptotic value $\log a$ on the same path and $|\log a| > \delta$, $j = 1, 2, 3, 4$.

The above remarks together with (3.57) and (3.58) imply that for the function f either

$$(3.59) \quad m(A(\{|w| > \delta\}) \cap \gamma') > 0 \quad \text{for arbitrary } \delta, \\ 0 < \delta < \infty, \text{ or}$$

(3.60) $A_\infty \cap \gamma'$ is uncountable.

Since $\gamma' \subset \gamma$, then the theorem follows.

The proof of each of the following corollaries closely follows the proof of the corresponding corollary to Theorem 2.

Corollary 4.1. Let f satisfy the hypotheses of Theorem 4. Then for any real λ , $0 < \lambda < \infty$, $A'(\{|w| \leq \lambda\}) \cap \gamma$ is uncountable.

Corollary 4.2. Let f satisfy the hypotheses of Theorem 4 and let $a \neq \infty$ be a complex number. Then $A'_a \cap \gamma$ is uncountable.

Corollary 4.3. Let f satisfy the hypotheses of Theorem 4. Suppose also that $A_\infty \cap \gamma$ is countable. Then for any real λ , $0 < \lambda < \infty$, $m(A'(\{|w| \leq \lambda\}) \cap \gamma) > 0$.

4. Arc Tracts for Finite Values and ∞ .

The first theorem we state is actually a Corollary of a theorem of McMillan [8, Theorem 2].

Theorem 5. Let f be holomorphic and non-constant in $\{|z| < 1\}$, and suppose that f has the asymptotic value $a \neq \infty$ on a curve Γ which tends to the total circumference $\{|z| = 1\}$. Then either f has infinitely many point tracts for ∞ at one point of $\{|z| = 1\}$ or f has an arc tract for ∞ .

Proof: We will first show that A^* can contain at most two points. Suppose ζ_1, ζ_2 , and ζ_3 are distinct points of A^* , and let Γ_1, Γ_2 , and Γ_3 be arcs in $\{|z| < 1\}$ ending at ζ_1, ζ_2 and ζ_3 respectively such that $f \rightarrow a_i$ on Γ_i , $i = 1, 2, 3$.

Since the arcs Γ_1, Γ_2 , and Γ_3 are eventually disjoint, we may assume that their only common point is the origin.

Now, if Γ failed to intersect two of these arcs, say Γ_1 and Γ_2 , infinitely many times then the curve Γ would not tend to one of the arcs $\widehat{\zeta_1 \zeta_2}$. Thus we assume that Γ intersects Γ_1 and Γ_2 in infinitely many points. From this, we obtain a sequence of cross-cuts $\{\gamma_n\}$ of the sector S determined by Γ_1, Γ_2 and an arc $\widehat{\zeta_1 \zeta_2}$ such that

$$(4.1) \quad \gamma_n \rightarrow \widehat{\zeta_1 \zeta_2} \quad \text{as } n \rightarrow \infty, \text{ and}$$

$$(4.2) \quad \sup_{z \in \gamma_n} |f(z) - a| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

These conditions follow from the following facts: the cross-cuts $\gamma_n \subset \Gamma$, Γ tends to $\{|z|=1\}$, and $f \rightarrow a$ on Γ .

Let M_1, M_2 , and M be bounds of f on Γ_1, Γ_2 and Γ respectively and let $B = \max \{M_1, M_2, M\}$. If $z \in S$, then from (4.1) there exists an arc γ_n which separates z from the arc $\widehat{\zeta_1 \zeta_2}$ in S . Therefore, by the maximum principle we have $|f(z)| < B$. Hence f is bounded on S , but then (4.2) contradicts Koebe's Lemma. Thus the set A^* consists of at most two points.

Since $m(A^*) = 0$, it follows from a theorem of McMillan [8, Theorem 2], that either f has finitely many arc tracts for ∞ whose ends cover $\{|z|=1\}$, or f has infinitely many tracts for ∞ .

If f has an arc tract for ∞ , then the conclusion of Theorem 5 is true. Otherwise, all the tracts for ∞ are

point tracts and we will show that they must be for the same point.

Suppose f has the asymptotic value ∞ at the points ζ_1 and ζ_2 . Let Γ_1 and Γ_2 be arcs ending at ζ_1 and ζ_2 respectively on which $f \rightarrow \infty$. The curve Γ must intersect one of these arcs, say Γ_1 , in infinitely many points, for otherwise Γ would not tend to one of the arcs $\widehat{\zeta_1\zeta_2}$ or $\widehat{\zeta_2\zeta_1}$. Therefore, we get a sequence $\{z_n\} \subset \Gamma$ such that $|z_n| \rightarrow 1$ and $f(z_n) \rightarrow \infty$. But this contradicts the fact that $f \rightarrow a \neq \infty$ on Γ . Thus the infinitely many point tracts for ∞ end at the same point. This concludes the proof of Theorem 5.

If we notice that for any asymptotic tract we can find a curve Γ which belongs to the tract and tends to the end of the tract, then we can replace the hypothesis that $f \rightarrow a \neq \infty$ on a curve Γ which tends to the total circumference $\{|z|=1\}$ by the hypothesis that f has an arc tract for $a \neq \infty$ with end $\{|z|=1\}$.

In order to get a result analogous to Theorem 5 for the curve Γ tending to an arc $\gamma \subset \{|z|=1\}$ where $\gamma \neq \{|z|=1\}$, example 3 in section 8 shows that the condition $f \rightarrow a \neq \infty$ on Γ is insufficient. However, the following theorem does give sufficient conditions for the analogous result.

Theorem 6. Let f be holomorphic in $\{|z|<1\}$, and

suppose f has the asymptotic value $a \neq \infty$ on a curve Γ which tends to an arc $\gamma \subset \{|z|=1\}$, $\gamma \neq \{|z|=1\}$. Suppose further that there exist points $\zeta_1 \in \gamma$ and $\zeta_2 \in \gamma$ and arcs Γ_1 and Γ_2 ending at ζ_1 and ζ_2 respectively on which f is bounded. Then either f has infinitely many point tracts for ∞ at some point of γ , or f has an arc tract for ∞ with end $K' \subset \gamma$.

Remark. Here again we could replace the hypothesis that f has the asymptotic value $a \neq \infty$ on a curve Γ which tends to an arc $\gamma \subset \{|z|=1\}$, $\gamma \neq \{|z|=1\}$ by the hypothesis that f has an arc-tract for $a \neq \infty$ with end $\gamma \subset \{|z|=1\}$, $\gamma \neq \{|z|=1\}$.

Proof: We first show that both Γ_1 and Γ_2 can not intersect Γ infinitely often. For suppose they do and let S be the subset of $\{|z|<1\}$ bounded by $\Gamma_1 \cup \Gamma_2$ and the arc $\widehat{\zeta_1 \zeta_2} \subset \gamma$. Then Γ provides a sequence of cross-cuts $\{\gamma_n\}$ of S such that

$$(4.3) \quad \gamma_n \rightarrow \widehat{\zeta_1 \zeta_2} \text{ as } n \rightarrow \infty, \text{ and}$$

$$(4.4) \quad \sup_{z \in \gamma_n} |f(z) - a| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then just as in Theorem 5, we obtain a contradiction to Koebe's Lemma since f is bounded on $\Gamma_1 \cup \Gamma_2$. It also follows that at least one of the points ζ_1 or ζ_2 is an end point of γ .

We may as well assume that Γ_1 does not intersect Γ , since we may replace that portion which does (only a finite

number of intersections) by a curve interior to $\{|z| < 1\}$ and maintain the property that f is bounded on Γ_1 . As a matter of fact, we assume that $\Gamma \cap \Gamma_1 = \{0\}$, $\Gamma_1 \cap \Gamma_2 = \{0\}$.

Our next step is to find a domain Δ and an arc $\gamma' \subset \gamma$ (by considering cases) such that f is locally unbounded in Δ near γ' .

Suppose first that ζ_2 is an interior point of γ . Let Δ be that domain of $\{|z| < 1\}$ which

is the intersection of the domain bounded by $\Gamma_1 \cup \Gamma_2 \cup \widehat{\zeta_1 \zeta_2}$, where $\widehat{\zeta_1 \zeta_2} \subset \gamma$. Then Δ is bounded by Γ_1 , some portion of Γ , some arcs of Γ_2 with end-points on Γ and the arc $\widehat{\zeta_1 \zeta_2} \subset \gamma$. We shall now show that f is locally unbounded in Δ near the arc $\widehat{\zeta_1 \zeta_2}$ (that is, for any point $\zeta \in \widehat{\zeta_1 \zeta_2}$, $\limsup |f(z)| = \infty$ as $z \rightarrow \zeta$, $z \in \Delta$).

Let $\zeta \in \widehat{\zeta_1 \zeta_2}$ and let N be an open disc centered at ζ . Let D be the domain bounded by $\Gamma_1 \cup \Gamma_2 \cup \widehat{\zeta_1 \zeta_2}$.

Now f is unbounded in $D \cap N$ or we again have a contradiction of Koebe's Lemma since we can find a sequence of arcs $\{\gamma_n\} \subset D \cap N$, $\gamma_n \rightarrow \gamma \subset \{|z|=1\}$ such that $\sup_{\gamma_n} |f(z)-a| \rightarrow 0$ as $n \rightarrow \infty$. The arcs $\{\gamma_n\}$ come from Γ , of course. On the other hand, f is uniformly bounded in those subdomains of D which are bounded only by arcs of Γ and arcs of Γ_2 . Thus f must be unbounded in $N \cap \Delta$.

Note that the above case also includes the possibility that Γ_2 intersects Γ infinitely often even though ζ_2 is an end-point of γ .

We now consider the case when ζ_2 is an end-point of γ and Γ_2 does not intersect Γ infinitely often. Let D_1 be the domain bounded by $\Gamma_1 \cup \Gamma \cup \gamma$ and D_2 be the domain bounded by $\Gamma_2 \cup \Gamma \cup \gamma$. We will show that f is locally unbounded in one of these domains near some arc $\gamma' \subset \gamma$.

Let ζ_3 be an interior point of γ and let N be an open disc centered at ζ_3 . The function f is unbounded in $N \cap \{|z| < 1\}$ since otherwise we again have a contradiction of Koebe's Lemma. Since N was arbitrary, it follows that there exists a sequence of points $\{z_n\} \subset \{|z| < 1\}$ such that $z_n \rightarrow \zeta_3$ as $n \rightarrow \infty$ and $|f(z_n)| \rightarrow \infty$ as $n \rightarrow \infty$. Either D_1 or D_2 must contain an infinite subsequence $\{z_{n_K}\}$ of $\{z_n\}$. We may as well assume (relabel if necessary) that D_1 contains the infinite subsequence $\{z_{n_K}\}$. We will show that f is locally unbounded in D_1 near the arc $\widehat{\zeta_1 \zeta_3}$.

Let ζ be a point on the arc $\widehat{\zeta_1 \zeta_3} \subset \gamma$. Let N' be an open disc centered at ζ . Suppose f is bounded in $N' \cap D_1$ and let R be the radius to ζ . It follows that f is bounded on $R \cap D_1$ and $\zeta \neq \zeta_3$. This implies that f is uniformly bounded in those subdomains of D_1 bounded by arcs of Γ and segments of R . Thus f is bounded in that portion of D_1 which is exterior to the sector S bounded by $R \cup \Gamma_1 \cup \widehat{\zeta_1 \zeta}$. However, this contradicts the fact that $z_{n_K} \rightarrow \zeta_3$, as $K \rightarrow \infty$, $\{z_{n_K}\} \subset D_1$ and $|f(z_{n_K})| \rightarrow \infty$ as $K \rightarrow \infty$, since we can find a neighborhood of ζ_3 disjoint from S . Thus f is unbounded in $N' \cap D_1$, and it follows that f is locally unbounded in D_1 near the arc $\widehat{\zeta_1 \zeta_3}$.

In either of the two cases above, we have a domain Δ bounded by Γ_1 , some portion of Γ , possibly some arcs

of Γ_2 with end-points on Γ , and the arc $\widehat{\zeta_1 \zeta_2}$ (by relabeling if necessary). Also, there is an arc $\widehat{\zeta_1 \zeta_3} \subset \widehat{\zeta_1 \zeta_2}$ such that f is locally unbounded in Δ near the arc $\widehat{\zeta_1 \zeta_3}$.

In the remainder of the proof, we will show that each component of $\{|f| > n\}$ contained in Δ , where n is sufficiently large, belongs to a tract for ∞ and then proceed to the desired conclusion.

Let $E \subset \Delta$ be a component of $\{|f| > n\}$ where n is chosen to be larger than the bound of f on $\Gamma_1 \cup \Gamma \cup \Gamma_2$. The following argument shows that f is unbounded in E .

By the choice of n , all boundary points of E interior to $\{|z| < 1\}$ belong to the level set $\{|f| = n\}$. If ζ_1 were not a boundary point of E , we could find a disc $D(\zeta_1)$ centered at ζ_1 such that the closed disc $\overline{D}(\zeta_1)$ does not intersect E . Using the fact that $\Gamma_1 \rightarrow \zeta$ and $\Gamma \rightarrow \gamma$ we see that the closure of each component of D_1 exterior to $D(\zeta_1)$ is contained in $\{|z| < 1\}$, and the closure of E must be a subset of one of these components. This implies that the boundary of E consists only of level curves (components of $\{|f| = n\}$). But this is impossible since $|f| > n$ in E . Therefore, the boundary of E contains ζ_1 and possibly a portion of γ .

Suppose f is bounded in E , say $|f(z)| \leq M$ for $z \in E$. Let $z_0 \in E$. For fixed $\epsilon > 0$, consider the function

$$(4.5) \quad \varphi_\epsilon(z) = \left(\frac{z-\zeta_1}{2}\right)^\epsilon f(z).$$

Choose $r > 0$ so small that $r < |z_0 - \zeta_1|$ and

$$(4.6) \quad \left(\frac{r}{2}\right)^\epsilon M \leq n.$$

Now z_0 belongs to a component E_1 of $E \cap \{|z-\zeta_1| > r\}$.

Since $E \subset D_1$ and $\Gamma \rightarrow \gamma$, it follows that E_1 is bounded

by a portion of $\{|f|=n\}$ and some arcs of the circle

$\{|z-\zeta_1|=r\}$. Note that

$$(4.7) \quad \left|\left(\frac{z-\zeta_1}{2}\right)^\epsilon\right| \leq 1 \quad \text{for } z \in \{|z| < 1\}.$$

Thus, using (4.6), (4.7) and the fact that $|f(z)| \leq M$ in

E we have

$$(4.8) \quad |\varphi_\epsilon(z)| \leq n,$$

On the boundary of E_1 . Therefore, by the maximum principle,

$$(4.9) \quad \left|\left(\frac{z_0-\zeta_1}{2}\right)^\epsilon f(z_0)\right| \leq n.$$

Since $\epsilon > 0$ was arbitrary, it follows that $|f(z_0)| \leq n$.

But this contradicts the fact that $z_0 \in E$ where E is

a component of $\{|f| > n\}$. Thus we have proved that f is

unbounded in E . It follows that each component of $\{|f| > n\}$

contained in Δ belongs to a tract for ∞ .

Suppose f has only N distinct tracts for ∞ in Δ

where N is a positive integer. Denote these tracts by

$\{D_1(\epsilon), \infty\}, \dots, \{D_N(\epsilon), \infty\}$. Choose n_0 so large that $\{|f| > n_0\}$

has N components in Δ . It follows from the preceding

paragraph that no matter how large we take n , we can get

no more than N components of $\{|f| > n\}$.

From the definition of asymptotic tract, we note that the components of $\{|f|>n\}$ in Δ are $D_i(\frac{1}{n})$, $i=1,\dots,N$. For each $n>n_0$, at least one of the above components must have diameter no smaller than $|\zeta_1-\zeta_3|>0$ since f is locally unbounded in Δ near the arc $\widehat{\zeta_1\zeta_3}$ and $\zeta_1 \in \overline{D}_i(\frac{1}{n})$, $i=1,\dots,N$. It follows that at least one of the N tracts, say $\{D_j(\epsilon), \infty\}$, has the property that $\text{diam } \overline{D}_j(\frac{1}{n}) \geq |\zeta_1-\zeta_3|$ for $n > n_0$ (recall that $D_j(\frac{1}{n}) \supset D_j(\frac{1}{n+1})$, $j=1,\dots,N$, all $n \geq 1$). Therefore, $K_j = \bigcap_{\epsilon>0} \overline{D}_j(\epsilon)$ is an arc of γ . Therefore we have an arc tract for ∞ with end contained in γ .

Finally, if f has infinitely many tracts for ∞ in Δ and one of these is an arc tract then we are finished. If they are all point tracts, then they must all be point tracts for the point ζ_1 since f clearly can not have the asymptotic value ∞ at an interior point of γ . This completes the proof of Theorem 6.

The following lemma will be useful to us in the remainder of the section.

Lemma 1. Let f be holomorphic in $\{|z|<1\}$. Let $\{D_1(\epsilon), a_1\}$ and $\{D_2(\epsilon), a_2\}$ be distinct arc tracts of f with ends K_1 and K_2 respectively. Suppose $K_1 \cap K_2$ contains an arc γ . Then f has an arc tract for ∞ with end γ' where γ' contains a subarc of γ .

To facilitate the proof, we assume some basic facts from the theory of prime ends (see Piranian [9]).

Proof. Let Γ_1 and Γ_2 be simple arcs belonging respectively to $\{D_1(\epsilon), a_1\}$ and $\{D_2(\epsilon), a_2\}$ such that

$\Gamma_1 \rightarrow K_1$, $\Gamma_2 \rightarrow K_2$ and $\Gamma_1 \cap \Gamma_2 = \{0\}$. Let Δ_1 and Δ_2 denote the two simply connected domains into which $\{|z| < 1\}$ is partitioned by $\Gamma_1 \cup \Gamma_2$. We will show that one of these domains Δ has the property that if Γ is an arc contained in Δ such that Γ tends to some portion of $\{|z|=1\}$ then in fact $\Gamma \rightarrow K$, where K contains a fixed arc of $\{|z|=1\}$.

We can find radii R_1 and R_2 to interior points ζ_1 and ζ_2 of γ such that the sector S bounded by R_1, R_2 and an arc $\zeta_1 \zeta_2 \subset \gamma$ has two infinite sequences of cross-cuts $\{\gamma_n^1\}$, $\gamma_n^1 \subset \Gamma_1$, $n \geq 1$ and $\{\gamma_n^2\}$, $\gamma_n^2 \subset \Gamma_2$, $n \geq 1$ such that each γ_n^i , $i = 1, 2$, $n \geq 1$ joins a point of R_1 to a point of R_2 .

From the above, we can find an infinite sequence $\{S_n\}$ of subdomains of S with the following properties. For each n , $S_n \subset \Delta_1$ or $S_n \subset \Delta_2$ and the boundary of S_n consists of: one arc of Γ_1 which joins a point of R_1 to a point of R_2 , one arc of Γ_2 which joins a point of R_1 to a point

of R_2 , some segments of R_1 and R_2 , and perhaps some cross-cuts of S from Γ_1 and Γ_2 (note that none of these can join a point of R_1 to a point of R_2).

The sequence $\{S_n\}$ "tends to" the arc $\widehat{\zeta_1 \zeta_2}$ in the sense that for arbitrary $\epsilon > 0$, there exists a positive integer N such that $S_n \subset S \cap \{1-\epsilon < |z| < 1\}$ for $n \geq N$. It follows from the properties of $\{S_n\}$ that one of the domains Δ_1 or Δ_2 must contain infinitely many S_n 's. Suppose Δ_1 (relabel if necessary) contains an infinite subsequence of $\{S_n\}$.

Let Γ be an arc in Δ_1 which begins at 0 and tends to $\{|z|=1\}$. Now Γ separates Γ_1 from Γ_2 in Δ_1 since any arc $c \subset \Delta_1$ joining a point of Γ_1 to a point of Γ_2 must intersect Γ . Thus, from the choice of the S_n 's, it follows that in each $S_n \subset \Delta_1$ we have a cross-cut $\gamma_n \subset \Gamma$ joining a point of R_1 to a point of R_2 . Therefore, Γ tends to an arc K , where $K \supset \widehat{\zeta_1 \zeta_2}$.

Now it follows readily that Δ_1 has exactly one prime-end P whose impression is contained in $\{|z|=1\}$ and whose set of principal points contains the arc $\widehat{\zeta_1\zeta_2}$. Let ϕ be a one-one conformal mapping of Δ_1 onto $\{|w|<1\}$. Let $\xi \in \{|w|=1\}$ be the point which corresponds to P under the mapping ϕ .

Note that there is no harm in assuming that a_1 and a_2 are finite; for otherwise, we would already have an arc tract for ∞ .

The function $f(\phi^{-1}(w))$ is holomorphic in $\{|w|<1\}$ and continuous on $\{|w|\leq 1\} - \{\xi\}$. Let ξ_0 be the point of $\{|w|=1\}$ which corresponds to 0 under ϕ . Then one of the arcs $\widehat{\xi_0\xi}$ corresponds to Γ_1 and the other to Γ_2 . Since Γ_1 and Γ_2 belong to distinct tracts, it follows from a theorem of Lindelöf (see Titchmarsh [11, p.179]) that $f(\phi^{-1})$ is unbounded in $\{|w|<1\}$. Then, a technique of Valiron [12, p.124] shows that $f(\phi^{-1})$ must have the asymptotic value ∞ on an arc Γ' which tends to ξ . The image of Γ' under ϕ^{-1} is an arc Γ'' in Δ_1 which tends to $\{|z|=1\}$. Hence, from the choice of Δ_1 , $\Gamma'' \rightarrow K$, where K is an arc of $\{|z|=1\}$, and $\widehat{\zeta_1\zeta_2} \subset K$. Therefore, f has an arc tract for ∞ with end $K \supset \widehat{\zeta_1\zeta_2}$ where $\widehat{\zeta_1\zeta_2} \subset \gamma$. This concludes the proof of Lemma 1.

We now extend Theorem 5 slightly.

Theorem 7. Let f be holomorphic and non-constant in $\{|z|<1\}$. Suppose further that the ends of arc tracts

of f for finite values cover $\{|z|=1\}$. Then either f has an arc tract for ∞ or infinitely many point tracts for ∞ .

Remarks. Theorem 7 differs from Theorem 5 in two respects. In Theorem 7, there may not be an arc tract for a finite value whose end is the whole circumference $\{|z|=1\}$. Also, in Theorem 7 we do not guarantee infinitely many tracts for ∞ at one point of $\{|z|=1\}$.

Proof. Suppose f has no arc tracts for ∞ . From Lemma 1 and the hypothesis of the theorem, it follows that $\{|z|=1\}$ is a union of a countable collection of closed intervals $\{C_i\}$, $i \in I$, where the interior of each C_i is non-void and $C_i \cap C_j$ consists of at most two points. The C_i 's are, of course, the ends of arc tracts of f for finite values.

It follows as in the proof of Theorem 5 that the interior of each C_i contains at most one asymptotic point of f . Therefore, A^* is countable since $A^* \cap C_i$ contains at most three points (one interior point and the two end-points) for each $i \in I$. Thus $m(A^*) = 0$ and again it follows from a theorem of McMillan [8, Theorem 2] and the assumption that f has no arc tract for ∞ that f has infinitely many point tracts for ∞ . Thus, if f has no arc tracts for ∞ , then f has infinitely many point tracts for ∞ , and Theorem 7 is proved.

Lemma 2. Let $f \in \mathcal{L}^*$ and let $\{D_1(\epsilon), a_1\}$ and

$\{D_2(\epsilon), a_2\}$ be two distinct arc tracts of f with ends K_1 and K_2 respectively. Then $K_1 \cap K_2$ contains at most two points.

Proof. Suppose first that $a_1 = \infty$. Then choose $\epsilon > 0$ so small that $D_1(\epsilon) \cap D_2(\epsilon) = \emptyset$. Let $C(\frac{1}{\epsilon})$ be the component of $\{|f| = \frac{1}{\epsilon}\}$ on the boundary of $D_1(\epsilon)$ which separates $D_1(\epsilon)$ from $D_2(\epsilon)$ in $\{|z| < 1\}$. Since $f \in \mathcal{L}^*$, $C(\frac{1}{\epsilon})$ is a cross-cut of $\{|z| < 1\}$ (that is, $C(\frac{1}{\epsilon})$ joins two points ζ_1 and ζ_2 of $\{|z| = 1\}$). Therefore, $\overline{D}_1(\epsilon) \cap \overline{D}_2(\epsilon) \cap \{|z| = 1\}$ contains at most the two points ζ_1 and ζ_2 .

If a_1 and a_2 are both finite and $K_1 \cap K_2$ contains an arc then by Lemma 1 f has an arc tract for ∞ with end K where $K \cap K_1 \cap K_2$ is an arc. But this is impossible by the above case. Therefore Lemma 2 is proved.

We can now state the following special cases of Theorem 6 and 7.

Theorem 8. Let $f \in \mathcal{L}^*$ and satisfy the hypotheses of Theorem 6. Then f has infinitely many point tracts for ∞ at one point of γ .

Proof. By Theorem 6, either f has an arc tract for ∞ with end $K \subset \gamma$ or f has infinitely many point tracts for ∞ at one point of γ . By Lemma 2, since $f \in \mathcal{L}^*$, it follows that f does not have an arc tract for ∞ with end $K \subset \gamma$. Therefore, Theorem 8 is proved.

Theorem 9. Let $f \in \mathcal{L}^*$ and satisfy the hypotheses of Theorem 7. Then f has infinitely many point tracts for ∞ .

Proof. Since ends of arc tracts for finite values cover $\{|z|=1\}$, it follows from Lemma 2 that f has no arc tract for ∞ . Thus from Theorem 7, f has infinitely many point tracts for ∞ .

5. A geometric Technique

In this section, we apply a technique of Collingwood and Cartwright [4] to obtain geometric proofs of some basic theorems about level sets.

Let f be holomorphic in $\{|z| < 1\}$. Recall that $A_a = \{\zeta \mid |\zeta| = 1 \text{ and } f \text{ has the asymptotic value } a \text{ at } \zeta\}$.

Theorem 10. Let f be holomorphic and non-constant in $\{|z| < 1\}$. Suppose further that for any arc $\gamma \subset \{|z| = 1\}$ either

$$(5.1) \quad A_\infty \cap \gamma \neq \emptyset, \text{ or}$$

$$(5.2) \quad A_{w_1} \cap \gamma \neq \emptyset \text{ and } A_{w_2} \cap \gamma \neq \emptyset \text{ where } w_1 \neq \infty, \\ w_2 \neq \infty \text{ and } w_1 \neq w_2.$$

Then $f \in \mathfrak{L}$.

Remarks. Theorem 10 is clearly a consequence of the fact that $\mathcal{A} = \mathfrak{B} = \mathfrak{L}$ (MacLane [7, Theorem 1], since the hypotheses immediately imply that $f \in \mathcal{A}$. But MacLane's proof of $\mathcal{A} = \mathfrak{B} = \mathfrak{L}$ depends on a lemma [7, Lemma 1] the proof of which is essentially analytic. The main virtue of Theorem 10 is that it has a fairly simple geometric proof and MacLane's Lemma 1 is an immediate corollary (Corollary 10.1).

Note that condition (5.2) merely says that f has asymptotic points for two distinct finite asymptotic values on γ .

Proof. Suppose $f \notin \mathfrak{L}$. Then there exists a sequence

of arcs $\{\gamma_n\}$ in $\{|z| < 1\}$ such that

$$(5.3) \quad \gamma_n \rightarrow \gamma, \text{ where } \gamma \text{ is an arc of } \{|z|=1\}, \text{ and}$$

$$(5.4) \quad |f(z)| = \lambda \text{ for } z \in \gamma_n, n = 1, 2, \dots,$$

where λ is a positive real number.

It follows immediately that $A_\infty \cap \gamma = \emptyset$. Then, by (5.2), let $\zeta_1 \in A_{w_1} \cap \gamma$ and $\zeta_2 \in A_{w_2} \cap \gamma$ and let Γ_1 and Γ_2 be arcs tending to ζ_1 and ζ_2 respectively such that $f \rightarrow w_i$ on Γ_i , $i = 1, 2$. Note that $\zeta_1 \neq \zeta_2$ since $A_\infty \cap \gamma = \emptyset$ (see section 2). Also, we may assume that ζ_1 and ζ_2 are interior points of γ , since the arc γ in the hypothesis is arbitrary.

Let $\overline{w_1 w_2}$ be the line segment joining w_1 to w_2 in the w -plane and let L be a line intersecting $\overline{w_1 w_2}$ orthogonally such that $w_1 \notin L$, $w_2 \notin L$. Choose $\rho > 0$ such that if $S_1 = \{w \mid |w - w_1| < \rho\}$ and $S_2 = \{w \mid |w - w_2| < \rho\}$ then $S_1 \cap L = \emptyset$ and $S_2 \cap L = \emptyset$.

Let $r_0 > 0$ be such that $z \in \Gamma_i$, $r_0 < |z| < 1$, implies $f(z) \in S_i$, $i = 1, 2$. Let S denote the sector of $\{|z| < 1\}$ bounded by Γ_1, Γ_2 and a subarc of γ . Let $C_r \subset \{|z| = r\}$, $r_0 < r < 1$, be a circular arc in S joining a point of Γ_1 to a point of Γ_2 such that $f' \neq 0$ on C_r . Then $f(C_r)$ is an analytic arc in the w -plane joining a point of S_1 to a point of S_2 . Let $S_r = \{z \in S \mid r < |z| < 1\}$.

Since $f(C_r)$ is analytic, we may assume (by moving L slightly if necessary) that L contains no multiple points of $f(C_r)$. It also follows that $f(C_r)$ can have at most a finite number of intersections with the line L . We shall ignore intersections which are not crossings since the corresponding curves in $\{|z| < 1\}$ do not cross in this case. The fact that one end-point of C_r lies in S_1 and the other in S_2 implies that the number of crossings is an odd number.

Let a and b denote the extreme points of intersection of $f(C_r)$ and L . Let $L_a \subset L$ and $L_b \subset L$ denote the half-lines from a and b respectively to ∞ which do not contain any points of $f(C_r) \cap L$. Let $\mathfrak{g}$ denote the Riemann Surface over the w -plane onto which f maps $\{|z| < 1\}$ and let $P_a \in \mathfrak{g}$ and $P_b \in \mathfrak{g}$ be the points over a and b respectively which correspond to points of C_r under f . If the curve in $\{|z| < 1\}$ corresponding to the lifting of L_a into $\mathfrak{g}$ starting at P_a does not lie in S_r , then since the number of crossings is odd, the curve in $\{|z| < 1\}$ corresponding to the lifting of L_b into $\mathfrak{g}$ starting at P_b must lie in S_r . Since f is holomorphic, $\mathfrak{g}$ has no points over ∞ . It follows that there are maximum half-open pieces of L_a and L_b which may be lifted into $\mathfrak{g}$ starting at P_a and P_b respectively.

From the above, we can obtain a curve $\Gamma \subset S_r$ (corresponding to the lifting of L_a or L_b) such that f has an asymptotic value on Γ . Then Γ tends to some portion of $\{|z|=1\}$. However, this contradicts the fact that any curve in S_r on which $|z| \rightarrow 1$ must intersect infinitely many γ_n ($|f| = \lambda$ on γ_n , $n \geq 1$), since the line L and hence the rays L_a and L_b can intersect $\{|w| = \lambda\}$ in at most two points. This completes the proof of Theorem 10.

As a corollary of this theorem we obtain MacLane's Lemma 1 [7,p.10].

Corollary 10.1. Let f be holomorphic, non-constant

and bounded in $\{|z| < 1\}$. Then $f \in \mathfrak{L}$.

Proof. It follows from the Theorems of Fatou and F. and M. Riesz that f satisfies (5.2) on any arc $\gamma \subset \{|z|=1\}$. Therefore $f \in \mathfrak{L}$ by Theorem 10 and we have proved Corollary 10.1.

By further exploiting the same basic technique, we are led to the following. Let w_1 and w_2 be distinct finite points in the plane. Denote by $G(\infty; w_1, w_2)$ a family of simple closed analytic curves on the sphere such that

(5.5) if $\gamma_1 \in G$, $\gamma_2 \in G$ and $\gamma_1 \neq \gamma_2$, then $\gamma_1 \cap \gamma_2 = \{\infty\}$, and

(5.6) if $\gamma \in G$, then γ separates w_1 from w_2 on the sphere.

Here, we say that γ separates w_1 from w_2 on the sphere if each of the two domains into which the simple closed curve γ partitions the sphere contains exactly one of the two points w_1, w_2 . Then we have the following theorem.

Theorem 11. Let f be holomorphic in $\{|z| < 1\}$.

Suppose further that f has the asymptotic value w_1 on an arc Γ_1 , and the asymptotic value w_2 on an arc Γ_2 , where $\Gamma_1 \cap \Gamma_2 = \{0\}$. Let S denote either subdomain of $\{|z| < 1\}$ bounded by $\Gamma_1 \cup \Gamma_2$ and an arc $K \subset \{|z|=1\}$.

Then, corresponding to each $\gamma \in G$, there exists a curve $\Gamma \subset S$ such that f has an asymptotic value on Γ and $f(\Gamma) \subset \gamma$.

Remark. The condition $\Gamma_1 \cap \Gamma_2 = \{0\}$ is merely a

convenience since Γ_1 and Γ_2 are necessarily eventually disjoint ($w_1 \neq w_2$).

Proof. Let $\gamma \in \mathcal{G}$, and choose $\rho > 0$ so that if $S_1 = \{w \mid |w - w_1| < \rho\}$ and $S_2 = \{w \mid |w - w_2| < \rho\}$ then $S_1 \cap \gamma = \emptyset$ and $S_2 \cap \gamma = \emptyset$. Now γ takes the place of the line L in the proof of Theorem 10. Then the proof of Theorem 11 is a straightforward generalization of the proof of Theorem 10. The only significant difference is that we must consider the possibility that $f(C_\gamma)$ has multiple points on γ . However, this case offers no particular difficulty. Thus we assume that the proof of Theorem 11 is complete.

The following theorem is another generalization of Theorem 10.

Theorem 12. Let f be holomorphic and non-constant in $\{|z| < 1\}$. Suppose further that for any arc $\gamma \subset \{|z| = 1\}$ there is a cross-cut K_γ of $\{|z| < 1\}$ with end-points on γ and a simple analytic arc C_γ on the w -sphere with $\infty \in C_\gamma$ such that $f(K_\gamma) \cap C_\gamma = \emptyset$. Then $f \in \mathcal{L}$.

Proof. Suppose $f \notin \mathcal{L}$ and let $\{\gamma_n\}$ be a sequence of arcs in $\{|z| < 1\}$ tending to an arc $\gamma \subset \{|z| = 1\}$ such that $|f| = \lambda$ on γ_n , $n \geq 1$. Let K_γ be a cross-cut of $\{|z| < 1\}$ joining interior points of γ and let D_γ be the subdomain of $\{|z| < 1\}$ bounded by K_γ and the subarc of γ determined by the end-points of K_γ . (We may assume the end-points of K_γ are interior points of γ since γ is arbitrary in the hypothesis.)

Suppose first that $f(D_Y) \cap C_Y = \emptyset$. In particular, it follows that f omits two finite values in D_Y . Let ψ be a one-one conformal mapping of D_Y onto $\{|w| < 1\}$. Then $\psi(\gamma_n)$ is a sequence of arcs in $\{|w| < 1\}$ tending to an arc $\gamma' \subset \{|w| = 1\}$ and $|f(\psi^{-1})| = \lambda$ on γ_n . But $f(\psi^{-1})$ omits two finite values in $\{|w| < 1\}$ and this implies $f(\psi^{-1}) \in \mathfrak{L}$ in $\{|w| < 1\}$ by results of MacLane [7, p.43]. This contradiction establishes the first case.

If $f(D_Y) \cap C_Y \neq \emptyset$, let $w \in f(D_Y) \cap C_Y$. Let $\mathfrak{g}$ be the Riemann Surface over the w -plane onto which f maps $\{|z| < 1\}$, then lift w into a point $P_w \in \mathfrak{g}$ which corresponds to a point of D_Y under f . Since $\mathfrak{g}$ has no points over ∞ (f is holomorphic) we can obtain an asymptotic value of f by lifting the piece of C_Y between w and ∞ onto $\mathfrak{g}$ starting at P_w . On the curve $\Gamma \subset D_Y$ corresponding to this maximal lifting of C_Y , we have $|z| \rightarrow 1$, since $f(K_Y) \cap C_Y = \emptyset$.

Thus Γ must intersect infinitely many of the arcs γ_n on which $|f|=\lambda$. However, $f(\Gamma) \subset C_\gamma$ and therefore C_γ intersects $\{|w|=\lambda\}$ in infinitely many points. This contradicts the fact that C_γ is analytic. Therefore, we have proved Theorem 12.

We will have a special case of Theorem 12 after some preliminary observations.

Let u be a real-valued harmonic function in $\{|z|<1\}$. For any real λ , let $L(\lambda) = \{z \mid u(z)=\lambda\}$ denote the level set of u for the level λ .

Definition. $u \in \mathcal{L}_H$ if and only if u is a non-constant, real-valued harmonic function in $\{|z|<1\}$ and all level sets $L(\lambda)$ of u end at points of $\{|z|=1\}$.

Let $f = u + iv$ be a holomorphic function in $\{|z|<1\}$ where $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$. Then we have the following lemma.

Lemma 3. Let $f \in \mathcal{L}$ and have no arc tracts. Then $u \in \mathcal{L}_H$ and $v \in \mathcal{L}_H$.

Proof. To prove that $u \in \mathcal{L}_H$, it suffices to show that $e^f \in \mathcal{L}$, since for any real number λ ,

$$(5.7) \quad \{z \mid u(z)=\lambda\} = \{z \mid |e^f|=e^\lambda\}.$$

Let γ be an arc of $\{|z|=1\}$. From the fact that $\mathcal{A} = \mathcal{B} = \mathcal{L}$ [7, Theorem 1] we have $f \in \mathcal{A}$ and hence $A \cap \gamma \neq \emptyset$ where A is the set of asymptotic points of f on $\{|z|=1\}$. Let $\zeta \in A \cap \overset{\circ}{\gamma}$, where $\overset{\circ}{\gamma}$ denotes the interior of γ . Using the fact that f has no arc tracts, choose a domain $D(\epsilon)$ belonging to the asymptotic tract for f at

ζ such that $\overline{D(\epsilon)} \cap \{|z|=1\} \subset \overset{\circ}{\gamma}$. The boundary of $D(\epsilon)$ in the interior of $\{|z|<1\}$ consists of components of either $\{|f|=\frac{1}{\epsilon}\}$ or $\{|f-a|=\epsilon\}$, depending respectively on whether $D(\epsilon)$ belongs to a tract for ∞ or a tract for a finite value a . In either case, since $f \in \mathcal{L}$ we can obtain a curve C (one of the boundary components of $D(\epsilon)$) tending to a point of $\overset{\circ}{\gamma}$ on which f (and hence e^f) is bounded. The arc γ was arbitrary, therefore it follows that $e^f \in \mathcal{B}$ and from the theorem quoted above ($\mathcal{A} = \mathcal{B} = \mathcal{L}$) we have $e^f \in \mathcal{L}$. Thus $u \in \mathcal{L}_H$.

We observe that v is the real part of $-if$. It is clear that $-if \in \mathcal{L}$ and has no arc tracts. Therefore, the above proof yields $v \in \mathcal{L}_H$. The proof of Lemma 3 is complete.

Remark. MacLane has shown [7,p.43] that if f is holomorphic, non-constant, and omits two finite values in $\{|z|<1\}$ then $f \in \mathcal{A}$ and f has no arc tracts. This fact will be useful to us in the following theorem.

Theorem 13. Let $f = u + iv$ be a holomorphic function in $\{|z|<1\}$. Then $u \in \mathcal{L}_H$ if and only if $v \in \mathcal{L}_H$.

Remark. It should be clear that $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

Proof. Suppose $u \notin \mathcal{L}_H$. We will show that this implies $v \notin \mathcal{L}_H$. Let $\{\gamma_n\}$ be a sequence of arcs in $\{|z|<1\}$ tending to an arc $\gamma \subset \{|z|=1\}$ such that $u(z)=\lambda$ for $z \in \gamma_n$, $n=1,2,\dots$. Let $e^{i\theta_1}$ and $e^{i\theta_2}$ be interior points of the arc γ , and let $e^{i\theta_0}$ be an interior point of the subarc of γ determined by $e^{i\theta_1}$ and $e^{i\theta_2}$.

Let N be a neighborhood of $e^{i\theta_0}$. Using the fact that

$u \notin \mathcal{L}_H$, it follows from Lemma 3 and the remark following the proof of Lemma 3 that f can omit at most one finite value in $N \cap \{|z| < 1\}$. Therefore, we can choose a sequence of points $\{z_n\} \subset \{|z| < 1\}$ such that $z_n \rightarrow e^{i\theta_0}$ and $f(z_n) = \lambda + \epsilon$, $\epsilon > 0$, $n \geq 1$. (Note that $\lambda + \epsilon$ is a real number.)

Let R_j be the radius to $e^{i\theta_j}$, $j=1,2$ and select $\eta > 0$ so that $e^{i(\theta_0+\eta)}$ and $e^{i(\theta_0-\eta)}$ are interior points of the arc of γ determined by $e^{i\theta_1}$ and $e^{i\theta_2}$. Let R_+ and R_- denote the radii to $e^{i(\theta_0+\eta)}$ and $e^{i(\theta_0-\eta)}$ respectively.

Choose N so that $n \geq N$ implies $z_n \in S_\eta$ where S_η is the sector bounded by R_+ , R_- and the arc $(e^{i(\theta_0-\eta)}, e^{i(\theta_0+\eta)})$ containing $e^{i\theta_0}$.

Denote by $\mathcal{S}$ the Riemann Surface over the w -plane onto which f maps $\{|z| < 1\}$. Let P_n denote the point of $\mathcal{S}$ over $w = \lambda + \epsilon$ which corresponds to z_n under f , $n \geq 1$. For

each $n \geq N$, lift the ray $\{u \geq \lambda + \epsilon\}$ into S starting at P_n . Since S has no points over ∞ , f has an asymptotic value on the curve $C_n \subset \{|z| < 1\}$ corresponding to the maximal lifting of $\{u \geq \lambda + \epsilon\}$ starting at P_n , $n \geq N$. It is clear from the construction that each C_n , $n \geq N$, begins at z_n and tends to some part of $\{|z| = 1\}$. Also, $C_n \cap \gamma_j = \emptyset$ for all n and j since $z \in C_n$ implies $u(z) \geq \lambda + \epsilon > \lambda$.

Let $S = \{re^{i\theta} | 0 < r < 1, \theta_1 < \theta < \theta_2\}$. Then for each $n \geq N_1$, for some positive integer N_1 , we can find arcs γ_p and γ_q belonging to the sequence $\{\gamma_n\}$ such that γ_p separates z_n from 0 in S and γ_q separates z_n from $\{|z| = 1\}$ in S . Then C_n lies in the subdomain of S bounded by γ_p , γ_q and segments of R_1 and R_2 . Therefore, given $\epsilon > 0$ we can find an N_2 such that $n \geq N_2$ implies $C_n \subset \{1 - \epsilon < |z| < 1\}$ since $\gamma_n \rightarrow \gamma$. It also follows that each C_n must furnish a cross-cut of either S_1 or S_2 where $S_1 = \{re^{i\theta} | \theta_1 < \theta < \theta_0 - \eta\}$ and $S_2 = \{re^{i\theta} | \theta_0 + \eta < \theta < \theta_2\}$.

Using the fact that $z \in C_n$ implies $v(z) = 0$, it is clear from the above facts that we can find a sequence $\{K_m\}$ of arcs in $\{|z| < 1\}$ tending to an arc $K \subset \{|z| = 1\}$ such that $v = 0$ on K_m (each K_m is a subarc of some C_n , $v = 0$ on C_n). Therefore, $v \notin \mathcal{L}_H$.

Applying the same argument to $g = if = v - iu$, we obtain $v \notin \mathcal{L}_H \Rightarrow u \notin \mathcal{L}_H$. This proves Theorem 13.

By altering the above proof slightly, we can prove the following theorem.

Theorem 14. Let u be real-valued, harmonic, and non-constant in $\{|z| < 1\}$. If there exists a real number λ such that $L(\lambda)$ ends at points of $\{|z| = 1\}$, then $u \in \mathcal{L}_H$.

Remark. Recall that in the definition of $\mathcal{L}_H$ we required that $L(\lambda) = \{z \mid u(z) = \lambda\}$ end at points of $\{|z| = 1\}$ for every real number λ .

Proof. The proof is carried out by supposing that $u \notin \mathcal{L}_H$ and showing that for every real number λ , $L(\lambda)$ fails to end at points of $\{|z| = 1\}$. We will simply note the necessary alterations to the proof of Theorem 13.

Let v be a harmonic function conjugate to u and consider the holomorphic function $f = u + iv$. As in the preceding proof, we let $\{\gamma_n\}$ be a sequence of arcs in $\{|z| < 1\}$ tending to an arc $\gamma \subset \{|z| = 1\}$ such that $u = \lambda$ on γ_n , $n \geq 1$. Let μ be any real number such that $\mu \neq \lambda$. The first change in the proof of Theorem 13 is to choose the sequence $\{z_n\}$ tending to $e^{i\theta_0}$ such that $f(z_n) = \mu + iv$ (a constant) for all $n \geq 1$. The only other change is that we lift the ray $\{w = \mu + iv \mid v \geq v\}$ instead of $u \geq \lambda + \epsilon$. Then we obtain a sequence $\{K_n\}$ of arcs in $\{|z| < 1\}$ tending to an arc $K \subset \{|z| = 1\}$ such that each K_n belongs to the level set $\{z \mid u(z) = \mu\}$. Theorem 14 is proved.

Corollary 14.1. Let f holomorphic and non-constant in $\{|z| < 1\}$. Suppose also that $f \neq 0$ in $\{|z| < 1\}$. If there exists a $\lambda > 0$ such that $L(\lambda) = \{z \mid |f(z)| = \lambda\}$

ends at points of $\{|z|=1\}$, then $f \in \mathfrak{L}$.

Proof. Let $u(z) = \log|f(z)|$ and apply Theorem 14.

This proves Corollary 14.1.

With the aid of the proof of Theorem 13, we can establish a sufficient condition for $e^f \in \mathfrak{L}$.

Denote by $\{u = \lambda\}$ and $\{v = \mu\}$ the lines $\{w = u + iv | u = \lambda\}$ and $\{w = u + iv | v = \mu\}$ respectively in the w -plane. We shall say that an arc C on the w -sphere has property (S) if

(5.8) C is a simple analytic arc on the w -sphere,

(5.9) $\infty \in C$,

and either

(5.10) $C \cap \{u=\lambda\}$ is finite for any real number λ , or

(5.11) $C \cap \{v=\mu\}$ is finite for any real number μ .

What we mean here is that C has property (S) if C satisfies either (5.8), (5.9) and (5.10) or (5.8), (5.9) and (5.11).

Theorem 15. Let f be holomorphic and non-constant in $\{|z|<1\}$. Suppose further that for any arc $\gamma \subset \{|z|=1\}$ there exists a cross-cut K_γ of $\{|z|<1\}$ with end points on γ and an arc C_γ on the w -sphere with property (S) such that $f(K_\gamma) \cap C_\gamma = \emptyset$. Then $e^f \in \mathfrak{L}$.

Remark. It is clear from Theorem 12 that $f \in \mathfrak{L}$. But it is not known to the author at this time whether $f \in \mathfrak{L} \Rightarrow e^f \in \mathfrak{L}$.

Proof. Suppose $e^f \notin \mathfrak{L}$. Then it follows from (5.8) that $u \notin \mathfrak{L}_H$ and from the proof of Theorem 13, we can find an arc $\gamma \subset \{|z|=1\}$ and sequences of arcs $\{\gamma_n^1\}$ and $\{\gamma_n^2\}$ in $\{|z|<1\}$ such that

$$(5.12) \quad u(z) = \lambda \quad \text{on} \quad \gamma_n, \quad n = 1, 2, \dots,$$

$$(5.13) \quad v(z) = \mu \quad \text{on} \quad \gamma_n, \quad n = 1, 2, \dots, \text{ and}$$

$$(5.14) \quad \gamma_n^i \rightarrow \gamma \quad \text{as} \quad n \rightarrow \infty, \quad i = 1, 2.$$

We have set $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

It follows from the hypothesis that we can find a cross-cut K joining interior points of γ and an arc C in the w -plane with property (S) such that $f(K) \cap C = \emptyset$. Let D_γ be the subdomain of $\{|z| < 1\}$ bounded by K and a subarc of γ .

If $f(D_\gamma) \cap C = \emptyset$, then from the remark following the proof of Lemma 3 and Lemma 3 itself we have a contradiction. Therefore, let $w \in f(D_\gamma) \cap C$ and denote by $\mathcal{S}$ the Riemann Surface over the w -plane which is the image of $\{|z| < 1\}$ under f . Starting at a point $P_w(\in \mathcal{S})$ over w which corresponds to a point of D_γ under f , we can lift a maximal portion of C between w and ∞ obtaining an asymptotic value for f ($\mathcal{S}$ has no points over ∞ since f is holomorphic). Let Γ denote the curve in D_γ which corresponds to this lifted portion of C . Then Γ runs to $\{|z|=1\}$ in D_γ ($f(K) \cap C = \emptyset$) and intersects infinitely many arcs $\{\gamma_n^1\}$ and $\{\gamma_n^2\}$. From (5.12) and (5.13), this implies $C \cap \{u=\lambda\}$ is infinite and $C \cap \{v=\mu\}$ is infinite since $f(\Gamma) \subset C$. This contradiction of the fact that C has property (S) establishes Theorem 15.

6. Asymptotic Classes for Real Harmonic Functions.

In the preceding section, we proved some simple statements about level sets of real harmonic functions. This section is simply a collection of facts about the asymptotic behavior of real harmonic functions which follow immediately from MacLane's work [7].

Let u be a real-valued harmonic function in $\{|z| < 1\}$. Recall that the level set $L(\lambda)$ is $\{z | u(z) = \lambda\}$ where λ is any real number. A level curve $C(\lambda)$ is any component of $L(\lambda)$.

We define the classes $\mathcal{A}_H$, $\mathcal{B}_H$, $\mathcal{L}_H$ and $\mathcal{L}_H^*$ for real-valued harmonic functions by simply replacing the word "holomorphic" by "real harmonic" in MacLane's definitions (see Section 2). Observe that u has the asymptotic value ∞ on a curve $\Gamma \subset \{|z| < 1\}$ if $|u| \rightarrow \infty$ on Γ ; that is, either $u \rightarrow +\infty$ or $u \rightarrow -\infty$ on Γ .

In the following discussion, we assume that $f = u + iv$ is a non-constant holomorphic function in $\{|z| < 1\}$ with $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$.

Lemma 4. (a) $u \in \mathcal{L}_H$ if and only if $e^f \in \mathcal{L}$.
 (b) $u \in \mathcal{L}_H^*$ if and only if $e^f \in \mathcal{L}^*$.

Remark. Part (a) of Lemma 4 has been used in section 5. The proof is obvious from relation (5.7), so we omit it.

Theorem 16. $\mathcal{A}_H = \mathcal{B}_H = \mathcal{L}_H \subset \mathcal{L}_H^*$ and the inclusion is proper.

Proof. It is not surprising that we rely heavily on the fact that $\mathcal{A} = \mathcal{B} = \mathcal{L}$ for holomorphic functions [7, Theorem 1].

The relation $\mathcal{A}_H \subset \mathcal{B}_H$ is clear from the definition of the classes.

To prove that $\mathcal{B}_H \subset \mathcal{L}_H$, let $u \in \mathcal{B}_H$. Then since $|e^f| = e^u$, it follows that $e^f \in \mathcal{B}$. Therefore $e^f \in \mathcal{L}$ by MacLane's theorem quoted above. Thus, by Lemma 4, $u \in \mathcal{L}_H$.

Finally, if $u \in \mathcal{L}_H$ then $e^f \in \mathcal{L}$ by Lemma 4, and again by MacLane's theorem, $e^f \in \mathcal{A}$. If e^f has the asymptotic value 0 or ∞ , then u has the asymptotic value $-\infty$ or ∞ respectively on the same path. If e^f has the finite non-zero asymptotic value a , then u has the asymptotic value $\log|a|$ on the same path. Thus, $u \in \mathcal{A}_H$ and we have proved that $\mathcal{L}_H \subset \mathcal{A}_H$.

Combining the above facts, we have shown that $\mathcal{A}_H \subset \mathcal{B}_H \subset \mathcal{L}_H \subset \mathcal{A}_H$ and thus $\mathcal{A}_H = \mathcal{B}_H = \mathcal{L}_H$. Example 4 in section 8 shows that the inclusion $\mathcal{L}_H \subset \mathcal{L}_H^*$ is proper. This concludes the proof of Theorem 16.

Theorem 17. Let u be a real-valued, non-constant, harmonic function in $\{|z| < 1\}$. Suppose further that

$$(6.1) \quad \int_0^1 (1-r)u^+(re^{i\theta})dr < \infty, \quad (\theta \in \Theta),$$

where Θ is dense in $[0, 2\pi]$. Then $u \in \mathcal{A}_H$.

Proof. Let $F = e^f$. Then $\log|F| = u$, and $\log^+|F| = u^+$.

From (6.1), we have

$$(6.2) \quad \int_0^1 (1-r)\log^+|F(re^{i\theta})|dr < \infty, \quad (\theta \in \Theta),$$

where Θ is dense in $[0, 2\pi]$. Therefore, it follows from a theorem of MacLane [7, Theorem 14] that $F \in \mathcal{A}$. Therefore $F \in \mathcal{L}$ [7, Theorem 1] and from Lemma 4, $u \in \mathcal{L}_H$. By Theorem 16, $u \in \mathcal{A}_H$ and Theorem 17 is proved.

We conclude this section by noting that we can use the above results to replace $\log^+ |f|$ by $\arg^+ f$ in MacLane's condition (I) [7, p. 35] if $f \neq 0$.

Theorem 18. Let f be holomorphic and non-constant in $\{|z| < 1\}$. Suppose further that $f \neq 0$ in $\{|z| < 1\}$ and

$$(6.3) \quad \int_0^1 (1-r) \arg^+ f(re^{i\theta}) dr < \infty, \quad (\theta \in \Theta),$$

where Θ is dense in $[0, 2\pi]$. Then $f \in \mathcal{A}$.

Remark. We naturally assume that we have selected some determination for $\arg f$ in $\{|z| < 1\}$. It makes no difference which determination we choose, since any two determinations simply differ by a constant and the convergence of the integral in (6.3) is unaffected by the addition of a constant to $\arg f$.

Proof. Since $f \neq 0$, $\arg f$ is a real harmonic function in $\{|z| < 1\}$. Using (6.3) it follows from Theorem 17 that $\arg f \in \mathcal{A}_H$. Then, from Theorem 16, we have $\arg f \in \mathcal{L}_H$. From Theorem 13 in Section 5, it follows that $\log |f| \in \mathcal{L}_H$. Finally, $\log |f| \in \mathcal{L}_H$ implies $f \in \mathcal{L}$ by Lemma 4, and MacLane's theorem ($\mathcal{A} = \mathcal{B} = \mathcal{L}$) finishes the proof.

7. $\mathcal{A}$, $\mathcal{B}$, $\mathcal{L}$ and $\mathcal{L}^*$ for Continuous Functions.

Barth [3] defined classes $\mathcal{A}_m$, $\mathcal{B}_m$, $\mathcal{L}_m$ and $\mathcal{L}_m^*$ for meromorphic functions in $\{|z| < 1\}$ corresponding to MacLane's classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{L}$ and $\mathcal{L}^*$ for holomorphic functions. We can obtain Barth's definitions by replacing the word "holomorphic" by the word "meromorphic" in MacLane's definitions (see Section 2). Barth proved [3, Theorem 3] that $\mathcal{A}_m \subset \mathcal{B}_m$, $\mathcal{L}_m \subset \mathcal{B}_m$, and no other inclusion relations between $\mathcal{A}_m$, $\mathcal{B}_m$, and $\mathcal{L}_m$ are valid.

In the present section, we will define classes $\mathcal{A}_c$, $\mathcal{B}_c$, $\mathcal{L}_c$ and $\mathcal{L}_c^*$ for continuous functions whose domain is $\{|z| < 1\}$ and whose range is a subset of the plane $\{|w| < \infty\}$. We will proceed to show that $\mathcal{A}_c \subset \mathcal{B}_c$, $\mathcal{L}_c \subset \mathcal{B}_c$, $\mathcal{L}_c \subset \mathcal{L}_c^*$ and no other inclusion relations between $\mathcal{A}_c$, $\mathcal{B}_c$, $\mathcal{L}_c$ and $\mathcal{L}_c^*$ are valid. In fact, the examples we give will be homeomorphisms of $\{|z| < 1\}$ so that no more inclusion relations can be obtained by requiring that the continuous functions be homeomorphisms.

Let f be a continuous function in $\{|z| < 1\}$ whose range is a subset of $\{|w| < \infty\}$. The level set $L(\lambda)$ of f is $\{z \mid |f(z)| = \lambda\}$ where λ is a non-negative real number. The level curve $C(\lambda)$ is any component of $L(\lambda)$. Note that these level curves are not necessarily arcs.

We now define the classes $\mathcal{A}_c$, $\mathcal{B}_c$, $\mathcal{L}_c$ and $\mathcal{L}_c^*$ by replacing the word "holomorphic" by "continuous and finite-valued" in MacLane's definitions (see Section 2).

Theorem 19. $A_c \subset B_c$, $L_c \subset B_c$, $L_c \subset L_c^*$ and no other
inclusion relations between the classes A_c , B_c , L_c and L_c^*
are valid.

Remark. The following Venn diagram illustrates the
class relations given by Theorem 19.

Proof. The fact that $A_c \subset B_c$ is obvious from the
definition of the classes. Also, $L_c \subset L_c^*$ is an immediate
consequence of the definition.

The proof of $L_c \subset B_c$ closely follows the proof of the
corresponding fact for holomorphic functions given by MacLane
[7,p.14]. Only slight alterations are necessary in MacLane's
proof to make it work in this case.

In the remainder of the proof, we will indicate how to
construct homeomorphisms of $\{|z| < 1\}$ as examples to show
that we can have no further inclusion relations.

Example 1. The function

$$f(re^{i\theta}) = re^{i(\theta + \frac{1}{1-r})}, \quad 0 \leq \theta < 2\pi,$$

is a homeomorphism of $\{|z| < 1\}$ onto $\{|w| < 1\}$ and $f \in \mathcal{B}_c \cap \mathcal{L}_c - \mathcal{A}_c$. The fact that $f \in \mathcal{B}_c$ is clear since f is bounded. Observing that $|f(re^{i\theta})| = r$, we conclude that $L(\lambda) = \{z \mid |z| = \lambda\}$, $0 \leq \lambda < 1$. Thus it is clear that $f \in \mathcal{L}_c$. To show that $f \notin \mathcal{A}_c$, let Γ be an arc in $\{|z| < 1\}$ which tends to a point $e^{i\theta_0}$. Choose θ_1 and θ_2 so that θ_0 is contained in an arc (θ_1, θ_2) . Let R_1 be the radius to θ_1 and R_2 be the radius to θ_2 . Then Γ is eventually in the sector S bounded by R_1 , R_2 and the arc (θ_1, θ_2) which contains θ_0 . Since $f(R_1)$ and $f(R_2)$ are spirals in $\{|w| < 1\}$, it is clear that f can not have an asymptotic value on Γ . Thus f has no asymptotic points on $\{|z| = 1\}$ and $f \notin \mathcal{A}_c$.

FIGURE 1.

The remaining examples will be constructed to have domain a square S instead of the disc $\{|z| < 1\}$ since this facilitates the descriptions. It is clear in each example that a one-one conformal map ϕ of S onto $\{|\xi| < 1\}$ will give us the desired behavior in $\{|\xi| < 1\}$ for the function f of the example composed with ϕ^{-1} .

Example 2. In this example, we construct a homeomorphism $f \in \mathcal{A}_C \cap \mathcal{B}_C - \mathcal{L}_C^*$.

Let $S = \{z = x + iy | 0 < x < 1, -2 < y < 2\}$ have the sequence of cross-cuts $\{\gamma_n\}$ defined by $\gamma_0 = \{z = x + iy | x=0, -2 \leq y \leq 2\}$, $\gamma_n = \{z = x + iy | x=1 - \frac{1}{n}, -2 \leq y \leq 2\}$, $n \geq 1$. For each $n \geq 0$, let E_n be the subdomain of S bounded by γ_n , γ_{n+1} and segments from the upper and lower sides of S . Let Γ be the curve $\{z = x + iy | y = \sin \frac{2\pi}{1-x}\}$ in S .

FIGURE 2.

Let $S' = \{w \mid |w-1| < 1\}$ and let C be the arc $\{w = u+iv \mid |w|=1, v > 0, |w-1| < 1\}$ in S' . Note that the end-point of C on $\{|w-1|=1\}$ is the point $e^{\frac{\pi i}{3}}$ and let $\{\gamma'_n\}$ be a sequence of cross-cuts of S' defined by $\gamma'_n = \{w \mid |w - e^{\frac{\pi i}{3}}| = \frac{1}{n}, |w-1| \leq 1\}$, $n \geq 1$. For each $n \geq 1$, let F_n be the intersection of the annulus $\{\frac{1}{n+1} < |z - e^{\frac{\pi i}{3}}| < \frac{1}{n}\}$ with S' and let $F_0 = S' \cap \{|w - e^{\frac{\pi i}{3}}| > 1\}$.

Let f_0 be a homeomorphism of $\bar{E}_0$ onto $\bar{F}_0$ so that $f_0(\gamma_1) = \gamma'_1$, and $f_0(\Gamma \cap \gamma_1) = 1$. Let f_1 be a homeomorphism of $\bar{E}_1$ onto $\bar{F}_1$ such that $f_1(\Gamma \cap \bar{E}_1) = C_1 \cap \bar{F}_1$, $f_1(\gamma_2) = \gamma'_2$ and $f_1 = f_0$ on γ_1 . Inductively, we define f_n to be a homeomorphism of $\bar{E}_n$ onto $\bar{F}_n$ such that $f_n(\Gamma \cap \bar{E}_n) = C \cap \bar{F}_n$, $f_n(\gamma_{n+1}) = \gamma'_{n+1}$ and $f_n = f_{n-1}$ on γ_n .

Let f be the function defined by $f(z) = f_n(z)$ if $z \in \bar{E}_n$. It is clear from the construction that f is a homeomorphism of $\bar{S} - \{x=1\}$ onto $\bar{S}' - \{e^{\frac{\pi i}{3}}\}$. Since f is bounded, it follows immediately that $f \in \mathcal{B}_c$. If Γ' is any arc in S tending to a point of $\{z \mid x=1, -2 \leq y \leq 2\}$ then $f \rightarrow e^{\frac{\pi i}{3}}$ on Γ' by the construction. Therefore $f \in \mathcal{A}_c$, since f is continuous on the remaining boundary of S . However, $f \notin \mathcal{L}_c^*$ since $|f(z)|=1$ for $z \in \Gamma$ and Γ tends to the segment $\{z \mid x=1, -1 \leq y \leq 1\}$. Thus $f \in \mathcal{A}_c \cap \mathcal{B}_c - \mathcal{L}_c^*$.

Example 3. This example will be a homeomorphism $f \in (\mathcal{B}_c \cap \mathcal{L}_c^*) - (\mathcal{A}_c \cup \mathcal{L}_c)$.

Let $S = \{z \mid 0 < x < 1, 0 < y < 1\}$ and $\{\gamma_n\}$ be the sequence of cross-cuts of S defined by $\gamma_0 = \{z \mid x=0, 0 \leq y \leq 1\}$,

$\gamma_n = \{z | x = 1 - \frac{1}{n}, 0 \leq y \leq 1\}$, $n \geq 1$. Let E_n be the subdomain of S bounded by γ_n , γ_{n+1} and segments from the upper and lower sides of S .

Consider the "pipe-system" P suggested by Figure 3. This system is constructed so that it spirals about the origin shrinking rapidly enough to allow each of its horizontal sections (except the first one) to have two cross-cuts by $\{|w|=1\}$, and the entire system is contained in $\{|w|<2\}$. We label the cross-cuts $\{\gamma'_n\}$ of P by $\{|w|=1\}$

FIGURE 3.

so that γ'_n separates γ'_{n+1} from 0 in P . For each $n \geq 1$, let F_n be the subdomain of P bounded by γ'_n , γ'_{n+1} and two curves from the boundary of P , and let F_0 be the subdomain bounded by γ'_1 having 0 on its boundary.

Let f_0 be a homeomorphism of $\bar{E}_0$ onto $\bar{F}_0$ such that $f_0(\gamma_1) = \gamma'_1$. Inductively, if f_{n-1} has been defined, let f_n be a homeomorphism of $\bar{E}_n$ onto $\bar{F}_n$ such that $f_n(\gamma_{n+1}) = \gamma'_{n+1}$ and $f_n = f_{n-1}$ on γ_n .

Let the function f be defined by $f(z) = f_n(z)$ if $z \in \bar{E}_n$. It is clear that f is a homeomorphism of $\bar{S} - \{x=1\}$ onto P . The boundedness of f again implies $f \in \mathcal{B}_C$. Since each level curve of f is simply a cross-cut of S , it is clear that $f \in \mathcal{L}_C^*$. However, $f \notin \mathcal{L}_C$ since $|f(z)|=1$, for $z \in \gamma_n$, and γ_n tends to the segment $\{z|x=1, 0 \leq y \leq 1\}$. If Γ is a curve in S tending to a point of $\{z|x=1, 0 \leq y \leq 1\}$ then $f(\Gamma)$ spirals out the pipe system P and thus f can not have an asymptotic value on Γ . Therefore $f \notin \mathcal{A}_C$ and we have the conclusion $f \in (\mathcal{B}_C \cap \mathcal{L}_C^*) - (\mathcal{A}_C \cup \mathcal{L}_C)$.

Example 4. We modify Example 3 to obtain a homeomorphism $f \in \mathcal{L}_C^* - \mathcal{B}_C$.

We alter the pipe system P as indicated in Figure-4. The system spirals about the origin shrinking rapidly enough to permit each horizontal section to have a cross-cut by $\{|w|=1\}$, and the n^{th} loop circuits the point $w=n$.

FIGURE 4.

Constructing the homeomorphism f as in Example 3, we see that we have maintained the property $f \in \mathcal{L}_c^*$. But if Γ is any curve in S tending to a point of $\{z | x=1, 0 \leq y \leq 1\}$ then $f(\Gamma)$ spirals out the system P and thus f has no limit on Γ and f is unbounded on Γ . Therefore, $f \notin \mathcal{B}_c$.

Example 5. Next, we construct an example of a homeomorphism $f \in (\mathcal{L}_c^* \cap \mathcal{A}_c) - \mathcal{L}_c$.

Again, consider the square S with cross-cuts $\{\gamma_n\}$. The "pipe system" P is considerably simpler this time, as indicated in Figure 5. The system oscillates about the curve $\{|w|=1\}$ so that any vertical section contains a

cross-cut by $\{|w|=1\}$, and shrinks rapidly enough to "tend to" $w=i$ in the sense that any curve C running through the pipe system must tend to $w=i$.

FIGURE 5.

Constructing our homeomorphism f inductively again so that $f(\gamma_n) = \gamma_n'$, $n = 1, 2, \dots$, we obtain $f \in \mathcal{A}_c$ since on any curve Γ tending to an interior point of $\{z|x=1, 0 \leq y \leq 1\}$, $f \rightarrow i$. Also, $f \in \mathcal{L}_c^*$ since any level curve is simply a cross-cut of S . But $f \notin \mathcal{L}_c$ since $|f(z)| = 1$ on γ_n and γ_n tends to the segment $\{z|x=1, 0 \leq y \leq 1\}$.

Example 6. Finally, we observe that we can combine the constructions given in Examples 2 and 3 to obtain a homeomorphism $f \in \mathcal{B}_c - (\mathcal{A}_c \cup \mathcal{L}_c^*)$.

Let $S = \{z|0 < x < 2, -2 < y < 2\}$, and let the sequence $\{\gamma_n\}$ be defined by $\gamma_n = \{z|x = 2 - \frac{1}{n}, -2 \leq y \leq 2\}$, $n \geq 1$.

Let Γ be the curve $\{z | y = \sin \frac{2\pi}{x}, 0 < x \leq 1\}$ in S .

FIGURE 6.

Let P be the "pipe system with bulb" indicated by Figure 6. Let $F_0 = \{w \mid |w-1| < \frac{1}{5}\}$, and let C be the arc $\{w = u + iv \mid v > 0, |w|=1, |w-1| \leq \frac{1}{5}\}$ in F_0 . Let f_0 be a homeomorphism of E_0 onto F_0 such that $f_0(\Gamma) = C$ and $f_0(\gamma_1) = \gamma_1'$ constructed as in Example 2. Let f_1 be a homeomorphism of $S - \bar{E}_0$ onto the remainder of the pipe

system such that $f_1(\gamma_n) = \gamma'_n$, $n \geq 1$, constructed as in Example 3. Now, using the fact that $f_0(\gamma_1) = \gamma'_1$ and $f_1(\gamma_1) = \gamma'_1$, alter one of the functions near γ_1 so that $f_0 = f_1$ on γ_1 . Now, define $f(z) = f_0(z)$ if $z \in E_0$ and $f(z) = f_1(z)$ if $z \in S - \overline{E}_0$. Making the requirement that the pipe system P be contained in $\{|w| < 2\}$, we have $f \in \mathcal{B}_c$. But $f \notin \mathcal{A}_c$ since any curve Γ' tending to a point of $\{z | x=2, -2 \leq y \leq 2\}$ is mapped by f into a curve which spirals out the pipe system, and thus f can not have an asymptotic value on Γ' . Since $|f|=1$ on Γ , it also follows that $f \notin \mathcal{L}_c^*$. This concludes Example 6.

It is clear that $\mathcal{A}_c \cap \mathcal{B}_c \cap \mathcal{L}_c \neq \emptyset$ since $\mathcal{A} = \mathcal{B} = \mathcal{L}$ [7, Theorem 1] for holomorphic functions.

Comparing the above examples with the Venn diagram following Theorem 19, we see that we have demonstrated that no further inclusion relations are possible in the classes $\mathcal{A}_c$, $\mathcal{B}_c$, $\mathcal{L}_c$ and $\mathcal{L}_c^*$.

8. Examples

Example 1. The modular function. Let w map $\{|z| < 1\}$ onto the universal covering surface S of the w -sphere with the three points $\{0, 1, \infty\}$ removed. It is known that $\log M(r) = O((1-r)^{-1})$ [12, p.154]. It is also known [7, Example 1] that w has only a countable set of asymptotic points on $\{|z|=1\}$. This example shows that the set A may be nothing more than countable even though $w \neq 0, 1$ and $\log M(r) = O((1-r)^{-1})$.

Example 2. MacLane has shown [7, Example 3] that there exists a function $f \in \mathcal{A}$ such that

$$(8.1) \quad \min_{z \in J_n} |f(z)| \rightarrow \infty$$

where $\{J_n\}$ is an expanding sequence of Jordan curves tending to $\{|z|=1\}$, and such that

$$(8.2) \quad M(r) < \frac{1}{(1-r)^{\frac{1}{2}}}, \quad (0 < r < 1).$$

It follows from (8.1) and a theorem of MacLane [7, Theorem 3] that $A_\infty = \{|z|=1\}$. It is obvious from (8.1) that f can have no finite asymptotic values. Since $M(r) = O((1-r)^{-1})$, this example shows that a function f satisfying the hypotheses of Theorem 3 may fail to have any finite asymptotic values (that is, $A^* = \emptyset$).

Example 3. Let Γ be the curve $\{re^{i\theta} \mid \theta = \sin \frac{2\pi}{1-r}\}$ in $\{|z| < 1\}$. Let

$$s_n = \{z \mid x = \frac{4n-3}{4n+1} \cos 1, \quad y \geq \frac{4n-3}{4n+1} \sin 1\} \cap \{|z| < 1\}, \quad n \geq 1,$$

and $t_n = \{z \mid x = \frac{4n-1}{4n+3} \cos 1, \quad y \leq -\frac{4n-1}{4n+3} \sin 1\} \cap \{|z| < 1\}, \quad n \geq 1.$

Let $T = \Gamma \cup \left(\bigcup_{n=1}^{\infty} s_n \right) \cup \left(\bigcup_{n=1}^{\infty} t_n \right) \cup \{z \mid y = 0, -1 < x \leq 0\}.$

Let $\varphi(z)$ be a continuous function in $\{|z| < 1\}$ such that

$$\varphi(z) = 0 \quad \text{for } z \in \Gamma \cup \left(\bigcup_{n=1}^{\infty} s_n \right) \cup \left(\bigcup_{n=1}^{\infty} t_n \right), \text{ and}$$

$$\varphi(z) = 1 \quad \text{for } z \in \{z \mid y = 0, -1 < x \leq -\frac{1}{2}\}.$$

It follows from the tress argument of Bagemihl and Seidel [2, Theorem 1] applied to the "modified tress" T that there exists a holomorphic f in $\{|z| < 1\}$ such that

$$|f(z) - \varphi(z)| \rightarrow 0 \text{ as } |z| \rightarrow 1 \text{ (} z \in T \text{)} .$$

It is clear that f is non-constant since $\varphi = 0$ on Γ and $\varphi = 1$ on $\{z \mid y = 0, -1 < x \leq -\frac{1}{2}\}$. Let C_1 be the circle containing the three points $\{0, e^i, e^{-i}\}$, and let D_1 be the disc bounded by C_1 . Let $w(z)$ be a conformal map of $D_1 \cap \{|z| < 1\}$ onto $\{|w| < 1\}$ and let $z(w)$ be the inverse map. Then the function $f(z(w))$ has an arc tract for the value 0 and has no asymptotic value ∞ . This example shows that the hypothesis of Theorem 6 can not be weakened to " f has an arc tract for a finite value." The above example was suggested to the author by J. E. McMillan.

Example 4. In this example, we construct a real harmonic function $u \in \mathcal{L}_H^* - \mathcal{L}_H$.

Let Γ_1 be the spiral $\{re^{i\theta} \mid \theta = \frac{1}{1-r}\}$, and Γ_2 the spiral $\{re^{i\theta} \mid \theta = \pi + \frac{1}{1-r}\}$. Let φ be a continuous function in $\{|z| < 1\}$ such that

$$\begin{aligned} \varphi(z) &= 0 \text{ for } z \in \Gamma_1, \frac{1}{2} < |z| < 1, \text{ and} \\ \operatorname{Re} \varphi(z) &\rightarrow +\infty \text{ as } |z| \rightarrow 1, z \in \Gamma_2 . \end{aligned}$$

If $T = \Gamma_1 \cup \Gamma_2$, then it follows from a theorem of Bagemihl and Seidel [2, Theorem 1] that there exists a holomorphic f in $\{|z| < 1\}$ such that

$$|\operatorname{Re} f(z) - \operatorname{Re} \varphi(z)| \rightarrow 0 \text{ as } |z| \rightarrow 1, z \in T .$$

Let $u = \operatorname{Re} f$. Then it is clear that $u \rightarrow 0$ on Γ_1 and $u \rightarrow +\infty$ on Γ_2 .

Let $S = \{re^{i\theta} \mid -\frac{\pi}{2} < \theta < 0\}$, $R_1 = \{re^{i\theta} \mid \theta = 0, r < 1\}$ and $R_2 = \{re^{i\theta} \mid \theta = -\frac{\pi}{2}, r < 1\}$. It is clear from the construction that there are no level curves of u in S which can tend to an arc or segment of the boundary of S . Now choose $r > 0$ such that $u(z) > 1$ if $r < |z| < 1$, $z \in \Gamma_2$, and $u(z) < 1$ for $r < |z| < 1$, $z \in \Gamma_1$. Then every subdomain of S contained in $\{r < |z| < 1\}$ and bounded by an arc of Γ_1 , an arc of Γ_2 and segments of the radii R_1 and R_2 contains a cross-cut by the level set $L(1) = \{z \mid u(z) = 1\}$ joining a point of R_1 to a point of R_2 . Thus we have a sequence of arcs $\{\gamma_n\}$

tending to the arc $\gamma = \{e^{i\theta} \mid -\frac{\pi}{2} \leq \theta \leq 0\}$ such that $u(z) = 1$ for $z \in \gamma_n$, $n \geq 1$. Let $w(z)$ map S onto $\{|w| < 1\}$ and let $z(w)$ be the inverse map. Then $u(z(w))$ is a real harmonic function in $\{|w| < 1\}$ and $u(z(w)) \in \mathcal{L}_H^* - \mathcal{L}_H$.

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