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by

Wayne Timothy Ford

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Thesis Director's Signature

A handwritten signature in cursive script, appearing to read "James Douglas, Jr.", is written over a horizontal line.

Houston, Texas

May, 1964

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Wayne T. Ford

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1. Introduction - Statement of the Problem.

Pucci [10] considers the integro-differential equation

$$u(x, y) = g(x, y) + \int_0^y K(y, \eta) \frac{\partial}{\partial x} u(x, \eta) d\eta \quad (1.1)$$

where g and K are prescribed functions such that $\frac{\partial^n}{\partial x^n} g$, $n=0, 1, 2, \dots$, is continuous in the half-strip

$$A_p = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y\}, \quad (1.2)$$

and K is continuous for $y > \eta \geq 0$. He also assumes the existence of constants M and α such that

$$|K(y, \eta)| \leq M (x - \eta)^{\alpha-1}, \quad \alpha > 0 \quad (1.3)$$

and that for each value of c ,

$$\lim_{n \rightarrow \infty} \frac{\frac{\partial^n g(x, y)}{\partial x^n} \frac{c^n}{\Gamma(n\alpha+1)}}{1} = 0 \quad (1.4)$$

uniformly in bounded subsets of A_p .

If we restricted our attention to $y \leq l$, it would be sufficient to require (1.4) whenever $|c| \leq C_0$, C_0 being a constant depending on M , α , and l .

Denoting the subclass of $C_x^\infty[A_p]$ satisfying (1.4) by $\mathcal{H}_\alpha$, and assuming $g \in \mathcal{H}_\alpha$, Pucci [10] shows the existence and uniqueness of $u \in \mathcal{H}_\alpha$ satisfying (1.1)

The solution has the representation

$$u(x, y) = \sum_{n=0}^{\infty} \mathcal{K}^n \frac{\partial^n}{\partial x^n} g(x, y) \quad (1.5)$$

where $(\mathcal{K}\varphi)(y) = \int_0^y K(y, \eta) \varphi(\eta) d\eta . \quad (1.6)$

Pucci indicates a numerical method of obtaining $u(x, y)$, but he suggests that the method studied by Douglas [1] "could give better results". Following this suggestion, we shall construct a numerical scheme using mathematical programming methods based on inequalities derived from (1.4).

However, this will be done in terms of the more general setting

$$u(x, y) = g(x, y) + \int_0^y K(y, \eta) (Au)(x, \eta) d\eta \quad (1.7)$$

where A is a linear operator depending only on the first variable, x . That is, A is assumed to depend on the function $U_\eta(x) = u(x, \eta)$ where η enters simply as a parameter to select the function U_η of the single variable x .

2. Kernels and Integrals.

$$\text{Let } K(y, \eta) = G(y, \eta)(y - \eta)^{\alpha-1}, \quad \alpha > 0, \quad (2.1)$$

where we assume

$$0 \leq G(y, \eta) \leq M, \quad 0 \leq \eta \leq y \leq t; \quad (2.2)$$

$G(y, \eta)$ is continuous on $0 \leq \eta \leq y \leq t$; and

$$|G(y + \Delta y, \eta) - G(y, \eta)| \leq L \Delta y, \quad 0 \leq y < y + \Delta y \leq t. \quad (2.3)$$

$$\text{Let } (\mathcal{K}\varphi)(y) = \int_0^y K(y, \eta)\varphi(\eta)d\eta, \quad (2.4)$$

where $\varphi(y)$ is assumed continuous on $[0, t]$.

$$\text{Let } |\varphi|_0 = \sup_{0 \leq y \leq t} |\varphi(y)|$$

$$\text{and } |\varphi|_{0,y} = \sup_{0 \leq \eta \leq y} |\varphi(\eta)|.$$

$$\text{Let } (\mathcal{K}^0\varphi)(y) = \varphi(y)$$

$$\text{and } (\mathcal{K}^{n+1}\varphi)(y) = (\mathcal{K}\mathcal{K}^n\varphi)(y).$$

$$\text{Lemma 2.1: } |(\mathcal{K}\varphi)(y + \Delta y) - (\mathcal{K}\varphi)y| \leq C \max(\Delta y, \overline{\Delta y}^\alpha) |\varphi|_0,$$

where C depends on α , M , L , and t .

$$\text{Proof: } (\mathcal{K}\varphi)(y + \Delta y) - (\mathcal{K}\varphi)(y)$$

$$= \int_0^{y+\Delta y} K(y+\Delta y, \eta)\varphi(\eta)d\eta - \int_0^y K(y, \eta)\varphi(\eta)d\eta$$

$$= I_1 + I_2 + I_3 \quad \text{where}$$

$$I_1 = \int_y^{y+\Delta y} (y+\Delta y - \eta)^{\alpha-1} G(y+\Delta y, \eta)\varphi(\eta)d\eta,$$

$$I_2 = \int_0^y \{G(y+\Delta y, \eta) - G(y, \eta)\} (y-\eta)^{\alpha-1} \varphi(\eta) d\eta, \text{ and}$$

$$I_3 = \int_0^y \{(y+\Delta y-\eta)^{\alpha-1} - (y-\eta)^{\alpha-1}\} G(y+\Delta y, \eta) \varphi(\eta) d\eta.$$

$$\text{Now, } |I_1| \leq M |\varphi|_0 \int_y^{y+\Delta y} (y+\Delta y-\eta)^{\alpha-1} d\eta = M |\varphi|_0 \overline{\Delta y}^\alpha / \alpha,$$

$$\text{and } |I_2| \leq L |\varphi|_0 \Delta y \int_0^y (y-\eta)^{\alpha-1} d\eta = L \Delta y |\varphi|_0 y^\alpha / \alpha.$$

Next, since $(y+\Delta y-\eta)^{\alpha-1} - (y-\eta)^{\alpha-1}$ has the same sign as that of $\alpha-1$,

$$\begin{aligned} |I_3| &\leq M |\varphi|_0 \left| \int_0^y \{(y+\Delta y-\eta)^{\alpha-1} - (y-\eta)^{\alpha-1}\} d\eta \right| \\ &= \alpha^{-1} M |\varphi|_0 \left| -\overline{\Delta y}^\alpha + (y+\Delta y)^\alpha - y^\alpha \right| \\ &\leq M |\varphi|_0 \{ \overline{\Delta y}^\alpha + |(y+\Delta y)^\alpha - y^\alpha| \} / \alpha. \end{aligned}$$

Now,

$$|(y+\Delta y)^\alpha - y^\alpha| \leq \begin{cases} \alpha y_0^{\alpha-1} \Delta y \leq \alpha t^{\alpha-1} \Delta y, & \alpha \geq 1, \\ \alpha y_1^{\alpha-1} \Delta y \leq \alpha \overline{\Delta y}^\alpha, & \alpha < 1, \quad y \geq \Delta y, \\ (y+\Delta y)^\alpha + y^\alpha \leq (2^\alpha+1) \overline{\Delta y}^\alpha, & \alpha < 1, y < \Delta y, \end{cases}$$

where y_0 and y_1 are in the interval $[y, y+\Delta y]$.

$$\text{Thus, } |I_3| \leq \begin{cases} (\frac{\overline{\Delta y}^\alpha}{\alpha} + t^{\alpha-1} \Delta y) M |\varphi|_0, & \alpha \geq 1, \\ (\overline{\Delta y}^\alpha + (2^\alpha+1) \overline{\Delta y}^\alpha) \frac{M |\varphi|_0}{\alpha}, & \alpha < 1. \end{cases}$$

If
$$C = \begin{cases} (2M + L\ell^\alpha + \alpha M\ell^{\alpha-1})/\alpha, & \alpha \geq 1, \\ (5M + L\ell^\alpha)/\alpha, & \alpha < 1, \end{cases} \quad (2.5)$$

then,

$$|(\mathcal{K}\varphi)(y+\Delta y) - (\mathcal{K}\varphi)(y)| \leq C \max(\Delta y, \overline{\Delta y}^\alpha) |\varphi|_0,$$

and the desired result has been obtained.

Next we develop a kernel $K^{(n)}(y, \eta)$ for the expression of $\mathcal{K}^n$ as an integral operator rather than as an iterated integral.

Let $K^{(1)}(y, \eta) = K(y, \eta)$ and, for $n > 1$,

$$K^{(n)}(y, \eta) = \int_{\eta}^y K^{(q)}(y, \xi) K^{(n-q)}(\xi, \eta) d\xi, \quad 1 \leq q \leq n.$$

It is well-known (see Volterra, [14]) that the above definition is independent of the particular choice of q . Also, it is known that

$$(\mathcal{K}^n \varphi)(y) = \int_0^y K^{(n)}(y, \eta) \varphi(\eta) d\eta. \quad (2.6)$$

Next, we obtain

Lemma 2.2:
$$K^{(n)}(y, \eta) \leq \frac{M^n (y-\eta)^{n\alpha-1} \Gamma(\alpha)^n}{\Gamma(n\alpha)}.$$

Thus,
$$|\mathcal{K}^n \varphi|_{0,y} \leq \frac{M^n y^{n\alpha} \Gamma(\alpha)^n}{\Gamma(n\alpha+1)} |\varphi|_{0,y}.$$

Proof: $K^{(1)}(y, \eta) \leq M(y-\eta)^{\alpha-1}$ by (2.1) and (2.2). Assume the results for a particular $n \geq 1$. Then

$$K^{(n+1)}(y, \eta) = \int_{\eta}^y K^{(1)}(y, \xi) K^{(n)}(\xi, \eta) d\xi$$

$$\begin{aligned}
&\leq \int_{\eta}^y M(y-\xi)^{\alpha-1} \cdot \frac{M^n(\xi-\eta)^{n\alpha-1} \Gamma(\alpha)^n}{\Gamma(n\alpha)} d\xi \\
&= \frac{M^{n+1} \Gamma(\alpha)^n}{\Gamma(n\alpha)} \int_{\eta}^y (y-\xi)^{\alpha-1} (\xi-\eta)^{n\alpha-1} d\xi .
\end{aligned}$$

Since (see [15], p. 254)

$$\int_{\eta}^y (y-\xi)^{\alpha-1} (\xi-\eta)^{n\alpha-1} d\xi = \frac{(y-\eta)^{n\alpha+\alpha-1} \Gamma(n\alpha) \Gamma(\alpha)}{\Gamma(n\alpha+\alpha)} ,$$

we have

$$K^{(n+1)}(y, \eta) \leq \frac{[M \Gamma(\alpha)]^{n+1}}{\Gamma(n\alpha+\alpha)} (y-\eta)^{n\alpha+\alpha-1} .$$

Thus, the first part of the lemma has been inductively verified.

Finally, we have

$$\begin{aligned}
|\mathcal{K}^n \varphi|_{0,y} &\leq |\varphi|_{0,y} \int_0^y K^{(n)}(y, \eta) d\eta \\
&\leq |\varphi|_{0,y} \int_0^y \frac{M^n \Gamma(\alpha)^n (y-\eta)^{n\alpha-1}}{\Gamma(n\alpha)} d\eta \\
&= \frac{M^n \Gamma(\alpha)^n y^{n\alpha}}{\Gamma(n\alpha+1)} |\varphi|_{0,y} .
\end{aligned}$$

3. Gevrey Class Topologies.

Let $I = [0,1]$. Let $C^m(I)$ denote the continuous real valued functions on the closed interval I having m continuous derivatives on I . For $\varphi \in C^m(I)$, define the linear differential operator with constant coefficients,

$$A\varphi = \sum_{k=0}^m a_{k,1} D^k \varphi(x) . \quad (3.1)$$

We use D^k to denote k -fold partial differentiation with respect to x . When $\varphi = \varphi(x,y)$, the operator A will be independent of y .

For $\varphi \in C^{mn}(I)$, we have

$$A^n \varphi = \sum_{k=0}^{mn} a_{k,n} D^k \varphi(x) , \quad (3.2)$$

where, for example, $a_{02} = a_{01}^2$, $a_{12} = 2a_{01}a_{11}$, $a_{22} = 2a_{01}a_{21} + a_{11}^2$, etc.

$$\text{Let } a_n = \sum_{k=0}^{mn} |a_{k,n}| , \quad (3.3)$$

$$a = a_1 , \text{ and } b = |a_{m,1}| .$$

$$\text{Then, } a^n \geq a_n \geq b^n . \quad (3.4)$$

Next, define

$$\|\varphi\|_0 = \sup_{x \in I} |\varphi(x)| , \quad \varphi \in C(I) , \quad (3.5)$$

$$\text{and } \|\varphi\|_n = \max_{0 \leq k \leq mn} \|D^k \varphi\|_0 , \quad \varphi \in C^{mn}(I) . \quad (3.6)$$

We observe that

$$\begin{aligned}
 \|A^n \varphi\|_0 &\leq \sum_{k=0}^{mn} |a_{k,n}| \cdot \|D^k \varphi\|_0 \\
 &\leq \left(\sum_{k=0}^{mn} |a_{k,n}| \right) \|\varphi\|_n = a_n \|\varphi\|_n \\
 &\leq \|\varphi\|_n a^n.
 \end{aligned} \tag{3.7}$$

Now, if $\varphi \in C^\infty(I)$ and if a constant C exists such that

$$\|\varphi\|_n \leq C t^n a^{-n} \Gamma(n\alpha+1), \quad n = 0, 1, \dots, \tag{3.8}$$

we will say that $\varphi \in \mathcal{H}_t$, and we set

$$\|\varphi\|_{\infty, t} = \inf \{C: (3.8) \text{ is valid} \}. \tag{3.9}$$

For $\varphi \in \mathcal{H}_t$, (3.7) may be written as

$$\|A^n \varphi\|_0 \leq \|\varphi\|_{\infty, t} t^n \Gamma(n\alpha+1) \tag{3.10}$$

which looks like the classical requirement that φ belong to a Gevrey class determined by the operator A . Of course, the class of functions satisfying (3.10) might be enlarged if the definition of $\|\cdot\|_{\infty, t}$ were less demanding. However, the choice of definition of $\|\cdot\|_{\infty, t}$ given by (3.9) is useful in assuring convergence of the numerical processes developed later.

We will write $\mathcal{Q}^n(I)$ to denote the normed linear space consisting of $C^{mn}(I)$ provided with the norm $\|\cdot\|_n$.

Also, let $\mathcal{H}$ represent the linear space defined by

$$\mathcal{H} = \bigcup_{t>0} \mathcal{H}_t. \tag{3.11}$$

Next, let $J = [0, t]$. If, for each $y \in J$, we have an associated function, $\phi_y(x) \in C(I)$, then we can construct a function $\phi(x, y)$ with domain $I \times J$ and range in R^1 by the relation

$$\phi(x, y) = \phi_y(x). \quad (3.12)$$

We define $\mathcal{A}^n(I, J)$ by the requirements that $\phi \in \mathcal{A}^n(I, J)$ if and only if

$$\left. \begin{aligned} 1. \quad \{\phi_y : y \in J\} &\subset \mathcal{A}^n(I); \\ 2. \quad \{\phi_y : y \in J\} &\text{ is bounded} \\ &\text{ in } \mathcal{A}^n(I); \\ 3. \quad D^s \phi &\text{ is continuous on } I \times J \\ &\text{ for } s = 0, 1, \dots, mn. \end{aligned} \right\} \quad (3.13)$$

Then, we set

$$\|\phi\|_0 = \sup_{(x, y) \in I \times J} |\phi(x, y)|. \quad (3.14)$$

Now, we continue to use (3.6) and (3.9). Thus, (3.7), (3.8), and (3.10) are valid. We let G_t denote the collection of functions, ϕ , satisfying (3.8) and define

$$G = \bigcup_{t>0} G_t. \quad (3.15)$$

Lemma 3.1: $\mathcal{A}^n(I, J)$ is a Banach space.

Proof: It is well-known that $\mathcal{A}^n(I)$ is complete. Suppose $\{\phi^k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathcal{A}^n(I, J)$.

Then, for each $y \in J$, $\{\phi_y^k\}_{k=1}^\infty$ is a Cauchy sequence in $\mathcal{A}^n(I)$. Since $\mathcal{A}^n(I)$ is complete, there exists $\phi_y \in \mathcal{A}^n(I)$ such that $\phi_y^k \rightarrow \phi_y$ as $k \rightarrow \infty$ in $\mathcal{A}^n(I)$. This defines

$\phi(x,y)$ by (3.12) as a real valued function with domain $I \times J$. It remains to show that $\phi(x,y)$ satisfies (3.13). The first part of (3.13) is clearly satisfied from the definition of ϕ_y as a limit point in $\mathcal{Q}^n(I)$.

For each $y \in J$, we have

$$\|\phi_y\|_n \leq \|\phi\|_n \leq \sup_k \|\phi^k\|_n,$$

which is bounded by the boundedness of Cauchy sequences, and the second part of (3.13) is satisfied.

Finally, as $\phi^k \in \mathcal{Q}^n(I,J)$, $\|\phi_y^k\|_n$ is a uniformly continuous function of y for each value of k . We need to show that $\|\phi_y^k\|_n \rightarrow \|\phi_y\|_n$ as $k \rightarrow \infty$ uniformly in y .

Select k_0 so that $k, j \geq k_0$ implies that $\|\phi^k - \phi^j\|_n < \frac{1}{2} \epsilon$. Then, for $y \in J$,

$$\begin{aligned} \|\phi_y - \phi_y^k\|_n &\leq \|\phi_y - \phi_y^j\|_n + \|\phi_y^j - \phi_y^k\|_n \\ &\leq \|\phi_y - \phi_y^j\|_n + \frac{1}{2} \epsilon. \end{aligned}$$

Fix $k \geq k_0$. For each $y \in J$, there exists $j \geq k_0$ (depending on y , of course) so that

$$\|\phi_y - \phi_y^j\|_n \leq \frac{1}{2} \epsilon.$$

Thus, $|\|\phi_y\|_n - \|\phi_y^k\|_n| \leq \|\phi_y - \phi_y^k\|_n \leq \epsilon$

for all $y \in J$, $k \geq k_0$; the third part of (3.13) is satisfied; and $\mathcal{Q}^n(I,J)$ is complete.

4. Integrals in Gevrey Classes.

Now, if $\phi \in C(I, J)$, we can apply the integral operator $\mathcal{K}$ to ϕ for each fixed $x \in I$ to obtain a function $\mathcal{K}\phi$ with domain $I \times J$ and range in $\mathbb{R}^1$.

We shall first study a sequence of interpolation operators on $C(I, J)$. Let

$$\begin{aligned} (\Pi_n \phi)(x, y) &= \frac{n}{\ell} \left\{ \phi(x, \frac{(i+1)\ell}{n}) (y - \frac{i\ell}{n}) \right. \\ &\quad \left. + (\frac{(i+1)\ell}{n} - y) \phi(x, \frac{i\ell}{n}) \right\}, \\ i\ell &\leq ny \leq (i+1)\ell. \end{aligned} \quad (4.1)$$

Then, we have

$$\begin{aligned} (\Pi_n \phi)(x, y) &= \frac{n}{\ell} \sum_{j=0}^{n-1} \psi_{n,j}(y) \phi(x, \frac{j\ell}{n}) \\ &= \frac{n}{\ell} \sum_{j=0}^{n-1} \psi_{n,j}(y) \phi_{j\ell/n}(x), \end{aligned} \quad (4.2)$$

where each $\psi_{n,j}$ is the continuous function on J given by

$$\psi_{n,j}(y) = \begin{cases} \frac{(j+1)\ell}{n} - y, & j\ell \leq ny < (j+1)\ell, \\ y - \frac{j\ell}{n}, & (j-1)\ell \leq ny < j\ell, \\ 0, & \text{otherwise.} \end{cases} \quad (4.3)$$

For y fixed, $(\Pi_n \phi)_y$ is a function of x . Moreover, if $\phi \in C^{mr}(I, J)$, then for this fixed y , $(\Pi_n \phi)_y$ is a linear combination of a finite number of functions in $C^{mr}(I)$. Thus, $(\Pi_n \phi)_y \in C^{mr}(I)$ and for $s \leq mr$,

$$\begin{aligned} D^s \Pi_n \phi &= \frac{n}{\ell} D^s \sum_{j=0}^{n-1} \psi_{n,j} \phi_{j\ell/n} \\ &= \frac{n}{\ell} \sum_{j=0}^{n-1} \psi_{n,j} D^s \phi_{j\ell/n} = \Pi_n D^s \phi. \end{aligned} \quad (4.4)$$

Further, we apply $\mathcal{K}$ to obtain

$$\begin{aligned} \mathcal{K} \Pi_n D^s \phi &= \mathcal{K} D^s \Pi_n \phi = \frac{n}{\ell} \sum_{j=0}^{n-1} \mathcal{K} \psi_{n,j} D^s \phi_{j\ell/n} \\ &= \frac{n}{\ell} D^s \sum_{j=0}^{n-1} \mathcal{K} \psi_{n,j} \cdot \phi_{j\ell/n} = D^s \mathcal{K} \Pi_n \phi. \end{aligned} \quad (4.5)$$

Now, by (4.1),

$$\|\Pi_n \phi\|_0 \leq \|\phi\|_0 \quad (4.6)$$

since $\frac{n}{\ell} [(y - \frac{i\ell}{n}) + (\frac{(i+1)\ell}{n} - y)] = 1$. Then, by (4.4),

$$\|D^s \Pi_n \phi\|_0 = \|\Pi_n D^s \phi\|_0 \leq \|D^s \phi\|_0. \quad (4.7)$$

We compare (4.5), (4.7), and Lemma 2.2,

$$\begin{aligned}
 \|D^s \mathcal{K}_{\Pi_n \Phi}\|_0 &= \|\mathcal{K} D^s \Pi_n \Phi\|_0 \\
 &\leq \frac{M\Gamma(\alpha)\iota^\alpha}{\Gamma(\alpha+1)} \|D^s \Pi_n \Phi\|_0 \\
 &\leq \frac{M\Gamma(\alpha)\iota^\alpha}{\Gamma(\alpha+1)} \|D^s \Phi\|_0
 \end{aligned} \tag{4.8}$$

We summarize these results as

Lemma 4.1: Let $\Phi \in \mathcal{A}^r(I, J)$. Then, $\Pi_n \Phi$ and $\mathcal{K}_{\Pi_n \Phi}$ belong to $\mathcal{A}^r(I, J)$; for $s \leq mr$, $\mathcal{K}_{\Pi_n} D^s \Phi = \mathcal{K} D^s \Pi_n \Phi = D^s \mathcal{K}_{\Pi_n \Phi}$; and, moreover,

$$\|\mathcal{K}_{\Pi_n \Phi}\|_r \leq \frac{M\Gamma(\alpha)\iota^\alpha}{\Gamma(\alpha+1)} \|\Phi\|_r.$$

Proof: All the lemma has been proved above except that some remarks on the y -continuity of $\mathcal{K}_{\Pi_n \Phi}$ are in order for our later use.

Since $\Phi \in \mathcal{A}^r(I, J)$ it is sensible to define

$$\Omega_s(\delta, \Phi) = \sup_{|u-v| \leq \delta} \|D^s \Phi_u - D^s \Phi_v\|_0, \quad s \leq mr. \tag{4.9}$$

By (3.13), $\|D^s \Phi_u - D^s \Phi_v\|_0$ is a continuous function of v (for u fixed) for v in J (and $u \in J$).

Lemma 4.2: Let $\Phi \in \mathcal{A}^r(I, J)$, $s \leq mr$. Then

$$\Omega_s(\delta, \Pi_n \Phi) \leq \delta \Omega_s(\iota/n, \Phi),$$

and $\Omega_s(\delta, \mathcal{K}_{\Pi_n \Phi}) \leq C \max(\delta, \delta^\alpha) \|D^s \Phi\|_0$ where C depends on α , M , L , and ι .

Proof: Since $\mathcal{K}_{\Pi_n} D^s \Phi = D^s \mathcal{K}_{\Pi_n} \Phi$, we apply Lemma 2.1 to obtain

$$\begin{aligned} & \| (D^s \mathcal{K}_{\Pi_n} \Phi)_u - (D^s \mathcal{K}_{\Pi_n} \Phi)_v \|_0 \\ & \leq C \max(|u-v|, |u-v|^\alpha) \|D^s \Pi_n \Phi\|_0 \\ & \leq C \max(|u-v|, |u-v|^\alpha) \|D^s \Phi\|_0, \end{aligned}$$

for $s \leq mr$ by (4.7). Thus, for $|u-v| \leq \delta$, we have

$$\Omega_s(\delta, \mathcal{K}_{\Pi_n} \Phi) \leq C \max(\delta, \delta^\alpha) \|D^s \Phi\|_0.$$

The other inequality in this lemma is a special case of Lemma 4.3: Let $u_0, u_1, \dots, u_q$ be given real numbers, and let the function $u(s)$ be defined by

$$u(s) = u_j(j+1-s) + u_{j+1}(s-j), \quad j \leq s \leq j+1.$$

Then, $|u(s) - u(r)| \leq \eta |s-r|$,

where $\eta = \max\{|u_{j+1} - u_j| : j = 0, 1, \dots, q-1\}$.

Proof: If s and r are integers, we have $|u(s) - u(r)| = |u_s - u_r| \leq \eta |s - r|$. Next, for $j \leq r < j+1 = s$, we have

$$\begin{aligned} |u(s) - u(r)| &= |u_{j+1} - u_j(j+1-r) - u_{j+1}(r-j)| \\ &= |(u_{j+1} - u_j)(j+1-r)| \\ &= |u_{j+1} - u_j| \cdot |s-r| \leq \eta |s-r|. \end{aligned}$$

Similarly, for $r = j < s \leq j+1$,

$$|u(s) - u(r)| = |u_{j+1} - u_j| \cdot |s-r| \leq \eta |s-r|.$$

Combination of these results gives the lemma.

To complete the proof of Lemma 4.2, simply set $u(s) = D^k_{\Pi_n} \Phi(x, s\ell/n)$, and we have, for $k \leq mr$,

$$\begin{aligned}
& |D^k \Pi_n \phi(x, s\ell/n) - D^k \Pi_n \phi(x, s'\ell/n)| \\
& \leq |s-s'| \ell n^{-1} \Omega_k(\ell/n, \phi) \\
& = |u-v| \Omega_k(\ell/n, \phi),
\end{aligned}$$

where $u = \frac{s\ell}{n}$ and $v = \frac{s'\ell}{n}$.

Theorem 4.1: Let $\phi \in \mathcal{A}^r(I, J)$. Then $\Pi_n \phi \rightarrow \phi$ as $n \rightarrow \infty$ in $\mathcal{A}^r(I, J)$; $\mathcal{K}\phi \in \mathcal{A}^r(I, J)$; $D^s \mathcal{K}\phi = \mathcal{K} D^s \phi$, $s \leq mr$; and $\mathcal{K} \Pi_n \phi \rightarrow \mathcal{K}\phi$ as $n \rightarrow \infty$ in $\mathcal{A}^r(I, J)$.

Proof: $D^s(\Pi_n \phi - \phi)(x, y) = (\Pi_n D^s \phi - D^s \phi)(x, y)$

$$\begin{aligned}
& = \frac{n}{\ell} \left\{ \left(y - \frac{i\ell}{n}\right) D^s \phi_{(i+1)\ell/n}(x) + \left(\frac{(i+1)\ell}{n} - y\right) D^s \phi_{i\ell/n}(x) \right\} \\
& \quad - D^s \phi_y(x), \quad i\ell \leq ny \leq (i+1)\ell,
\end{aligned}$$

which may be written

$$\begin{aligned}
& \frac{n}{\ell} \left\{ \left(y - \frac{i\ell}{n}\right) (D^s \phi_{(i+1)\ell/n}(x) - D^s \phi_y(x)) \right. \\
& \quad \left. + \left(\frac{(i+1)\ell}{n} - y\right) (D^s \phi_{i\ell/n}(x) - D^s \phi_y(x)) \right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|D^s(\Pi_n \phi - \phi)\|_0 & \leq \sup_{|u-v| \leq \frac{\ell}{n}} |D^s \phi_u(x) - D^s \phi_v(x)| \\
& \leq \sup_{|u-v| \leq \frac{\ell}{n}} \|D^s \phi_u - D^s \phi_v\|_0 = \Omega_s\left(\frac{\ell}{n}, \phi\right),
\end{aligned}$$

$$\text{and } \|\Pi_n \phi - \phi\|_r \leq \max_{0 \leq s \leq mr} \Omega_s\left(\frac{\ell}{n}, \phi\right). \quad (4.10)$$

That is, $\Pi_n \phi \rightarrow \phi$ in $\mathcal{A}^r(I, J)$.

By Lemma 4.1 and Lemma 2.2,

$$\begin{aligned}
 & \|D^s \mathcal{K}_{\Pi_i \Phi} - D^s \mathcal{K}_{\Pi_j \Phi}\|_0 \\
 &= \|\mathcal{K} D^s \Pi_i \Phi - \mathcal{K} D^s \Pi_j \Phi\|_0 \\
 &\leq \frac{M\Gamma(\alpha)\iota^\alpha}{\Gamma(\alpha+1)} \|D^s \Pi_i \Phi - D^s \Pi_j \Phi\|_0 .
 \end{aligned}$$

Thus, $\|\mathcal{K}_{\Pi_i \Phi} - \mathcal{K}_{\Pi_j \Phi}\|_r \leq \|\Pi_i \Phi - \Pi_j \Phi\|_r$, and $\{\mathcal{K}_{\Pi_n \Phi}\}_{n=1}^\infty$ is a Cauchy sequence. Then $\Psi \in \mathcal{A}^r(I, J)$ exists so that $\mathcal{K}_{\Pi_n \Phi} \rightarrow \Psi$ in $\mathcal{A}^r(I, J)$. But in the uniform norm, $\|\cdot\|_0$, $\mathcal{K}_{\Pi_n \Phi} \rightarrow \mathcal{K}_\Phi$ as $n \rightarrow \infty$. Thus, $\mathcal{K}_\Phi = \Psi \in \mathcal{A}^r(I, J)$.

Finally, we have

$$\|D^s \mathcal{K}_{\Pi_n \Phi} - D^s \mathcal{K}_\Phi\|_0 \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

But, by the above proof,

$$\|\mathcal{K}_{\Pi_n} D^s \Phi - \mathcal{K} D^s \Phi\|_0 \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

Since $\mathcal{K}_{\Pi_n} D^s \Phi = D^s \mathcal{K}_{\Pi_n \Phi}$, we have

$$\mathcal{K} D^s \Phi = D^s \mathcal{K}_\Phi . \quad (4.11)$$

5. Existence and Uniqueness.

On the notation of the above sections, we write (1.7) in the form

$$u = g + \mathcal{K}Au, \quad (5.1)$$

where we assume $g \in G_t$. We shall study the existence and uniqueness (in G) of a solution u of (5.1).

We shall denote

$$\|\phi\|_{n,y} = \sup_{0 \leq \eta \leq y} \|\phi_\eta\|_n, \quad (5.2)$$

and
$$\|\phi\|_{\infty,t,y} = \sup_{0 \leq \eta \leq y} \|\phi_\eta\|_{\infty,t}.$$

Then $G_{t,y}$ is similar to G_t in the sense that $G_t = G_{t,t}$.

Lemma 5.1: If $\phi \in G_{t,r}$, then $\sum_{n=0}^{\infty} (\mathcal{K}^n A^n \phi)(x,y)$ is well defined for $0 \leq y \leq r$ when

$$M\Gamma(\alpha)r^\alpha t < 1.$$

Proof: By Lemma 2.2 and relation (3.10),

$$\begin{aligned} \left| \sum_{n=0}^{m-1} (\mathcal{K}^n A^n \phi)(x,y) \right| &\leq \sum_{n=0}^{m-1} \frac{M_{\Gamma}^{n,n}(\alpha)y^{n\alpha}}{\Gamma(n\alpha+1)} \|A^n \phi\|_{0,y} \\ &\leq \sum_{n=0}^{m-1} M_{\Gamma}^{n,n}(\alpha)y^{n\alpha} \|\phi\|_{\infty,t,y} t^n \\ &\leq \|\phi\|_{\infty,t,y} \sum_{n=0}^{m-1} M_{\Gamma}^{n,n}(\alpha)y^{n\alpha} t^n \\ &= \|\phi\|_{\infty,t,y} \left\{ \frac{1 - (M\Gamma(\alpha)y^\alpha t)^m}{1 - M\Gamma(\alpha)y^\alpha t} \right\} \end{aligned}$$

$$\rightarrow \frac{\|\phi\|_{\infty, t, y}}{1 - M\Gamma(\alpha)y^{\alpha t}} \quad \text{as } m \rightarrow \infty, \quad (5.3)$$

if $M\Gamma(\alpha)y^{\alpha t} < 1$.

The above result is not enough. We shall need estimates on $D^s F$ for $s = 0, 1, \dots$, where $F(x, y)$ represents the sum defined in the previous lemma.

$$\text{Let } F_j = \sum_{n=0}^{j-1} (\mathcal{K}^{nA^n\phi})(x, y).$$

By Theorem 4.1,

$$D^s F_j = \sum_{n=0}^{j-1} (\mathcal{K}^{nA^n D^s \phi})(x, y). \quad (5.4)$$

Now, by (3.7)

$$\|A^n D^s \phi\|_{0, y} \leq \|D^s \phi\|_{n, y} a^n, \quad (5.5)$$

and we note that

$$\|D^s \phi\|_{n, y} \leq \|\phi\|_{n, y}, \quad s = 0;$$

$$\|D^s \phi\|_{n, y} \leq \|\phi\|_{n+1, y}, \quad 1 \leq s \leq m;$$

$$\|D^s \phi\|_{n, y} \leq \|\phi\|_{n+2, y}, \quad m+1 \leq s \leq 2m;$$

and, in general,

$$\left. \begin{aligned} \|D^s \phi\|_{n, y} &\leq \|\phi\|_{n+\mu, y}, \\ \text{where } \mu &= \left[\frac{s+m-1}{m} \right]. \end{aligned} \right\} \quad (5.6)$$

(We are using $[x]$ for greatest integer in x .)

We combine (5.5), (5.6), and (3.8) to obtain

$$\begin{aligned}
\|A^n D^s \phi\|_{0,y} &\leq \|D^s \phi\|_{n,y} a^n \\
&\leq \|\phi\|_{n+\mu,y} a^n \\
&\leq \|\phi\|_{\infty,t,y} t^{n+\mu} a^{-\mu} \Gamma(n\alpha + \mu\alpha + 1) .
\end{aligned} \tag{5.7}$$

Using this result and Lemma 2.2 in (5.4), we have

$$\begin{aligned}
\|D^s F_j\|_{0,y} &\leq \left\{ \sum_{n=0}^{j-1} \frac{(M\Gamma(\alpha)y^\alpha t)^n \Gamma(n\alpha + \mu\alpha + 1)}{\Gamma(n\alpha + 1)} \right\} \\
&\quad \cdot \left(\frac{t}{a}\right)^\mu \|\phi\|_{\infty,t,y} .
\end{aligned} \tag{5.8}$$

We need an inequality for gamma functions (see [11], p.9):

$$\frac{\Gamma(u + v + 1)}{\Gamma(u + 1) \Gamma(v + 1)} \leq (1 - \epsilon)^{-u} \epsilon^{-v} \tag{5.9}$$

for $u \geq 0$, $v \geq 0$, and $0 < \epsilon < 1$. Then

$$\begin{aligned}
\frac{\Gamma(n\alpha + \mu\alpha + 1)}{\Gamma(n\alpha + 1)} &= \left[\frac{\Gamma(n\alpha + \mu\alpha + 1)}{\Gamma(n\alpha + 1) \Gamma(\mu\alpha + 1)} \right] \Gamma(\mu\alpha + 1) \\
&\leq (1 - \epsilon)^{-n\alpha} \epsilon^{-\mu\alpha} \Gamma(\mu\alpha + 1) ,
\end{aligned}$$

and (5.8) may be written

$$\begin{aligned}
\|D^s F_j\|_{0,y} &\leq \left(\frac{t}{a\epsilon^\alpha}\right)^\mu \|\phi\|_{\infty,t,y} \Gamma(\mu\alpha + 1) \sum_{n=0}^{j-1} \lambda^n \\
&= \left(\frac{t}{a\epsilon^\alpha}\right)^\mu \|\phi\|_{\infty,t,y} \Gamma(\mu\alpha + 1) \sigma_j ,
\end{aligned} \tag{5.10}$$

where $\lambda = \frac{M\Gamma(\alpha)y^\alpha t}{(1 - \epsilon)^\alpha}$, (5.11)

and $\sigma_j = \frac{1 - \lambda^j}{1 - \lambda}$.

Now, we wish to simplify (5.10). First, set $t_\epsilon = t\epsilon^{-\alpha}$. Next, if $t_\epsilon < a$, increase t_ϵ so that $t_\epsilon \geq a$, thus increasing the right-hand side of (5.10). As the maximum for μ as s ranges over $0, 1, \dots, mn$ is n , we have (see (3.6))

$$\|F_j\|_{n,y} \leq \sigma_j \left(\frac{t_\epsilon}{a}\right)^n \|\Phi\|_{\infty,t,y} \max_{0 \leq \mu \leq n} \Gamma(\mu\alpha+1). \quad (5.12)$$

For $\alpha \geq 1$, we have

$$\|F_j\|_{n,y} \leq \sigma_j t_\epsilon^n a^{-n} \|\Phi\|_{\infty,t,y} \Gamma(n\alpha+1). \quad (5.13)$$

This result is also valid if $\alpha < 1$ and $n\alpha+1 \geq 2$. However, as $\Gamma(\mu\alpha+1) < 1$ when $0 < \mu\alpha < 1$, we note that $\Gamma(\mu\alpha+1) > 0.8$ for all μ . Thus, for $\alpha < 1$ and $n\alpha+1 < 2$,

$$\|F_j\|_{n,y} \leq \frac{5}{4} \sigma_j t_\epsilon^n a^{-n} \|\Phi\|_{\infty,t,y} \Gamma(n\alpha+1), \quad (5.14)$$

which holds in any case as the right hand side of (5.14) exceeds that of (5.13).

Choose r so that for $0 \leq y \leq r$

$$\lambda \leq \frac{\nu - 1}{\nu} = \frac{M\Gamma(\alpha)r^\alpha t}{(1-\epsilon)^\alpha} < 1. \quad (5.15)$$

Then $\sigma_j < \nu$ and (5.14) may be written

$$\|F_j\|_{n,r} \leq \frac{5}{4} \nu t_\epsilon^n a^{-n} \|\Phi\|_{\infty,t,r} \Gamma(n\alpha+1). \quad (5.16)$$

Thus,

$$\|F_j\|_{\infty,t_\epsilon,r} \leq \frac{5}{4} \nu \|\Phi\|_{\infty,t,r} \quad (5.17)$$

The above work provides the proof of

Lemma 5.2: If $\Phi \in G_{t,r}$ and $F = \sum_{k=0}^{\infty} \chi^k A^k \Phi$, then $F \in G_{t(\epsilon),r}$

for $0 < \epsilon < 1$ and r such that (5.15) is valid, where
 $t(\epsilon) = t_\epsilon = \max \left(\frac{t}{\epsilon^\alpha}, a \right)$.

Theorem 5.1: Let $g \in G_t$, where t is such that
 $M\Gamma(\alpha)t^\alpha < 1$, and let

$$u = \sum_{n=0}^{\infty} \mathcal{K}^{nA^n} g.$$

Then u is a solution of (5.1) such that $u \in G_{t(\epsilon)}$ if

$M\Gamma(\alpha)t^\alpha < (1-\epsilon)^\alpha < 1$. Moreover,

$$\frac{4}{5} \|u\|_{\infty, t(\epsilon)} \leq \frac{(1-\epsilon)^\alpha \|g\|_{\infty, t}}{(1-\epsilon)^\alpha - M\Gamma(\alpha)t^\alpha},$$

and u is the only solution of (5.1) in the class of functions

$$G = \bigcup_{\eta} G_{\eta}.$$

Proof: Much of the theorem is immediate from Lemma 5.2.

First, u is a solution of (5.1) since

$$\begin{aligned} u &= g + \sum_{n=1}^{\infty} \mathcal{K}^{nA^n} g \\ &= g + \mathcal{K}A \sum_{n=0}^{\infty} \mathcal{K}^{nA^n} g = g + \mathcal{K}Au. \end{aligned}$$

Suppose (5.1) has two solutions in G . Denote the difference by v and take $v \in G_r$. Then,

$v = \mathcal{K}Av = \mathcal{K}^2A^2v = \dots$, and Lemma 2.2 and (3.10) imply that

$$\begin{aligned} \|v\|_{0,y} &\leq \frac{M_{\Gamma}^{n\alpha}(\alpha)y^{n\alpha}}{\Gamma(n\alpha+1)} \|A^n v\|_{0,y} \\ &\leq (M\Gamma(\alpha)y^{\alpha r})^n \|v\|_{\infty, r, y} \end{aligned}$$

for all n . For y sufficiently small, namely $y < y_0$ with $M\Gamma(\alpha)y_0^\alpha \leq 1$, we have

$$\|v\|_{0,y} = 0. \quad (5.4)$$

If $y_0 \geq \ell$, we have proved uniqueness. If $y_0 < \ell$, continuity of v implies that (5.4) has the form

$$\|v\|_{0,y_0} = 0. \quad (5.5)$$

Select y such that $y_0 \leq y < 2y_0$.

$$\text{Then, } v(x, y) = \int_{y_0}^y K(y, \eta) A v(x, \eta) d\eta.$$

$$\text{Let } \bar{v}(x, y) = v(x, y+y_0), \quad 0 \leq y < y_0.$$

$$\begin{aligned} \text{Then, } \bar{v}(x, y) &= \int_{y_0}^{y+y_0} K(y+y_0, \eta) A v(x, \eta) d\eta \\ &= \int_0^y K(y+y_0, \eta+y_0) A v(x, \eta+y_0) d\eta \\ &= \int_0^y K(y+y_0, \eta+y_0) A \bar{v}(x, \eta) d\eta. \end{aligned}$$

Now, define $\bar{G}(y, \eta)$ and $\bar{K}(y, \eta)$ by

$$\begin{aligned} K(y+y_0, \eta+y_0) &= G(y+y_0, \eta+y_0) (y-\eta)^{\alpha-1} \\ &= \bar{G}(y, \eta) (y-\eta)^{\alpha-1} = \bar{K}(y, \eta). \end{aligned}$$

Then, all our work can be redone to show that

$$\|\bar{v}\|_{0,y_0} = 0,$$

which will show that $\|v\|_{o,2y_o} = 0$. Iteration of this process shows that $\|v\|_{o,t} = 0$, and the theorem is proved.

6. Finite Differences and Derivatives.

We need certain facts about finite differences.

We consider certain collections of continuous real-valued functions, $f(x)$, $0 \leq x \leq 1$. Let (see [5])

$$\left. \begin{aligned} \Delta_h f(x) &= f(x+h) - f(x) \\ \text{and} \quad \Delta_h^{s+1} f(x) &= \Delta_h (\Delta_h^s f(x)) . \\ \text{Thus,} \quad \Delta_h^s f(x) &= (-1)^s \sum_{j=0}^s \binom{s}{j} (-1)^j f(x+jh) . \end{aligned} \right\} \quad (6.1)$$

Further, if

$$\left. \begin{aligned} E_h f(x) &= f(x+h) , \\ E_h^0 f(x) &= f(x) , \\ \text{and} \quad E_h^{s+1} f(x) &= E_h (E_h^s f(x)) , \end{aligned} \right\} \quad (6.2)$$

$$\text{then} \quad \Delta_h^s f(x) = (E_h - 1)^s f(x) . \quad (6.3)$$

In case a particular value of h is understood, the subscripts on Δ and E will be omitted.

Theorem 6.1: Let $c_0, c_1, c_2, \dots, c_{q+1}$ be a sequence of non-negative numbers, $q \geq 0$; let f map I into the reals; and let

$$|\Delta_h^s f(x)| \leq |h|^s c_s, \quad s = 0, 1, 2, \dots, q+1,$$

whenever x and $x+sh$ are contained in I . Then $f \in C^q(I)$, and

$$|D^s f(x)| \leq c_s, \quad s = 0, 1, 2, \dots, q.$$

Proof: We shall show the existence and continuity of

$$\lim_{h \rightarrow 0} \frac{\Delta_h^s f(x)}{h^s}, \quad s = 0, 1, \dots, q.$$

Let $\Sigma_n = \{x \in I : x = k2^{-n} \text{ for integral } k\}$, $n = 1, 2, \dots$ (see [6]). Then, let $\Sigma = \bigcup_{n=1}^{\infty} \Sigma_n$, and enumerate Σ in the form, $\Sigma = \{x_1, x_2, \dots\}$. Now, fix $x_r \in \Sigma$ and note that

$$\left\{ \frac{\Delta_h^s f(x_r)}{h^s} : h = \pm 2^{-p}, x_r + sh \in \Sigma \right\}$$

is a bounded infinite set of real numbers. Thus, there is a set $\{p_1, p_2, \dots\}$ of integers, such that

$$\lim_{i \rightarrow \infty} \frac{\Delta_{h(i)}^s f(x_r)}{[h(i)]^s} = D^s f(x_r) \quad (6.4)$$

exists, where $h(i)$ is 2^{-p_i} or -2^{-p_i} , and we need to justify the notation, $D^s f(x_r)$.

With fixed h understood, we have

$$\begin{aligned} (E-1)^s E^i (E^r - 1)f(x) &= (E-1)^s E^i \sum_{j=0}^{r-1} E^j (E-1)f(x) \\ &= (E-1)^{s+1} E^i \sum_{j=0}^{r-1} E^j f(x). \end{aligned}$$

Thus,

$$\begin{aligned} \left| \frac{\Delta^s E^i f(x)}{h^s} - \frac{\Delta^s E^{i+r} f(x)}{h^s} \right| &\leq \sum_{j=0}^{r-1} |h| \left| \frac{\Delta^{s+1} E^{i+j} f(x)}{h^{s+1}} \right| \\ &\leq r|h|c_{s+1} \end{aligned} \quad (6.5)$$

Let $b_i \geq 0$, $i = 0, 1, \dots, sk-s$ be such that

$$\sum_{i=0}^{sk-s} b_i E^i = (1+E+\dots+E^{k-1})^s. \quad (6.6)$$

Then,
$$(E-1)^s \sum_{i=0}^{sk-s} b_i E^i = (E^{k-1})^s, \quad (6.7)$$

and
$$\sum_{i=0}^{sk-s} b_i = k^s. \quad (6.8)$$

Thus,
$$\begin{aligned} \Delta_{kh}^s f(x) &= \sum_{i=0}^{sk-s} b_i E^i (E-1)^s f(x) \\ &= \sum_{i=0}^{sk-s} b_i E^i \Delta^s f(x), \end{aligned}$$

and

$$\begin{aligned} &|\Delta_{kh}^s f(x) - \Delta_{kh}^s E^r f(x)| \\ &\leq |h|^s \sum_{i=0}^{sk-s} b_i \left| \frac{\Delta^s E^i f(x)}{h^s} - \frac{\Delta^s E^{r+i} f(x)}{h^s} \right| \\ &\leq |h|^s \sum_{i=0}^{sk-s} b_i r |h| c_{s+1} \leq |h|^s k^s r |h| c_{s+1} \end{aligned}$$

by (6.5) and (6.8). That is,

$$\left| \frac{\Delta_{kh}^s f(x)}{(kh)^s} - \frac{\Delta_{kh}^s f(x+rh)}{(kh)^s} \right| \leq r |h| c_{s+1}. \quad (6.9)$$

Let $h \rightarrow 0$ and $r \rightarrow \infty$ so that rh remains fixed. Let k be fixed. Then the continuity of $D^s f(x)$ follows (once the notation is justified). Finally,

$$\begin{aligned} &\left| \frac{\Delta_{kh}^s f(x)}{(kh)^s} - \frac{\Delta^s f(x)}{h^s} \right| \\ &= |k^{-s} \sum_{i=0}^{sk-s} b_i \left\{ \frac{\Delta^s f(x+ih)}{h^s} - \frac{\Delta^s f(x)}{h^s} \right\}| \\ &\leq k^{-s} \sum_{i=0}^{sk-s} b_i c_{s+1} |h| = \frac{1}{2} c_{s+1} |h|^s (k-1), \quad (6.10) \end{aligned}$$

since $\sum_{i=0}^{sk-s} b_i x^i = (1+x+\dots+x^{k-1})^s$, and hence

$$\sum_{i=0}^{sk-s} i b_i x^{i-1} = s (1+2x+\dots+(k-1)x^{k-2}) (1+x+\dots+x^{k-1})^{s-1},$$

and

$$k^{-s} \sum_{i=0}^{sp-s} i b_i = k^{-s} s \left[\frac{(k-1)k}{2} \right] k^{s-1} = \frac{1}{2} s(k-1).$$

Fix $x \in \Sigma$ and $hk = \Delta x$ such that $|\Delta x| c_{s+1}^s < \frac{1}{2}\epsilon$,

$$\text{and } |D^s f(x) - \frac{\Delta_{\Delta x}^s f(x)}{(\Delta x)^s}| < \frac{1}{2}\epsilon, \quad (6.11)$$

where $x+s\Delta x \in \Sigma$. Using this h in (6.10), we have

$$\begin{aligned} \left| \frac{\Delta_{kh}^s f(x)}{(kh)^s} - \frac{\Delta^s f(x)}{h^s} \right| &\leq \frac{1}{2} c_{s+1}^s (k-1) |\Delta x| k^{-1} \\ &< \frac{k-1}{k} \cdot \frac{\epsilon}{4} < \frac{\epsilon}{4} \text{ for } k = 1, 2, \dots \end{aligned}$$

That is,

$$\left| \frac{\Delta_{\Delta x}^s f(x)}{(\Delta x)^s} - \frac{\Delta_{\Delta x/k}^s f(x)}{(\Delta x/k)^s} \right| < \frac{\epsilon}{4}, \quad k = 1, 2, \dots \quad (6.12)$$

Similarly, let k be a positive integer so $hk = \Delta x$, and apply (6.10) to obtain

$$\begin{aligned} \left| \frac{\Delta_{j\Delta x/k}^s f(x)}{(j\Delta x/k)^s} - \frac{\Delta_{\Delta x/k}^s f(x)}{(\Delta x/k)^s} \right| &\leq \frac{1}{2} c_{s+1}^s \left| \frac{\Delta x}{k} \right| s(j-1) \\ &< \frac{1}{4}\epsilon, \quad j = 1, 2, \dots, k. \end{aligned} \quad (6.13)$$

Now, let $r \in \Sigma$, $r \neq 0$, $r = j/k$, $x \in \Sigma$, and $(x+sr\Delta x) \in \Sigma$.

We combine (6.11), (6.12), and (6.13) to obtain

$$\left| D^s f(x) - \frac{\Delta_{r\Delta x}^s f(x)}{(r\Delta x)^s} \right| < \epsilon ,$$

and we have

$$\lim_{\substack{h \rightarrow 0 \\ h \in \Sigma}} \frac{\Delta_h^s f(x)}{h^s} = D^s f(x) , \quad x \in \Sigma .$$

Thus, $D^s f(x)$ exists and is continuous. That $D^s f(x)$ satisfies the correct bound is clear from (6.4), and the theorem is proved.

7. Bernstein Polynomials and Interpolation.

Let $I_n = \{kn^{-1} : k = 0, 1, \dots, n\}$. If $\varphi(x)$ is a real-valued function whose domain contains I_n , we can define the n th Bernstein polynomial for φ (see [9], p. 88, and [7]) by the relation

$$(B_n \varphi)(x) = \sum_{v=0}^n \binom{n}{v} \varphi\left(\frac{v}{n}\right) x^v (1-x)^{n-v}. \quad (7.1)$$

Since $B_n \varphi$ is a polynomial, $B_n \varphi \in C^\infty(I)$, and $B_n \varphi \in \mathcal{H}_t$ for each t . In fact, if $s \leq n$,

$$(D^s B_n \varphi)(x) = \frac{n!}{(n-s)!} \sum_{v=0}^{n-s} \binom{n-s}{v} x^v (1-x)^{n-v-s} \Delta^s \varphi\left(\frac{v}{n}\right), \quad (7.2)$$

where the understood subscript on Δ is n^{-1} . If $s > n$, $D^s B_n \varphi \equiv 0$. Thus, if $\varphi \in C^k(I)$, $k \leq n$,

$$\begin{aligned} \|D^k B_n \varphi\|_0 &\leq \frac{n!}{(n-k)!} \max_{0 \leq v \leq n-k} \left| \Delta^k \varphi\left(\frac{v}{n}\right) \right| \\ &= \frac{n!}{n^k (n-k)!} \max_{0 \leq v \leq n-k} \left| n^k \Delta^k \varphi\left(\frac{v}{n}\right) \right| \\ &= \frac{n!}{n^k (n-k)!} \|D^k \varphi\|_0 \leq \|D^k \varphi\|_0 \end{aligned} \quad (7.3)$$

Now, set

$$\omega_x(\delta, \varphi) = \max \{ |\varphi(x) - \varphi(x')| : x, x' \in I, |x - x'| < \delta \}; \quad (7.4)$$

$\omega_x(\delta, \varphi)$, as a function of δ , is called the modulus of continuity of φ with respect to x (see [7] for this, as well

as for the results summarized below).

For $\varphi \in C(I)$, (Popoviciu, [7], p. 20)

$$|\varphi(x) - B_n \varphi(x)| \leq \frac{5}{4} \omega_x(n^{-\frac{1}{2}}, \varphi). \quad (7.5)$$

Moreover, for $\varphi \in C^1(I)$,

$$|\varphi(x) - B_n \varphi(x)| \leq \frac{3}{4} n^{-\frac{1}{2}} \omega_x(n^{-\frac{1}{2}}, D\varphi). \quad (7.6)$$

For $\varphi \in C^k(I)$, $k < n$, we have

$$(B_{n-k} D^k \varphi)(x) = \sum_{\nu=0}^{n-k} \binom{n-k}{\nu} x^\nu (1-x)^{n-k-\nu} D^k \varphi\left(\frac{\nu}{n-k}\right). \quad (7.7)$$

We compare (7.2) and (7.7) to obtain

$$\begin{aligned} & (D^k B_n \varphi - B_{n-k} D^k \varphi)(x) \\ &= \sum_{\nu=0}^{n-k} \binom{n-k}{\nu} x^\nu (1-x)^{n-k-\nu} \left\{ \frac{n!}{(n-k)!} \Delta^k \varphi\left(\frac{\nu}{n}\right) - D^k \varphi\left(\frac{\nu}{n-k}\right) \right\}. \end{aligned} \quad (7.8)$$

Now, for $\varphi \in C^k(I)$ and $0 \leq \nu \leq n-k$,

$$\begin{aligned} & \left| \frac{n!}{(n-k)!} \Delta^k \varphi\left(\frac{\nu}{n}\right) - D^k \varphi\left(\frac{\nu}{n-k}\right) \right| \\ &= \left| \frac{n!}{n^k (n-k)!} D^k \varphi\left(\frac{\nu}{n} + \frac{k}{n} \theta_\nu\right) - D^k \varphi\left(\frac{\nu}{n-k}\right) \right| \\ &\leq \left| D^k \varphi\left(\frac{\nu}{n} + \frac{k}{n} \theta_\nu\right) - D^k \varphi\left(\frac{\nu}{n-k}\right) \right| \frac{n!}{n^k (n-k)!} \\ &\quad + \left(1 - \frac{n!}{n^k (n-k)!}\right) \left| D^k \varphi\left(\frac{\nu}{n-k}\right) \right| \end{aligned}$$

$$\begin{aligned}
&\leq |D^k_{\varphi}(\frac{\nu}{n} + \frac{k}{n}\theta_{\nu}) - D^k_{\varphi}(\frac{\nu}{n-k})| + (1 - \frac{n!}{n^k(n-k)!}) \|D^k_{\varphi}\|_0 \\
&\leq \omega_x(\frac{k}{n}, D^k_{\varphi}) + (1 - \frac{n!}{n^k(n-k)!}) \|D^k_{\varphi}\|_0
\end{aligned}$$

since $|\frac{\nu}{n} + \frac{k}{n}\theta_{\nu} - \frac{\nu}{n-k}| = |\frac{k}{n}(\theta_{\nu} - \frac{\nu}{n-k})| \leq \frac{k}{n}$

for $0 \leq \theta_{\nu} \leq 1$, $0 \leq \nu \leq n-k$.

Thus, for $\varphi \in C^k(I)$, we have

$$\begin{aligned}
&\|D^k B_n \varphi - B_{n-k} D^k \varphi\|_0 \\
&\leq \omega_x(\frac{k}{n}, D^k_{\varphi}) + (1 - \frac{n!}{n^k(n-k)!}) \|D^k_{\varphi}\|_0 \\
&\leq \omega_x(\frac{k}{n}, D^k_{\varphi}) + \frac{k(k-1)}{2n} \|D^k_{\varphi}\|_0 , \tag{7.9}
\end{aligned}$$

as we shall show inductively that

$$0 \leq (1 - \frac{n!}{n^k(n-k)!}) \leq \frac{k(k-1)}{2n} , \quad 0 \leq k < n . \tag{7.10}$$

First, the result is clear for $k = 0$, $k = 1$. For $k = 2$, we have

$$1 - \frac{n!}{n^2(n-2)!} = 1 - \frac{n-1}{n} = \frac{1}{n} = \frac{k(k-1)}{2n} .$$

Assume the result for a particular k . Then

$$n^k - (n)(n-1) \dots (n-k+1) \leq \frac{k(k-1)}{2} n^{k-1} ,$$

or

$$n^k - \frac{k(k-1)}{2} n^{k-1} \leq n(n-1) \dots (n-k+1) .$$

$$\begin{aligned}
\text{Thus, } & n(n-1) \dots (n-k+1)(n-k) \\
& \cong (n-k) \left(n^k - \frac{k(k-1)}{2} n^{k-1} \right) \\
& = n^{k+1} - n^k \left[\frac{k(k-1)}{2} + k \right] + \frac{k^2(k-1)}{2} n^{k-1} \\
& \cong n^{k+1} - n^k \left[\frac{k(k-1)}{2} + k \right] = n^{k+1} - n^k \left[\frac{(k+1)k}{2} \right],
\end{aligned}$$

and (7.10) follows.

Let $\phi(x, y) \in C(I, J)$. Then, we can define

$$(B_n \phi)(x, y) = \sum_{v=0}^n \binom{n}{v} \phi\left(\frac{v}{n}, y\right) x^v (1-x)^{n-v}. \quad (7.11)$$

Lemma 7.1: Let $\phi \in \mathcal{A}^r(I, J)$. Then $B_n \phi \in G_t$ for every t , and

$$B_n \phi \rightarrow \phi \text{ in } \mathcal{A}^r(I, J) \text{ as } n \rightarrow \infty.$$

Proof: First, we introduce the notation,

$$w_x(\delta, \phi) = \sup_{y \in J} \omega_x(\delta, \phi). \quad (7.12)$$

For $\phi \in C^1(I, J)$, observe that

$$w_x(\delta, \phi) \leq \delta \|D\phi\|_0. \quad (7.13)$$

By (7.3), $\|B_n \phi\|_r \leq \|\phi\|_r$, $(r-1)m < n$. Of course, for $(r-1)m \geq n$,

$$\|B_n \phi\|_r = \|B_n \phi\|_{r-1}$$

since $D^k B_n \phi \equiv 0$ for $k > n$. Thus,

$$\|B_n \phi\|_r \leq \|\phi\|_r. \quad (7.14)$$

Let $u, v \in J$. Then,

$$\| (B_n \Phi)_u - (B_n \Phi)_v \|_r \leq \| \Phi_u - \Phi_v \|_r , \quad (7.15)$$

part 3 of (3.13) is satisfied, and

$$B_n \Phi \in \mathcal{A}^r(I, J) .$$

Finally, from (7.5) and (7.9),

$$\begin{aligned} & \| D^k B_n \Phi - D^k \Phi \|_0 \\ & \leq \| D^k B_n \Phi - B_{n-k} D^k \Phi \|_0 \\ & \quad + \| B_{n-k} D^k \Phi - D^k \Phi \|_0 \\ & \leq w_x \left(\frac{k}{n}, D^k \Phi \right) + \frac{k(k-1)}{2n} \| D^k \Phi \|_0 \\ & \quad + \frac{5}{4} w_x \left((n-k)^{-\frac{1}{2}}, D^k \Phi \right) . \end{aligned} \quad (7.16)$$

Let k fixed and $n \rightarrow \infty$. Then

$$\| D^k B_n \Phi - D^k \Phi \|_0 \rightarrow 0 .$$

That is, $B_n \Phi \rightarrow \Phi$ in $\mathcal{A}^r(I, J)$.

Recall the definition of the operator Π_n on $C(I, J)$, as given by relation (4.1). We note that

$$\begin{aligned} (B_n \Pi_n \Phi)(x, y) &= B_n \left\{ \frac{n}{\ell} \sum_{j=0}^{n-1} \Psi_{n,j}(y) \Phi \left(x, \frac{j\ell}{n} \right) \right\} \\ &= \frac{n}{\ell} \sum_{j=0}^{n-1} \Psi_{n,j}(y) (B_n \Phi) \left(x, \frac{j\ell}{n} \right) \\ &= (\Pi_n B_n \Phi)(x, y) . \end{aligned}$$

$$\text{That is, } \Pi_n B_n = B_n \Pi_n . \quad (7.17)$$

We compare $\Pi_n B_n$ and the identity operator on $\mathcal{Q}^r(I, J)$.

$$\begin{aligned}
 & \|\Pi_n B_n \phi - \phi\|_r \\
 & \leq \|\Pi_n B_n \phi - \Pi_n \phi\|_r + \|\Pi_n \phi - \phi\|_r \\
 & \leq \|B_n \phi - \phi\|_r + \|\Pi_n \phi - \phi\|_r
 \end{aligned} \tag{7.18}$$

Then, we can estimate the first term by (7.16) and the second by (4.10). Thus,

$$\Pi_n B_n \phi \rightarrow \phi \quad \text{in } \mathcal{Q}^r(I, J) \quad \text{as } n \rightarrow \infty.$$

8. Approximations.

Let $J_n = \{k\ell n^{-1} : k = 0, 1, \dots, n\}$. If $\phi(x, y)$ has a domain containing $I_n \times J_n$, we can define

$$B_n \Pi_n \phi = \Pi_n B_n \phi \in C(I, J). \quad (8.1)$$

We write the integro-differential equation (5.1) in the form

$$g = Tu = (1 - \mathcal{K}A)u, \quad (8.2)$$

and define an approximation to Tu by

$$T_n u = (1 - \mathcal{K}A) \Pi_n B_n u. \quad (8.3)$$

Now, for $\phi \in G_t$, we shall compare the operators T_n and T .

$$\begin{aligned} (T_n - T)\phi &= (\Pi_n B_n - \mathcal{K}A \Pi_n B_n - 1 + \mathcal{K}A)\phi \\ &= (\Pi_n B_n - 1)\phi - \mathcal{K}A(\Pi_n B_n - 1)\phi. \end{aligned} \quad (8.4)$$

Since we can estimate the first term in the right-hand side of (8.4) by (7.18), we consider the second term:

$$\begin{aligned} &\|D^S \mathcal{K}A(\Pi_n B_n - 1)\phi\|_0 \\ &= \|\mathcal{K}AD^S(\Pi_n B_n - 1)\phi\|_0 \\ &\leq \frac{M\Gamma(\alpha)\ell^\alpha}{\Gamma(\alpha+1)} \|AD^S(\Pi_n B_n - 1)\phi\|_0. \end{aligned}$$

Let $\frac{M\Gamma(\alpha)\ell^\alpha}{\Gamma(\alpha+1)} = C_1$. The right-hand side above is not more than

$$\begin{aligned}
& C_1 \sum_{i=0}^m |a_{i,1}| \|D^{s+i}(\Pi_n B_n - 1)\bar{\Phi}\|_0 \\
& \leq C_1 a \max_{0 \leq i \leq m} \|D^{s+i}(\Pi_n B_n - 1)\bar{\Phi}\|_0 \\
& \leq C_1 a \max_{0 \leq i \leq m} \{ \|D^{s+i} \Pi_n (B_n \bar{\Phi} - \bar{\Phi})\|_0 \\
& \quad + \|D^{s+i}(\Pi_n \bar{\Phi} - \bar{\Phi})\|_0 \} .
\end{aligned}$$

Now, if $s \leq (r-1)m$, and $\Psi \in \mathcal{Q}^r(I, J)$,

$$\max_{0 \leq i \leq m} \|D^{s+i} \Psi\|_0 \leq \|\Psi\|_r .$$

Thus, we have

$$\begin{aligned}
& \|D^s \mathcal{K} A(\Pi_n B_n - 1)\bar{\Phi}\|_0 \\
& \leq C_1 a \{ \|\Pi_n (B_n \bar{\Phi} - \bar{\Phi})\|_r + \|\Pi_n \bar{\Phi} - \bar{\Phi}\|_r \} \\
& \leq C_1 a \{ \|B_n \bar{\Phi} - \bar{\Phi}\|_r + \|\Pi_n \bar{\Phi} - \bar{\Phi}\|_r \}
\end{aligned}$$

for $0 \leq s \leq (r-1)m$. That is,

$$\begin{aligned}
& \|\mathcal{K} A(\Pi_n B_n - 1)\bar{\Phi}\|_{r-1} \\
& \leq C_1 a \{ \|B_n \bar{\Phi} - \bar{\Phi}\|_r + \|\Pi_n \bar{\Phi} - \bar{\Phi}\|_r \} . \tag{8.5}
\end{aligned}$$

We combine (8.4), (8.5), and (7.18) to obtain

$$\begin{aligned}
\|(T_n - T)\bar{\Phi}\|_{r-1} & \leq \|B_n \bar{\Phi} - \bar{\Phi}\|_{r-1} + \|\Pi_n \bar{\Phi} - \bar{\Phi}\|_{r-1} \\
& \quad + C_1 a \{ \|B_n \bar{\Phi} - \bar{\Phi}\|_r + \|\Pi_n \bar{\Phi} - \bar{\Phi}\|_r \} \\
& \leq C \{ \|B_n \bar{\Phi} - \bar{\Phi}\|_r + \|\Pi_n \bar{\Phi} - \bar{\Phi}\|_r \} \tag{8.6}
\end{aligned}$$

where $C = 1 + \frac{M \alpha_a}{\alpha}$.

Let u denote the solution in G of (8.2). Then,
 $Tu = g$. We have

$$\begin{aligned} \text{Lemma 8.1: } \|T_n u - \Pi_n B_n g\|_r \\ \leq C \|(B_n - 1)u\|_{r+1} + C \|(\Pi_n - 1)u\|_{r+1} \\ + \|(B_n - 1)g\|_r + \|(\Pi_n - 1)g\|_r \end{aligned}$$

Proof: The lemma is an immediate consequence of relations
 (8.4) and (7.18).

9. The Programming Problem.

First, we combine relation (3.6) and Theorem 5.1 to define

$$\beta_r = \frac{5t_e^r a^{-r} \Gamma(r\alpha+1) (1-\epsilon)^\alpha \|g\|_{\infty,t}}{4[(1-\epsilon)^\alpha - M^r(\alpha) t^\alpha]} . \quad (9.1)$$

In this section, the finite difference operator Δ^s will have an understood subscript n^{-1} .

We will define a collection $\mathfrak{F}_n$ of functions f satisfying a number of conditions. These conditions will be labeled (F1), (F2), etc., as they appear.

(F1) There is a function $F(x,y)$ whose domain contains $I_n \times J_n$ such that

$$f = \prod_n B_n F .$$

(F2) If $f = \prod_n B_n F$, then for all integers $p, q, s \geq 0$ such that

$$\Delta^s F(pn^{-1}, qn^{-1})$$

can be calculated using points in $I_n \times J_n$ it is true that

$$n^s \Delta^s F(pn^{-1}, qn^{-1}) \leq \beta_r(s) ,$$

where $r(s) = \min \{r : mr \geq s\}$ and m is the fixed order of the operator A .

We postpone the complete specification of $\mathfrak{F}_n$ to note Lemma 9.1: The G-solution u of $Tu = g$ satisfies condition (F2).

Proof: We shall denote $u_{pq} = u(pn^{-1}, qtn^{-1})$. Then,

$$\begin{aligned}
 |n^s \Delta^s u_{pq}| &= |D^s u(\xi, qtn^{-1})| \\
 &\leq \|D^s u\|_0 \leq \|u\|_{r(s)} \\
 &\leq \|u\|_{\infty, t_e} t_e^r a^{-r} \Gamma(r\alpha+1) \\
 &\leq \beta_{r(s)} .
 \end{aligned}$$

Now, our eventual goal will be to study

$$e_n(\mathfrak{U}_n) = \inf_{f \in \mathfrak{U}_n} e_n(f) , \quad (9.2)$$

$$\text{where } e_n(f) = \|Tf - \Pi_n B_n g\|_0 . \quad (9.3)$$

We shall show that $\Pi_n B_n u \in \mathfrak{U}_n$. Then, we shall estimate $e_n(\mathfrak{U}_n)$ by

$$e_n(\Pi_n B_n u) = \|T_n u - \Pi_n B_n g\|_0 \quad (9.4)$$

through the use of Lemma 8.1.

Since we want $\Pi_n B_n u \in \mathfrak{U}_n$, we shall study u to develop our final condition on $\mathfrak{U}_n$.

First, we observe, for $s = 0, 1, \dots$,

$$\begin{aligned}
 D^s u &= D^s g + \int_0^y K(y, \eta) A D^s u(x, \eta) d\eta , \\
 \|D^s u_y - D^s u_{y+\Delta y}\|_0 &\leq \|D^s g_y - D^s g_{y+\Delta y}\|_0 \\
 &\quad + C \max(\Delta y, \overline{\Delta y}^a) \|D^s A u\|_0
 \end{aligned}$$

by Lemma 2.1. Now, as before,

$$\|D^s A u\|_0 = \left\| \sum_{k=0}^m a_{k,1} D^{k+s} u \right\|_0$$

$$\begin{aligned} &\leq a \max_{0 \leq k \leq m} \|D^{k+s} u\|_0 \\ &\leq a \|u\|_{r+1} \text{ if } 0 \leq s \leq mr. \end{aligned}$$

Thus, (see (4.9)), for $0 \leq s \leq mr$,

$$\Omega_s(\delta, u) \leq \Omega_s(\delta, g) + aC \max(\delta, \delta^\alpha) \|u\|_{r+1}. \quad (9.5)$$

Now, we shall assume that g is such that

$$\Omega_s(\delta, g) \leq P\delta, \quad 0 \leq s \leq m, \quad (9.6)$$

for some constant P .

Finally, we state the last requirement on $\mathfrak{U}_n$:

(F3) If $f = \Pi_n B_n F$, then for all integers p and q such that (pn^{-1}, qn^{-1}) and $(pn^{-1}, (q+1)n^{-1})$ belong to $I_n \times J_n$, it is true that

$$|F_{p,q+1} - F_{p,q}| \leq C_1 \max(n^{-1}, n^{-\alpha}), \quad (9.7)$$

where $C_1 \geq P + aC \|u\|_2$. (We can use the results of Theorem 5.1 to estimate $\|u\|_2$ to be sure C_1 satisfies the above condition.) Moreover, if $\alpha < 1$, we choose a fixed number d such that $0 < d \leq 1$, and we require

$$|F_{p,q+s} - F_{p,q}| \leq C_1 s^\alpha n^{-\alpha} \ell^\alpha \quad (9.8)$$

for all integers s such that $s \leq nd/\ell$.

Now that $\mathfrak{U}_n$ is fully defined, we have

Lemma 9.2: $\Pi_n B_n u \in \mathfrak{U}_n$.

Proof: Lemma 9.1, relation (9.5), and relation (9.6).

Lemma 9.3: Let $f_n \in \mathfrak{U}_n$, $n = 2, 3, \dots$. Then the sequence

$\{f_n\}$ is equicontinuous.

Proof: By Lemma 4.3, for $\alpha \geq 1$, we have

$$\begin{aligned} |f_n(x, y) - f_n(x, y')| &\leq |y - y'| n C_1 \max(n^{-1}, n^{-\alpha}) \\ &\leq C_1 |y - y'|. \end{aligned} \quad (9.9)$$

Now, if $\alpha < 1$, we restrict y and y' so that $|y - y'| \leq d$. Then, as in the proof of Lemma 4.3, if y and y' are both in J_n , by (9.8) we have

$$|f_n(x, y) - f_n(x, y')| \leq C_1 |y - y'|^\alpha. \quad (9.10)$$

If $j\ell \leq ny \leq (j+1)\ell \leq i\ell \leq y'n < (i+1)\ell$, we have, by (9.8),

$$\begin{aligned} |f_n(x, y) - f_n(x, y')| &\leq |f_n(x, y) - f_n(x, \frac{(j+1)\ell}{n})| \\ &\quad + |f_n(x, \frac{(j+1)\ell}{n}) - f_n(x, \frac{i\ell}{n})| + |f_n(x, \frac{i\ell}{n}) - f_n(x, y')| \\ &\leq (\frac{(j+1)\ell}{n} - y) \frac{n}{\ell} (C_1 n^{-\alpha} \ell^\alpha) + C_1 [\frac{(i-j-1)\ell}{n}]^\alpha \\ &\quad + (y' - \frac{i\ell}{n}) \frac{n}{\ell} (C_1 n^{-\alpha} \ell^\alpha) \end{aligned} \quad (9.11)$$

$$\leq 2C_1 (\frac{\ell}{n})^\alpha + C_1 [\frac{(i-j-1)\ell}{n}]^\alpha. \quad (9.12)$$

Let $\ell n_0^{-1} \leq |y - y'| \leq \ell(n_0 - 1)^{-1}$. For $n \geq n_0$, (9.12) may be written as

$$|f_n(x, y) - f_n(x, y')| \leq 3C_1 |y - y'|^\alpha. \quad (9.13)$$

However, if $\ell n^{-1} > |y - y'|$, then we must have $i = j+1$, and (9.11) may be written as

$$\begin{aligned}
|f_n(x,y) - f_n(x,y')| &\leq |y'-y|C_1\left(\frac{1}{n}\right)^{\alpha-1} \\
&\leq C_1|y'-y|^{\alpha}.
\end{aligned} \tag{9.14}$$

Thus, in any case, if $|y-y'| < d$, we have

$$|f_n(x,y) - f_n(x,y')| \leq C_2 \max(|y-y'|, |y-y'|^\alpha). \tag{9.15}$$

Since, by (7.3),

$$\|DB_n\varphi\|_0 \leq \max_{0 \leq v \leq n-1} |n\Delta\varphi(\frac{v}{n})|,$$

we use property (F2) to write

$$\begin{aligned}
|f_n(x,y) - f_n(x',y)| &= |x-x'| \cdot |Df_n(\xi,y)| \\
&\leq \beta_1|x-x'|,
\end{aligned} \tag{9.16}$$

and the lemma is proved.

Finally, the programming problem consists of the following process:

We seek a function $U_n(x,y)$ whose domain is $I_n \times J_n$ where U_n is such that

$$u_n = \Pi_n B_n U_n \in \mathfrak{U}_n, \tag{9.17}$$

and such that

$$e_n(u_n) = e_n(\mathfrak{U}_n). \tag{9.18}$$

In fact, it will be sufficient for our needs if we find a function u_n such that

$$e_n(u_n) \leq e_n(\Pi_n B_n u) . \tag{9.19}$$

Thus, there is no need that u_n be unique. Of course, the particular algorithm used will produce a definite u_n .

We note that

$$\|u_n\|_0 \leq \beta_0, \quad (9.20)$$

and we apply Lemma 9.3 to obtain

Theorem 9.1: Let $u_n \in \mathfrak{U}_n$, $n = 2, 3, \dots$. Then there is a subsequence $\{u_{n_k}\}$ which is Cauchy in the norm $\|\cdot\|_0$.

Proof: Ascoli's theorem.

10. Convergence.

Let u_n denote the function in $\mathfrak{U}_n$ given by our programming problem. Let $v \in C(I, J)$ be the uniform limit of $\{u_{n_k}\} \subset \{u_n\}$ where we have selected a subsequence from $\{u_n\}$ by means of Theorem 9.1.

Lemma 10.1: $D^s v$ exists for all s , and, for $s \leq m_r$,

$$\|D^s v\|_0 \leq \|v\|_r \leq \beta_r.$$

Proof: Since every u_n satisfies condition (F2), v will satisfy the hypotheses of Theorem 6.1, and the lemma follows.

Our situation is this: we know that v satisfies parts 1 and 2 of (3.13), but we have to show that v satisfies part 3 of (3.13). That is, we want to show $v \in G$.

Lemma 10.2: $D^s v \in C(I, J)$ for $s = 0, 1, 2, \dots$. Thus, $v \in G$.

Proof: The result is known for $s = 0$. Also, by (9.15), for $|y - y'| < d$,

$$|v(x, y) - v(x, y')| \leq C_2 \max(|y - y'|, |y - y'|^\alpha).$$

$$\text{Thus, } \lim_{\epsilon \rightarrow 0} \Omega_0(\epsilon, v) = 0. \quad (10.1)$$

We shall show inductively that

$$\lim_{\epsilon \rightarrow 0} \Omega_s(\epsilon, v) = 0, \quad s = 0, 1, \dots. \quad (10.2)$$

Assume the result for some $s \geq 0$, and set

$$v(x, y) = D^s v(x, y).$$

$$\text{Then, } \|D^1 v\|_0 \leq \beta_{r(s+1)}. \quad \text{Also,}$$

$$v(x+h, y+k) - v(x, y+k) = hDv(\xi+\eta, y+k) ,$$

and

$$v(x+h, y) - v(x, y) = hDv(\xi, y)$$

where ξ and $\xi+\eta$ are somewhere between x and $x+h$. We shall denote θ_i , $i = 1, 2, 3, 4$ where $|\theta_i| \leq 1$. Now,

$$Dv(\xi+\eta, y+k) - Dv(x, y+k) = \theta_1 h \beta_{r(s+2)} ,$$

and

$$Dv(\xi, y) - Dv(x, y) = \theta_2 h \beta_{r(s+2)} .$$

Thus,

$$v(x+h, y+k) - v(x, y) = hDv(x, y+k) + \theta_1 h^2 \beta_{r(s+2)}$$

and

$$v(x+h, y) - v(x, y) = hDv(x, y) + \theta_2 h^2 \beta_{r(s+2)} .$$

$$\therefore, \quad Dv(x, y+k) - Dv(x, y) = h \beta_{r(s+2)} (\theta_2 - \theta_1)$$

$$\begin{aligned} &+ \frac{1}{h} \{ [v(x+h, y+k) - v(x+h, y)] \\ &\quad - [v(x, y+k) - v(x, y)] \} \\ &= h \beta_{r(s+2)} (\theta_2 - \theta_1) + \frac{\theta_4 - \theta_3}{h} \Omega_0(k, v) . \end{aligned}$$

$$\therefore, \quad \Omega_0(k, Dv) \leq 2|h|B + \frac{2}{|h|} \Omega_0(k, v) . \quad (10.3)$$

Given $\epsilon > 0$, select h so that

$$\frac{1}{2}\epsilon \leq 2|h|B \leq \epsilon ,$$

and we have

$$\Omega_0(k, Dv) \leq \epsilon + \frac{8B}{\epsilon} \Omega_0(k, v) . \quad (10.4)$$

Then, with ϵ and h fixed, we have

$$\lim_{k \rightarrow 0} \Omega_0(k, Dv) \leq \epsilon, \quad (10.5)$$

and the induction is complete.

Finally, combining (10.2) and Lemma 10.1, we have

$$\begin{aligned} & |D^s v(x+h, y+k) - D^s v(x, y)| \\ & \leq |D^s v(x+h, y+k) - D^s v(x+h, y)| \\ & \quad + |D^s v(x+h, y) - D^s v(x, y)| \\ & \leq \Omega_0(k, D^s v) + |h| \beta_{r(s+1)} \\ & \rightarrow 0 \text{ as } h, k \rightarrow 0, \end{aligned} \quad (10.6)$$

and the lemma is proved.

Lemma 10.3: There exists $\{u_{n_j}\} \subset \{u_{n_k}\}$ such that $u_{n_j} \rightarrow v$ in $\mathcal{A}^1(I, J)$ as $j \rightarrow \infty$.

Proof: We know that $\{u_n\}$ is equicontinuous. The method of proof in the above lemma shows that $\{D^s u_n\}$ is equicontinuous for each fixed s . As D^s is a closed operator, if $\{D^s u_{n_j}\}$ is a convergent subsequence of $\{D^s u_{n_k}\}$, then $D^s u_{n_j} \rightarrow D^s v$ in uniform norm. Thus, for each s , $D^s v$ is the uniform limit of $D^s u_{n_j}$, and since $u_{n_j} \in G_t$, we have $u_{n_j} \rightarrow v$ in $\mathcal{A}^1(I, J)$.

Theorem 10.1: $Tv = g$. Thus, $v = u$, and $u_n \rightarrow u$ in $C(I, J)$ as $n \rightarrow \infty$.

Proof: $\|Tv - g\|_0 \leq \|(T - T_{n_j})v\|_0 + \|T_{n_j}(v - u_{n_j})\|_0$

$$\leq \|T_{n_j} u_{n_j} - T_{n_j} B_{n_j} g\|_0 + \|T_{n_j} B_{n_j} g - g\|_0.$$

$$\begin{aligned} \text{By (9.19), } \|T_{n_j} u_{n_j} - \Pi_{n_j} B_{n_j} g\|_0 &= e_{n_j}(u_{n_j}) \\ &\leq e_{n_j}(\Pi_{n_j} B_{n_j} u) \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

By (8.6), (7.16), and (4.10),

$$\|(T - T_{n_j})v\|_0 \leq C\{\|B_{n_j} v - v\|_1 + \|\Pi_{n_j} v - v\|_1\} \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Similarly, by (7.18), $\|\Pi_{n_j} B_{n_j} g - g\|_0 \rightarrow 0$. Finally, as in the development of (8.5),

$$\|T_{n_j}(v - u_{n_j})\|_0 \leq C_1 \max_{0 \leq i \leq m} \|D^i(v - u_{n_j})\|_0 \rightarrow 0 \text{ as } j \rightarrow \infty.$$

Thus, $Tv = g$; and, by uniqueness, $v = u$, and $u_n \rightarrow u$ as $n \rightarrow \infty$.

11. Cauchy Problems.

There are several special cases in which our integro-differential problem is equivalent to a well-known Cauchy problem.

Example 1: Let

$$K(y, \eta) = (y - \eta) , \quad (11.1)$$

$$g(x, y) = yg_1(x) + g_2(x) , \quad (11.2)$$

$$\text{and} \quad A = D . \quad (11.3)$$

Then, the integro-differential problem is equivalent to

$$\left. \begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{\partial u}{\partial x} \\ u(x, 0) &= g_2(x) \\ \frac{\partial u}{\partial y} \Big|_{y=0} &= g_1(x) \end{aligned} \right\} \quad (11.4)$$

Example 2: Let (11.1) and (11.2) hold. Let

$$A = D^2 \quad (11.5)$$

and we have

$$\left. \begin{aligned} \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} &= 0 \\ u(x, 0) &= g_2(x) \\ \frac{\partial u}{\partial y} \Big|_{y=0} &= g_1(x) \end{aligned} \right\} \quad (11.6)$$

Example 3: Let (11.1), (11.2) hold, and let

$$A = -D^2 . \quad (11.7)$$

Then, we have

$$\left. \begin{aligned} \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} &= 0 \\ u(x, 0) &= g_2(x) \\ \frac{\partial u}{\partial y} \Big|_{y=0} &= g_1(x) \end{aligned} \right\} \quad (11.8)$$

Example 4: Let

$$K(y, \eta) \equiv 1 , \quad (11.9)$$

$$g(x, y) = g_2(x) , \quad (11.10)$$

and $A = D^2 . \quad (11.11)$

Then, the integro-differential problem is equivalent to

$$\left. \begin{aligned} \frac{\partial u}{\partial y} &= \frac{\partial^2 u}{\partial x^2} \\ u(x, 0) &= g_2(x) \end{aligned} \right\} \quad (11.12)$$

For these examples, the series

$$\sum_{s=0}^{\infty} \chi^{sA} g$$

has the well-known forms

$$\begin{aligned} \sum_{s=0}^{\infty} \left(\frac{y^{2s+1}}{(2s+1)!} g_1^{(s)}(x) + \frac{y^{2s}}{(2s)!} g_2^{(s)}(x) \right) , \\ \sum_{s=0}^{\infty} \left(\frac{y^{2s+1}}{(2s+1)!} g_1^{(2s)}(x) + \frac{y^{2s}}{(2s)!} g_2^{(2s)}(x) \right) , \end{aligned}$$

$$\sum_{s=0}^{\infty} (-1)^s \left(\frac{y^{2s+1}}{(2s+1)!} g_1^{(2s)}(x) + \frac{y^{2s}}{(2s)!} g_2^{(2s)}(x) \right) ,$$

and $\sum_{s=0}^{\infty} \frac{y^s}{s!} g_2^{(2s)}(x)$ for examples 1, 2, 3, and 4,

respectively.

If $g_1(x) = \sin \pi x$ and $g_2(x) = \cos \pi x$, these forms are

$$\cos \pi x \left\{ 1 - \frac{\pi y^3}{3!} - \frac{\pi^2 y^4}{4!} + \frac{\pi^3 y^7}{7!} + \frac{\pi^4 y^8}{8!} - \dots \right\}$$

$$+ \sin \pi x \left\{ y + \frac{\pi y^2}{2!} - \frac{\pi^2 y^5}{5!} - \frac{\pi^3 y^6}{6!} + \frac{\pi^4 y^9}{9!} + \dots \right\} ,$$

$$\frac{1}{\pi} \sin \pi x \sin \pi y + \cos \pi x \cos \pi y ,$$

$$\frac{1}{\pi} \sin \pi x \sinh \pi y + \cos \pi x \cosh \pi y ,$$

and $e^{-\pi^2 y} \cos \pi x$, respectively.

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