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MELTING WITHIN A SPHERICAL ENCLOSURE

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ABSTRACT

MELTING WITHIN A SPHERICAL ENCLOSURE

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The problem considered is that of the melting of a phase change material, initially at its saturation temperature, which is enclosed within a spherical shell whose surface temperature is suddenly raised to some fixed value. The density of the solid is assumed to exceed the density of the liquid, the implication being that the solid will continually drop toward the bottom of the shell as melting progresses. This bulk motion of the solid generates a flow field within the liquid, which gives rise to shear and pressure forces that must balance the weight of the solid.

The energy equation written for the liquid region is solved in conjunction with an interface heat balance equation, while the velocity field is calculated approximately by assuming a parabolic profile for the polar component of velocity and neglecting the radial component. After making a suitable variable transformation, the energy equation and the interface equation are expressed in finite difference form. The energy equation is solved by the alternating direction method, while the interface equation is solved using a modified Newton-Raphson procedure.

Experimental evidence confirms the dropping solid hypothesis, and has permitted a limited quantitative verification of the mathematical model.

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To the memory of my grandparents,

William Cullen Higdon

Bessie Smith Higdon

and to my sister,

Miriam Moore

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NOMENCLATURE

Dimensional Variables

T	Temperature
T_3	Shear force at interface
T_4	Shear force at wall
t	Time
r	Radial coordinate
θ	Polar coordinate
ϕ	Azimuthal coordinate
r^*	Interface position
x	Abscissa variable
y	Ordinate variable
R	Radius of shell
$\hat{U}_r$	Radial position unit vector
$\hat{U}_\theta$	Polar position unit vector
$\hat{n}$	Unit vector normal to interface
$\hat{U}_t$	Unit vector tangential to interface
$\hat{k}$	Unit vector in y direction
V_r	Radial velocity component
V_θ	Polar velocity component
V_D	Dropping speed
$\vec{V}_D$	Dropping velocity
$\vec{V}_{int}$	Interface velocity
V_z	Z component of velocity in local cylindrical coordinates
F	Surface function
A	Area
P	Pressure
h	Gap width at $\theta = \pi$
$\mathcal{Q}$	$R\Delta\theta$
B_{j+1}	Liquid volume bounded by θ_j , θ_{j+1} , and shifted interface.
C_{j+1}	Liquid volume bounded by θ_j , θ_{j+1} , and interface
J	Polar mesh descriptor
I	Radial mesh descriptor

ρ_L	Density of liquid
ρ_s	Density of solid
h_L	Enthalpy of liquid
h_s	Enthalpy of solid
h_{Ls}	Enthalpy of fusion
c_p	Specific heat
k	Thermal conductivity
α	Thermal diffusivity
μ	Dynamic viscosity
g	Acceleration due to gravity
dV_p/dt	Volumetric flow rate

Non Dimensional Variables

η'	Radial coordinate
η	Transformed radial coordinate
η^*	Interface position
F_o	Fourier number
Re	Reynolds number
Pr	Prandtl number
Ste	Stefan number
$v_{\eta'}$	Radial velocity component
v_{θ}	Polar velocity component
ψ	Temperature
VD	Dropping speed
EN, ENN, ENT, ETT, ET	Energy equation coefficients
$\eta^{*'} $	Shifted interface position
ψ^*	Intermediate value of ψ_{n+1}

Subscripts

n	Time level
j	θ descriptor
i	η descriptor
r	Radial direction
θ	Polar direction
t	Tangential direction
D	Dropping
sat	Saturation conditions
W	Wall
L	Liquid
s	Solid
int	Interface
S	Surface of solid

I. INTRODUCTION AND LITERATURE SURVEY

I.1 Introduction

Phase change thermal storage devices find application in cases where a large amount of energy must be transferred and stored but only a limited temperature difference is available as the driving potential. This equipment is already being marketed by elements of the solar industry as a space saving alternative to rock beds, and the Department of Energy has a multimillion dollar program underway to develop these devices and rate them. The growing number of uses for these systems has encouraged researchers to study the details of the phase change process itself, so that optimum designs can be developed in a systematic fashion.

The melting process is characterized by the presence of both a solid and liquid, separated by an interface, the position of which is not known a priori. The energy equation written for the solid and liquid regions is linear; the interface equation is not, and therein lies the chief difficulty associated with this class of problems.

This work investigates the melting of a substance contained within a spherical shell. The solid's density is assumed to exceed that of the liquid, the implication being that the solid will continually descend as material is melted away from its underside. This movement generates a fluid motion which contributes a convective term to the energy equation written for the liquid region.

The solid is assumed to be at the saturation temperature throughout. The energy equation written for the liquid region and

the interface equation are first transformed to a convenient domain and there solved by the method of finite differences.

1.2. Literature Survey

Murray and Landis [1] introduced two important techniques for accommodating the presence of the interface in a finite difference mesh. One technique was the variable space method. This technique employs equal size space increments which change in size as the fusion front moves. Lazaridis [2] presented a general numerical procedure for solving multidimensional melting and solidification problems. He used a fixed mesh and approximated the temperature near the interface with a quadratic equation. Both Murray and Landis and Lazaridis employed explicit differencing procedures which restrict the size of the time step.

Few studies have included the effect of convection. Sparrow, et al. [3] included the effects of natural convection in their numerical study of the melting of a phase change material surrounding a vertical cylindrical heating element. Kuiken [4] studied the solidification of a liquid onto a moving sheet. He adopted the assumptions of boundary layer theory and was able to find a similarity transformation. Kroeger and Ostrach [5] included free convection terms in their formulation of a continuous casting process. They concluded that although significant flow patterns could develop, the effect that convection had on the interface position was negligible. Vanier and Tien [6] experimentally determined Nusselt numbers for the melting of ice spheres in an essentially infinite medium. Boger and Westwater [7] performed

experiments on the freezing and melting of ice slabs and discovered that free convection was not present for Rayleigh numbers less than 1700. The authors give detailed instructions on how to properly choose the constants that make up the Rayleigh number. They found that free convection would be adequately accounted for in the one dimensional melting and freezing of a slab if an effective thermal conductivity was chosen, and found this to hold true for Rayleigh numbers up to 10^7 .

Solomon [8] presents a formula for the melting time of a phase change material (PCM) body of rather general shape subject to a constant wall temperature boundary condition. He developed the formula from a consideration of computer calculations, exact solutions, and quasi-stationary approximations. Riley, et al. [9] considered the one dimensional inward solidification of a sphere initially at the fusion temperature and suddenly subjected to a lowering of the wall temperature. They assumed that the liquid remained at the fusion temperature, and used perturbation techniques to solve the problem for the case of small Stefan numbers. Tao [10] presents early numerical results for the freezing of a sphere subject to a constant heat transfer coefficient boundary condition. Grimado [11] considered the case of symmetric melting and solidification of spheres.

Nicholas [27] and Nicholas and Bayazitoglu [28] were the first researchers to consider the effects of unequal solid and liquid densities in their study of melting within a cylindrical enclosure.

II. ANALYSIS

II.1 Description of the Problem

We consider a phase change material, initially solid and at its saturation temperature, T_{sat} , which completely fills a spherical shell. The shell is assumed to have negligible thermal resistance. At time $t=0$, the temperature of the shell wall is suddenly raised to some value T_w , and melting commences.

The geometry used to describe the problem is shown in figure (2.1.1.a). The boundary separating the liquid and solid phases will henceforth be referred to as the interface, and will be identified by the position vector $r\hat{U}_r$, as shown in figure (2.1.1.b). The interface is assumed to be a sharply defined surface; we assume melting occurs at a precise temperature and there is no dendritic structure or slush present at the interface. Work done by Thomas and Westwater [12] indicates that this is an idealization. Symmetry suggests that temperature, velocity, and interface position can only be functions of r , θ , and t . Constant thermophysical properties are assumed. The difference in density between the solid and liquid phases is considered only so far as it suggests a continual repositioning of the solid toward the bottom of the shell.

II.2. The Energy Equation

The energy equation written for spherical coordinates is

$$\rho_L C_P \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} \right) = k \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial T}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right) \quad (2.2.1)$$

(a) Defining the coordinate system.

(b) Defining r^* .

Figure 2.1.1.

The boundary equations are

$$T(R, \theta, t) = T_W \quad (2.2.2.a)$$

$$T(r^*, \theta, t) = T_{sat} \quad (2.2.2.b)$$

$$\partial T / \partial \theta (r, 0, t) = 0 \quad (2.2.2.c)$$

$$\partial T / \partial \theta (r, \pi, t) = 0 \quad (2.2.2.d)$$

and the initial condition is

$$T(r, \theta, 0) = T_{sat} \quad (2.2.2.e)$$

The following non-dimensional parameters are introduced.

$$\eta' = r/R; \eta^* = r^*/R; F_o = \alpha t/R^2; v_\theta = V_\theta/V_D; v_r = V_r/V_D;$$

$$\Psi = (T - T_{sat}) / (T_w - T_{sat}); Re = \rho_L V_D R / \mu; Pr = \mu Cp / k$$

All material properties refer to the liquid state.

Introducing these parameters into the energy equation and rearranging gives

$$\begin{aligned} \frac{\partial \Psi}{\partial F_o} + Re Pr \left(v_r \frac{\partial \Psi}{\partial \eta'} + \frac{v_\theta}{\eta'} \frac{\partial \Psi}{\partial \theta} \right) = \\ \frac{\partial^2 \Psi}{\partial \eta'^2} + \frac{\cot \theta}{\eta'^2} \frac{\partial \Psi}{\partial \theta} + \frac{1}{\eta'^2} \frac{\partial^2 \Psi}{\partial \theta^2} + \frac{2}{\eta'} \frac{\partial \Psi}{\partial \eta'} \end{aligned} \quad (2.2.3)$$

The boundary and initial conditions become

$$\Psi(1, \theta, F_o) = 1 \quad (2.2.4.a)$$

$$\Psi(0, \theta, F_o) = 0 \quad (2.2.4.b)$$

$$\partial \Psi / \partial \theta (\eta', 0, F_o) = 0 \quad (2.2.4.c)$$

$$\partial \Psi / \partial \theta (\eta', \pi, F_o) = 0 \quad (2.2.4.d)$$

$$\Psi(\eta', \theta, 0) = 0 \quad (2.2.4.e)$$

II.3. The Interface Equation

We apply a heat balance to the fixed control volume about the interface as it is drawn in figure (2.3.1).

$$\rho_s h_s \vec{V}_D \cdot \hat{n} + k \nabla T \cdot \hat{n} - \rho_L h_L \vec{V}_D \cdot \hat{n} = -\rho_s h_{Ls} \vec{V}_{int} \cdot \hat{n} \quad (2.3.1)$$

$\vec{V}_D$ is the velocity of the solid caused by the continual dropping of the solid due to gravity, and $\hat{n}$ is the unit normal to the surface directed into the liquid phase. The above heat balance neglects energy fluxes entering the upper and lower ends of the control volume. Since the temperature of the liquid in the region near the interface is everywhere T_{sat} , the net energy flux entering these sides is felt to be negligible.

Equation (2.3.1) can be rearranged.

$$-\rho_s h_{Ls} (\vec{V}_{int} - \vec{V}_D) \cdot \hat{n} = k \nabla T \cdot \hat{n} \quad (2.3.2)$$

Following a procedure suggested by Patel [13] and Ozisik [28], the interface equation can be cast into a more usable form.

Define the scalar function F such that

$$F = r - r^* (\theta, t) \quad (2.3.3)$$

The unit normal $\hat{n}$ can be expressed by

$$\hat{n} = \frac{\nabla F}{|\nabla F|} = \frac{\nabla F}{|\nabla F|} \quad (2.3.4)$$

The velocities can be written as

$$\vec{V}_{int} = \dot{r} \hat{U}_r + r \dot{\theta} \hat{U}_\theta \quad (2.3.5)$$

$$\vec{V}_D = v_D \hat{k} \quad (2.3.6)$$

Since both T and F are constant valued functions at the interface, we can write,

Figure 2.3.1. Control volume for heat balance on interface.

$$\frac{\partial T}{\partial r} dr + \frac{\partial T}{\partial \theta} d\theta + \frac{\partial T}{\partial t} dt = 0 \quad (2.3.7)$$

$$\frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial \theta} d\theta + \frac{\partial F}{\partial t} dt = 0 \quad (2.3.8)$$

Using equations (2.3.4) and (2.3.5) we can write

$$\begin{aligned} \vec{V}_{int} \cdot \hat{n} &= \vec{V}_{int} \cdot \frac{\nabla F}{|\nabla F|} \\ &= (\dot{r} \hat{U}_r + r\dot{\theta} \hat{U}_\theta) \cdot \left(\frac{\partial F}{\partial r} \hat{U}_r + \frac{1}{r} \frac{\partial F}{\partial \theta} \hat{U}_\theta \right) \frac{1}{|\nabla F|} \end{aligned} \quad (2.3.9)$$

or,

$$\vec{V}_{int} \cdot \hat{n} = \frac{\partial r^*}{\partial t} \frac{1}{|\nabla F|} \quad (2.3.9)$$

If we write

$$\hat{k} = \cos \theta \hat{U}_r - \sin \theta \hat{U}_\theta$$

then

$$\begin{aligned} \vec{V}_D \cdot \hat{n} &= V_D (\cos \theta \hat{U}_r - \sin \theta \hat{U}_\theta) \cdot \left(\frac{\partial F}{\partial r} \hat{U}_r + \frac{1}{r} \frac{\partial F}{\partial \theta} \hat{U}_\theta \right) \frac{1}{|\nabla F|} \\ &= V_D \left(\cos \theta + \frac{1}{r} \sin \theta \frac{\partial r^*}{\partial \theta} \right) \end{aligned} \quad (2.3.10)$$

Furthermore, we can write

$$\nabla T \cdot \hat{n} = \left(\frac{\partial T}{\partial r} \hat{U}_r + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{U}_\theta \right) \cdot \left(\frac{\partial F}{\partial r} \hat{U}_r + \frac{1}{r} \frac{\partial F}{\partial \theta} \hat{U}_\theta \right) \frac{1}{|\nabla F|} \quad (2.3.11)$$

We can develop an alternate expression for $\partial T / \partial \theta$. First we solve equation (2.3.8) for $d\theta$, then we substitute this result into equation (2.3.7). The result is

$$\left(\frac{\partial T}{\partial r} - \frac{\partial T}{\partial \theta} \frac{\partial F}{\partial r} / \frac{\partial F}{\partial \theta} \right) dr + \left(\frac{\partial T}{\partial t} - \frac{\partial F}{\partial t} \frac{\partial T}{\partial \theta} / \frac{\partial F}{\partial \theta} \right) dt = 0 \quad (2.3.12)$$

Since dr and dt are independent, one can write

$$\frac{\partial T}{\partial \theta} = \frac{\partial T}{\partial r} \frac{\partial F}{\partial \theta} / \frac{\partial F}{\partial r} \quad (2.3.13)$$

On the boundary, $\frac{\partial F}{\partial \theta} = -\frac{\partial r^*}{\partial \theta}$, so upon substitution of

equation (2.3.13) into equation (2.3.11), we conclude that

$$\nabla T \cdot \hat{n} = \frac{\partial T}{\partial r} \left(1 + \left(\frac{1}{r^*} \frac{\partial r^*}{\partial \theta} \right)^2 \right) \frac{1}{|\nabla F|} \quad (2.3.14)$$

Combining equations (2.3.9), (2.3.10), and (2.3.14) with equation (2.3.2), we arrive at the working relation

$$-\rho_s h_{Ls} \left[\frac{\partial r^*}{\partial t} - V_D \left(\cos \theta + \frac{\sin \theta}{r^*} \frac{\partial r^*}{\partial \theta} \right) \right] = k \frac{\partial T}{\partial r} \left(1 + \left(\frac{1}{r^*} \frac{\partial r^*}{\partial \theta} \right)^2 \right) \quad (2.3.15)$$

Equation (2.3.15) can be non dimensionalized using the parameters introduced for the energy equation. The result is

$$V_D \left(\cos \theta + \frac{\sin \theta}{\eta^*} \frac{\partial \eta^*}{\partial \theta} \right) - \frac{\partial \eta^*}{\partial F_o} = Ste \frac{\partial \Psi}{\partial \eta^*} \left(1 + \left(\frac{1}{\eta^*} \frac{\partial \eta^*}{\partial \theta} \right)^2 \right) \quad (2.3.16)$$

where

$$V_D = R/\alpha \quad V_D \quad (2.3.17)$$

$$Ste = \left(\frac{\rho_L}{\rho_s} \right) \frac{C_p (T_w - T_{sat})}{h_{Ls}} \quad (2.3.18)$$

Ste is the Stefan number.

The boundary conditions for the interface equation reflect the symmetry of the problem.

$$\left. \frac{\partial \eta^*}{\partial \theta} \right|_{\theta=0, \pi} = 0 \quad (2.3.19)$$

We should note the fact that the interface can never touch the shell interface. If it did, there would be a discontinuity in the temperature field. For this reason, the interface is given a false start at time $t=0$.

II.4. Approximate Fluid Mechanics

The equations developed for the temperature distribution and the interface position only apply so long as the origin of the coordinate system lies within the solid region; otherwise a single radial line would cut the interface in two locations and an identification problem would arise. Hence we can only hope to predict the interface position up to the time when $\eta^* (\theta=0)=0$. For much of this time the lower half of the solid remains roughly spherical, with the gap between the solid and the sphere nearly constant and of the order of 2% of the shell's inner radius. The gap width is seen to change slowly with time. In light of these observations, a quasi-steady-state approximation to the velocity field will be made. The width of the gap is small compared to its curvature, so we will approximate the flow within the gap as like unto a pressure induced flow between parallel plates. Since the radial velocity is never larger than V_D , and must decrease rapidly to zero at the shell wall, it is neglected [30].

We know that Poiseuille flow is characterized by a parabolic velocity profile. We will calculate the coefficients of this profile by assuming the velocity is zero at the interface and shell wall, and must satisfy mass continuity. Essentially, we are assuming that the solid and shell remain fixed, and fluid is injected into the gap with negligible momentum to produce the gap flow. In reality, the velocity of the fluid at the interface must be equal to the solid's velocity.

Knowledge of the velocity distribution will allow us to

calculate shear and pressure force distributions. The vertical components of these forces must combine to balance the weight of the solid, which must be corrected for buoyancy effects. The mechanism of achieving a force balance is simple. If the weight is in excess, the solid will drop closer to the shell. This will boost the heat flux and increase the volumetric flow, which must negotiate a more constricted gap than before. Hence the velocities will increase, and so will the shear and pressure forces. The converse applies if the fluid forces are in excess.

The first step in calculating the velocity profile at a particular value of θ , say θ_p , is to find the volumetric flow rate, dV_p/dt , across a surface A, illustrated in figure (2.4.1). Surface A is generated by revolving through one revolution in the ϕ direction the segment of the radial line which spans the gap between the interface and the shell. The angle θ is constant throughout the revolution. The infinitesimal volume of fluid, dV_p , is obviously equal to the amount of melted material within the volume described by $0 \leq \phi \leq 2\pi$, $\eta^* \leq \eta' \leq 1$, and $\theta_p \leq \theta \leq \pi$, which is displaced by the falling solid during the time dt .

Imagine that the motion of the interface during the time dt can be separated into two distinct phases. First, only melting occurs, and the solid velocity is zero. This phase lasts until the final instant of the time interval, whereupon, in the second phase, melting is arrested and the solid suddenly drops to its new equilibrium position. If we could find the volume of fluid that was present at the end of phase one, and subtract from it the volume present

Figure 2.4.1. Defining Surface A and the positive sense of the polar velocity.

Figure 2.4.2. Finite control volume showing forces acting on fluid elements.

at the end of phase two, the difference would be dV_p .

Suppose that the interface at the end of phase one could be described as

$$\eta^* = g_1(\theta) \quad (2.4.1)$$

and the interface at the end of phase two could be described by

$$\eta^* = g_2(\theta) \quad (2.4.2)$$

Then dV_p would be given by

$$dV_p = 2\pi R^3 \left[\int_{\theta_p}^{\pi} \int_{g_1(\theta)}^1 (\eta')^2 \sin\theta \, d\eta' \, d\theta - \int_{\theta_p}^{\pi} \int_{g_2(\theta)}^1 (\eta')^2 \sin\theta \, d\eta' \, d\theta \right] \quad (2.4.3)$$

If the velocity V_θ is given by

$$\frac{V_\theta}{V_D} = \alpha + \beta \eta' + \gamma (\eta')^2 \quad (2.4.4)$$

where α , β , and γ are functions of θ , then we could write

$$dV_p = -R^2 \Delta t \int_0^{2\pi} \int_{\eta^*}^1 V_D (\alpha + \beta \eta' + \gamma (\eta')^2) \eta' \sin\theta \, d\eta' \, d\phi \quad (2.4.5)$$

The minus sign occurs because the normal to surface A is taken to lie in the $-\hat{U}_\theta$ direction. The two other conditions on the velocity are given by equation (2.4.6)

$$V_\theta(\eta' = \eta^*) = 0 \quad (2.4.6a)$$

$$V_\theta(\eta' = 1) = 0 \quad (2.4.6b)$$

The detailed calculation of dV_p and the coefficients α , β , and γ is presented in Appendix A.

Once the velocity is known as a function of η' and θ , we

can calculate the shear forces exerted on the fluid by the wall. The shear forces are assumed to be positive if they are directed in the $\hat{U}_\theta$ direction. Figure (2.4.2, see pg. 13) shows a finite control volume, identifying areas and forces.

Presumably we know $V_\theta = f(\eta')$ at $\theta = \theta_j$ and $\theta = \theta_{j+1}$. Then a force balance on the control volume produces

$$T_3 + T_4 + A_1 P_1 - A_2 P_2 = \int_{A_2} \rho_L V_\theta^2 dA - \int_{A_1} \rho_L V_\theta^2 dA \quad (2.4.7)$$

where

$$T_3 = \tau_3 A_3 \quad (2.4.8.a)$$

$$T_4 = \tau_4 A_4 \quad (2.4.8.b)$$

The shear stresses are given approximately by

$$\tau_3 = (-\mu) \left(\frac{1}{2} \right) \left(\frac{V_D}{R} \right) \left(\frac{\partial v_\theta}{\partial \eta'} \right) \bigg|_{\substack{\theta = \theta_{j+1} \\ \eta^*_{j+1}}} + \frac{\partial v_\theta}{\partial \eta'} \bigg|_{\substack{\theta = \theta_j \\ \eta^*_j}} \quad (2.4.9.a)$$

$$\tau_4 = (\mu) \left(\frac{1}{2} \right) \left(\frac{V_D}{R} \right) \left(\frac{\partial v_\theta}{\partial \eta'} \right) \bigg|_{\substack{\theta = \theta_{j+1} \\ \eta' = 1}} + \frac{\partial v_\theta}{\partial \eta'} \bigg|_{\substack{\theta = \theta_j \\ \eta' = 1}} \quad (2.4.9.b)$$

The areas are given by

$$A_1 = \pi R^2 \sin \theta_j (1 - (\eta^*_j)^2) \quad (2.4.10.a)$$

$$A_2 = \pi R^2 \sin \theta_{j+1} (1 - (\eta^*_{j+1})^2) \quad (2.4.10.b)$$

$$A_3 = R^2 \int_0^{2\pi} \int_{\theta_j}^{\theta_{j+1}} (\eta^*)^2 \sin\theta \, d\theta \, d\phi \quad (2.4.10.c)$$

$$A_4 = 2\pi R^2 (\cos \theta_j - \cos \theta_{j+1}) \quad (2.4.10.d)$$

Detailed expressions for the momentum flux integral and the area A_3 are developed in Appendix A.

Notice that the shear forces and the momentum fluxes can be calculated independent of equation (2.4.7) on the strength of the assumed velocity profile. After these forces are calculated the pressure distribution is calculated from equation (2.4.7).

We assume we know the pressure at θ equals zero to be zero. Equation (2.4.7) fails to hold if $\theta_{j+1} = \pi$, since A_2 at that point is zero. An alternate expression for the pressure at $\theta = \pi$ must be developed. We will assume we know the pressure and velocity at $\theta = \pi - \Delta\theta$, and that $\Delta\theta$ is sufficiently small that local cylindrical coordinates apply to good approximation. The geometry is described in Figure (2.4.3).

The procedure used is a modification of the procedure developed by Stefan [26]. The continuity equation is

$$\frac{1}{r} \frac{\partial}{\partial r} (r V_r) + \frac{dV_z}{dz} = 0 \quad (2.4.11)$$

The momentum equation is

$$\frac{dP}{dr} = \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r V_r) \right) + \frac{\partial^2 V_r}{\partial z^2} \right] \quad (2.4.12)$$

Figure 2.4.3. Local cylindrical coordinate system employed to develop expression for $p(\theta=\pi)$.

We propose that

$$V_z = f(z) \quad (2.4.13a)$$

$$V_r = k(z, r) \quad (2.4.13b)$$

$$P = P(r) \quad (2.4.13c)$$

From continuity we find that

$$V_r = \frac{-r}{2} \frac{df}{dz} \quad (2.4.14)$$

If we integrate the continuity equation over the volume we get

$$\int_0^{2\pi} \int_0^H \int_0^r \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{-r^2}{2} \frac{df}{dz} \right) r dr dz d\phi + \int_0^{2\pi} \int_0^H \int_0^r r \frac{df}{dz} dr dz d\phi = 0 \quad (2.4.15)$$

or

$$-\pi r^2 \int_0^H \frac{df}{dz} dz + \pi r^2 \left(f(z) \Big|_0^H \right) = 0 \quad (2.4.16)$$

After noting that by equation (2.4.14) the term $\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r V_r) \right)$ is identically zero, we can integrate the remainder of equation (2.4.12) to get

$$\frac{1}{2} \frac{dP}{dr} z^2 + g(r)z + h(r) = V_r \quad (2.4.17)$$

where $g(r)$ and $h(r)$ are constants of integration. The boundary conditions are

$$V_r (z=0) = 0 \quad (2.4.18.a)$$

$$V_r (z=h) = 0 \quad (2.4.18.b)$$

$$V_z (z=0) = 0 \quad (2.4.18.c)$$

$$V_z (z=h) = V_D \quad (2.4.18.d)$$

The resulting expression for V_r is

$$V_r = \frac{h^2}{2\mu} \frac{dP}{dr} \left[\left(\frac{z}{h}\right)^2 - \frac{z}{h} \right] \quad (2.4.19)$$

If we substitute equation (2.4.19) into equation (2.4.16), and integrate, we arrive at

$$P \Big|_{\theta=\pi} = P \Big|_{\theta=\pi-\Delta\theta} - \frac{3\mu\Omega^2}{h^3} V_D \quad (2.4.20)$$

which is the desired expression for the pressure at $\theta=\pi$.

The last remaining task is to formulate the equilibrium expression between the gravity and fluid forces. The shear stress τ_3 is the magnitude of the shear stress the solid exerts on the fluid, and is positive in the $\hat{U}_\theta$ direction. τ_3 is also the magnitude of the shear stress the liquid exerts on the solid if we identify the positive direction to be $-\hat{U}_\theta$. The pressure force on the surface acts along the $-\hat{n}$ direction, since $\hat{n}$ is the normal to the surface, positive into the liquid. Keeping these sign conventions in mind, the force balance on the solid becomes

$$R^3(\rho_s - \rho_L) g \int_0^{2\pi} \int_0^\pi \int_0^{r^*} (\eta')^2 \sin\theta \, d\eta' \, d\theta \, d\phi = - \int_S \tau_3 \hat{U}_\theta \cdot \hat{k} \, dA - \int_S P \hat{n} \cdot \hat{k} \, dA \quad (2.4.21)$$

The details of how these integrals are approximated are given in Appendix A.

II.5. Variable Transformation

To facilitate calculations, we adopt a variable transformation used by Murray and Landis [1]. We replace the variable η' by the

variable η , defined as

$$\eta = \frac{\eta^* - \eta'}{\eta^* - 1} \quad (2.5.1)$$

The derivative expressed in the new coordinate system become

$$\frac{\partial}{\partial \eta'} = \frac{-1}{\eta^* - 1} \frac{\partial}{\partial \eta} \quad (2.5.2.a)$$

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} + \frac{1-\eta}{\eta^* - 1} \frac{\partial \eta^*}{\partial \theta} \frac{\partial}{\partial \eta} \quad (2.5.2.b)$$

$$\frac{\partial^2}{\partial \eta'^2} = \left(\frac{1}{\eta^* - 1} \right)^2 \frac{\partial^2}{\partial \eta^2} \quad (2.5.2.c)$$

$$\begin{aligned} \frac{\partial^2}{\partial \theta^2} = & \frac{\partial^2}{\partial \theta^2} + \frac{2(1-\eta)}{\eta^* - 1} \frac{\partial \eta^*}{\partial \theta} \frac{\partial^2}{\partial \eta \partial \theta} + \left(\frac{1-\eta}{\eta^* - 1} \frac{\partial \eta^*}{\partial \theta} \right)^2 \frac{\partial^2}{\partial \eta^2} + \\ & \left[\frac{1-\eta}{\eta^* - 1} \frac{\partial^2 \eta^*}{\partial \theta^2} - \frac{2(1-\eta)}{(\eta^* - 1)^2} \left(\frac{\partial \eta^*}{\partial \theta} \right)^2 \right] \frac{\partial}{\partial \eta} \end{aligned} \quad (2.5.2.d)$$

$$\frac{\partial}{\partial F_o} = \frac{\partial}{\partial F_o} + \frac{1-\eta}{\eta^* - 1} \frac{\partial \eta^*}{\partial F_o} \frac{\partial}{\partial \eta} \quad (2.5.2.e)$$

The transformed energy equation becomes

$$\frac{\partial \Psi}{\partial F_o} = EN \frac{\partial \Psi}{\partial \eta} + ENN \frac{\partial^2 \Psi}{\partial \eta^2} + ENT \frac{\partial^2 \Psi}{\partial \eta \partial \theta} + ETT \frac{\partial^2 \Psi}{\partial \theta^2} + ET \frac{\partial \Psi}{\partial \theta} \quad (2.5.3)$$

where

$$\begin{aligned} EN = & \left[\frac{\eta-1}{\eta^*-1} \frac{\partial \eta^*}{\partial F_o} + \frac{2}{\eta'(1-\eta^*)} + \frac{\cot \theta}{(\eta')^2} \left(\frac{1-\eta}{\eta^*-1} \frac{\partial \eta^*}{\partial \theta} \right) + \frac{1-\eta}{(\eta')^2 (\eta^*-1)} \frac{\partial^2 \eta^*}{\partial \theta^2} + \right. \\ & \left. \frac{2(\eta-1)}{(\eta')^2 (\eta^*-1)^2} \left(\frac{\partial \eta^*}{\partial \theta} \right)^2 - \frac{\text{Re Pr } v_r}{(1-\eta^*)} + \frac{\text{Re Pr } v_\theta (1-\eta)}{\eta' (1-\eta^*)} \frac{\partial \eta^*}{\partial \theta} \right] \end{aligned} \quad (2.5.4.a)$$

$$ENN = \frac{1}{(\eta^*-1)^2} + \frac{1}{(\eta')^2} \left(\frac{1-\eta}{\eta^*-1} \frac{\partial \eta^*}{\partial \theta} \right)^2 \quad (2.5.4.b)$$

$$ENT = \frac{1}{(\eta')^2} \frac{2(1-\eta)}{\eta^*-1} \frac{\partial \eta^*}{\partial \theta} \quad (2.5.4.c)$$

$$ET = \frac{\cot \theta}{(\eta')^2} - \frac{Re \, Pr \, v_\theta}{\eta'} \quad (2.5.4.d)$$

$$ETT = \frac{1}{(\eta')^2} \quad (2.5.4.e)$$

The boundary conditions do not change, because symmetry requires that $\frac{\partial \eta^*}{\partial \theta}$ be zero at $\theta=0$ and $\theta=\pi$.

The interface equation becomes

$$VD \left(\cos \theta + \frac{\sin \theta}{\eta^*} \frac{\partial \eta^*}{\partial \theta} \right) - \frac{\partial \eta^*}{\partial F_o} = Ste \frac{\partial \Psi}{\partial \eta} \left(\frac{1}{1-\eta^*} \right) \left(1 + \left(\frac{1}{\eta^*} \frac{\partial \eta^*}{\partial \theta} \right)^2 \right) \quad (2.5.5)$$

The boundary conditions for the interface equation do not change.

METHOD OF SOLUTION

III.1. Introduction

The energy and interface equations are solved by finite difference techniques. The purpose of the variable transformation is to fix the interface at $\eta=0$ and the inner shell surface at $\eta=1$. The transformed plane wherein the solution will be sought is a rectangular region with fixed boundaries and constant mesh spacing..

III.2. Solution of Energy Equation

The Crank Nicholson procedure is used to solve the energy equation. Crank Nicholson is used because it is a second order in time procedure and is unconditionally stable. The energy equation written in Crank Nicholson format is

$$\begin{aligned} \frac{\psi_{ijn+1} - \psi_{ijn}}{\Delta F_o} = & \frac{1}{2} EN_{ij} (\psi_{ijn+1} + \psi_{ijn})_{\eta} + \frac{1}{2} ENN_{ij} (\psi_{ijn+1} + \psi_{ijn})_{\eta\eta} \\ & + \frac{1}{2} ENT_{ij} (\psi_{ijn+1} + \psi_{ijn})_{\eta\theta} + \frac{1}{2} ET_{ij} (\psi_{ijn+1} + \psi_{ijn})_{\theta} \\ & + \frac{1}{2} ETT_{ij} (\psi_{ijn+1} + \psi_{ijn})_{\theta\theta} \end{aligned} \quad (3.2.1)$$

where the operators are defined as

$$(A_{ij})_{\eta} = (A_{i+1j} - A_{i-1j})/2\Delta\eta \quad (3.2.2.a)$$

$$(A_{ij})_{\eta\eta} = (A_{i+1j} - 2A_{ij} + A_{i-1j})/(\Delta\eta)^2 \quad (3.2.2.b)$$

$$(A_{ij})_{\eta\theta} = (A_{i+1j+1} - A_{i+1j-1} - A_{i-1j+1} + A_{i-1j-1})/(4\Delta\eta\Delta\theta) \quad (3.2.2.c)$$

$$(A_{ij})_{\theta\theta} = (A_{ij+1} - 2A_{ij} + A_{ij-1})/(\Delta\theta)^2 \quad (3.2.2.d)$$

$$(A_{ij})_{\theta} = (A_{ij+1} - A_{ij-1})/2\Delta\theta \quad (3.2.2.e)$$

A_{ij} can assume the identity of either ψ_{ijn+1} or ψ_{ijn} .

Equation (3.2.1) contains nine unknowns and would be expensive to solve in a straight forward manner, either by Gaussian elimination performed on a band matrix or any of the iterative methods. We avoid this problem by employing the alternating direction method first described by Peaceman and Rachford [14] and Douglas [15]. The particular adaptation employed here to accommodate the mixed derivative was first developed by Asgarpour [16].

The idea behind the alternating direction procedure is to advance forward one time step by employing one or more intermediate solutions, each of which is implicit in only one space variable. Let ψ^*_{ij} denote the first approximation, written implicitly in the η direction

$$\begin{aligned} \frac{\psi^*_{ij} - \psi_{ijn}}{\Delta F_o} = & \frac{1}{2} EN_{ij} (\psi^*_{ij} + \psi_{ijn})_{\eta} + \frac{1}{2} ENN_{ij} (\psi^*_{ij} + \psi_{ijn})_{\eta\eta} \\ & + \frac{1}{2} ENT_{ij} (b \psi^*_{ij} + a \psi_{ijn})_{\eta\theta} + ETT_{ij} (\psi_{ijn})_{\theta\theta} \\ & + ET_{ij} (\psi_{ijn})_{\theta} \end{aligned} \quad (3.2.3)$$

where

$$a + b = 2.$$

The peculiar treatment of the mixed derivative term arises because we do not know how to properly weigh ψ^* and ψ_n .

The second approximation is given by

$$\begin{aligned}
\frac{\psi_{ijn+1} - \psi_{ijn}}{\Delta F_o} &= \frac{1}{2} EN_{ij} (\psi_{ij}^* + \psi_{ijn})_{\eta} + \frac{1}{2} ENN_{ij} (\psi_{ij}^* + \psi_{ijn})_{\eta\eta} \\
&+ \frac{1}{2} ENT_{ij} (b \psi_{ij}^* + a(c\psi_{ijn+1} + d\psi_{ijn}))_{\eta\theta} \\
&+ \frac{1}{2} ETT_{ij} (\psi_{ijn+1} + \psi_{ijn})_{\theta\theta} + \frac{1}{2} ET_{ij} (\psi_{ijn+1} + \psi_{ijn})_{\theta}
\end{aligned}
\tag{3.2.4}$$

where

$$c + d = 1$$

Again, c and d arise because we do not know yet how to account for the θ influence in the mixed derivative.

We subtract equation (3.2.3) from equation (3.2.4)

$$\begin{aligned}
\frac{\psi_{ijn+1} - \psi_{ij}^*}{\Delta F_o} &= \frac{1}{2} ENT_{ij} (ac \psi_{ijn+1} + (ad-a) \psi_{ijn})_{\eta\theta} \\
&+ \frac{1}{2} ETT_{ij} (\psi_{ijn+1} - \psi_{ijn})_{\theta\theta} + \frac{1}{2} ET_{ij} (\psi_{ijn+1} - \psi_{ijn})_{\theta}
\end{aligned}
\tag{3.2.5}$$

We solve equation (3.2.5) for ψ_{ij}^* and substitute this result into equation (3.2.4).

$$\begin{aligned}
\frac{\psi_{ijn+1} - \psi_{ijn}}{\Delta F_o} &= \frac{1}{2} EN_{ij} (\psi_{ijn+1} + \psi_{ijn})_{\eta} + \frac{1}{2} ENN_{ij} (\psi_{ijn+1} + \psi_{ijn})_{\eta\eta} \\
&+ \frac{1}{2} ENT_{ij} (b \psi_{ijn+1} + ac \psi_{ijn+1} + ad \psi_{ijn})_{\eta\theta} \\
&+ \frac{1}{2} ET_{ij} (\psi_{ijn+1} + \psi_{ijn})_{\theta} + \frac{1}{2} ETT_{ij} (\psi_{ijn+1} + \psi_{ijn})_{\theta\theta} \\
&+ \mathcal{O}(\Delta F_o)
\end{aligned}
\tag{3.2.6}$$

If we require that $b+ac=1$ and $ad=1$, we will recover the Crank Nicholson equation with one additional term, a term which will

tend to zero as $(\Delta F_0)^2$. Collecting the equations in a,b,c, and d, we find

$$a + b = 2 \quad (3.2.7.a)$$

$$c + d = 1 \quad (3.2.7.b)$$

$$b + ac = 1 \quad (3.2.7.c)$$

$$ad = 1 \quad (3.2.7.d)$$

There are many solutions to these equations. One which can be used to advantage is to require that

$$a = 2 \quad (3.2.8.a)$$

$$b = 0 \quad (3.2.8.b)$$

$$c = \frac{1}{2} \quad (3.2.8.c)$$

$$d = \frac{1}{2} \quad (3.2.8.d)$$

The working equations are now

$$\begin{aligned} \frac{\psi_{ij}^* - \psi_{ijn}}{\Delta F_0} &= \frac{1}{2} EN_{ij} (\psi_{ij}^* + \psi_{ijn})_n + \frac{1}{2} ENN_{ij} (\psi_{ij}^* + \psi_{ijn})_{nn} \\ &+ ENT_{ij} (\psi_{ijn})_{n\theta} + ETT_{ij} (\psi_{ijn})_{\theta\theta} + ET_{ij} (\psi_{ijn})_{\theta} \end{aligned} \quad (3.2.9)$$

and

$$\begin{aligned} \frac{\psi_{ijn+1} - \psi_{ij}^*}{\Delta F_0} &= \frac{1}{2} ENT_{ij} (\psi_{ijn+1} - \psi_{ijn})_{n\theta} + \frac{1}{2} ETT_{ij} (\psi_{ijn+1} - \psi_{ijn})_{\theta\theta} \\ &+ \frac{1}{2} ET_{ij} (\psi_{ijn+1} - \psi_{ijn})_{\theta} \end{aligned} \quad (3.2.10)$$

The usual case finds $\psi_{ijn+1} = \psi_{ijn}$ to start with. Equation (3.2.9) is used to predict ψ_{ij}^* . Then equation (3.2.10) is applied to give a first approximation to the new ψ_{n+1} . This

predicted value of Ψ_{n+1} is introduced into the mixed derivative term on the right hand side of equation (3.2.10), which is then solved to give a corrected value of Ψ_{n+1} . This is compared with the previous prediction for Ψ_{n+1} and equation (3.2.10) is solved again until good agreement is reached. The advantage gained by our particular choice of a, b, c, and d is that Ψ^* need only be calculated once. Note that we do not use equation (3.2.4), but rather the difference between equation (3.2.3) and equation (3.2.4); the latter is much simpler algebraically to handle.

III.3. Solution of the Interface Equation

The interface equation is written in a finite difference form akin to the Crank Nicholson approach.

$$\begin{aligned} & VD \left(\cos \theta_j + \frac{2 \sin \theta_j}{(\eta_{jn+1}^* + \eta_{jn}^*)} \frac{\partial \eta^*}{\partial \theta} \right) - \left(\frac{\eta_{jn+1}^* - \eta_{jn}^*}{\Delta F_o} \right) \\ & + 2 \text{Ste} \frac{\partial \Psi}{\partial \eta} \left(\frac{1}{\eta_{jn+1}^* + \eta_{jn}^* - 2} \right) \left(1 + \frac{4}{(\eta_{jn+1}^* + \eta_{jn}^*)^2} \left(\frac{\partial \eta^*}{\partial \theta} \right)^2 \right) = 0 \end{aligned} \quad (3.3.1)$$

where

$$\frac{\partial \eta^*}{\partial \theta} \approx \frac{\eta_{j+1, n+1}^* - \eta_{j-1, n+1}^* + \eta_{j+1, n}^* - \eta_{j-1, n}^*}{4\Delta\theta} \quad (3.3.2)$$

This equation is solved using the Modified Quasilinearization Algorithm, a more robust version of the Newton Raphson method. The MQA procedure is described by Mielé and Iyer [17].

The most successful mesh size used was $\Delta\eta=\Delta\theta=0.1$. The value of ΔF_0 varied but was usually 0.001. A simplified flow chart of the program appears in Appendix B.

IV. RESULTS AND DISCUSSION

The position of the interface as a function of time for various Stefan numbers is depicted in figures (4.1) through (4.4). The characteristic shape of the interface is due primarily to the fact that most of the melting is occurring in the lower hemisphere, and that the melting rate is very nearly uniform there. The cause of this behavior can be directly traced to the influence of gravity. Initially, the melting rate is uniform everywhere. As time progresses, the upper portion of the solid is drawn further and further away from the shell surface, because the dropping velocity and the melting velocity reinforce one another. On the other hand, the lower portion of the solid is more or less held fixed to its initial position, because the dropping velocity essentially is the negative of the melting velocity. If gravity were not a factor, uniform melting would continue for the duration of the melting cycle.

Temperature profiles for various Stefan numbers at two values of time and polar angle are plotted in figures (4.5) and (4.6). A more detailed examination of typical temperature profiles is presented in figure (4.7) and (4.8). Note the sharp contrast between profiles drawn for the upper hemisphere and the lower hemisphere. This difference would be greatly magnified if the abscissa had been η' rather than η .

The heat flux at the outer wall is plotted as a function of time and polar angle in figure (4.9). The reason that the heat flux in the vicinity of $\theta=\pi$ increases with time is that the solid is settling to a lower portion in the shell than it occupied initially.

The heat flux at the interface is plotted as a function of time and polar angle in figure (4.10). As time progresses, the melting rate in the lower hemisphere increasingly dominates the melting rate in the upper hemisphere. Eventually, the finite difference code cannot successfully span the transition zone, and the code fails, as shown in the lower most curve.

Pressure as a function of time and polar angle is plotted in figure (4.11). The pressure at $\theta=0$ is set equal to zero, which would be the case if the top of the shell were vented. The pressure is seen to increase sharply as one nears $\theta=\pi$.

Each figure is supported by a table listing the values used to generate the curves.

Figure 4.1. Interface position at various stages of melting.
Ste. = 0.0552, $R = 3.272$ cm.

Figure 4.2. Interface position at various stages of melting.
Ste. = 0.255, $R = 3.272$ cm.

Figure 4.3. Interface position at various stages of melting
 $Ste. = 0.500$, $R = 3.272$ cm.

Figure 4.4. Interface position at various stages of melting.
 $Ste. = 0.769$, $R = 3.272$ cm.

Figure 4.5. Temperature as a function of radial position for $\theta=0$, $\pi/2$, and $F_o=0.002$.

Figure 4.6. Temperature as a function of radial position for $\theta=0$, $\pi/2$, and $F_o=0.008$.

Figure 4.7. Temperature as a function of radial position at $\theta=0, \pi/4, \pi/2, 3\pi/4$, for $F_O = 0.008$ and $F_O = 0.014$. Ste. = 0.255.

Figure 4.8. Temperature as a function of radial position at $\theta=0, \pi/4, \pi/2, 3\pi/4$, for $F_O = 0.022$ and $F_O = 0.030$. Ste. = 0.255

Figure 4.9. Heat flux at shell surface as a function of θ for $F_0 = 0.002, 0.024,$ and 0.050 . Ste. = 0.100 , $R = 3.272$ cm.

Figure 4.10. Heat flux at interface as a function of θ for $F_0 = 0.002$, 0.024 , and 0.050 . Ste. = 0.10 , $R = 3.272$ cm.

Figure 4.11. Pressure as a function of θ for $F_o = 0.002$, 0.024, 0.050. Ste. = 0.100, $R = 3.272$ cm.

Table 4.1.a. Interface position at various stages of melting
 Ste. = 0.0552, R = 3.272 cm, $F_o = 0.016$

θ	η^*	θ	η^*
0.101	0.847	1.723	0.967
0.304	0.851	1.925	0.981
0.507	0.860	2.128	0.988
0.709	0.872	2.331	0.991
0.912	0.888	2.534	0.992
1.115	0.907	2.736	0.993
1.317	0.927	2.939	0.993
1.520	0.948	3.1416	0.994

Table 4.1.b. Interface position at various stages of melting
 Ste. = 0.0552, R = 3.272 cm, $F_o = 0.032$

θ	η^*	θ	η^*
0.101	0.706	1.723	0.950
0.304	0.714	1.925	0.979
0.507	0.729	2.128	0.988
0.709	0.752	2.331	0.991
0.912	0.782	2.534	0.992
1.115	0.819	2.736	0.993
1.317	0.861	2.934	0.993
1.520	0.906	3.1416	0.993

Table 4.1.c. Interface position at various stages of melting
 Ste. = 0.0552, R = 3.272 cm, $F_o = 0.048$

θ	η^*	θ	η^*
0.101	0.570	1.723	0.922
0.304	0.580	1.925	0.976
0.507	0.599	2.128	0.987
0.709	0.628	2.331	0.990
0.912	0.668	2.534	0.992
1.115	0.719	2.736	0.993
1.317	0.780	2.939	0.993
1.520	0.850	3.1416	0.993

Table 4.1.d. Interface position at various stages of melting
 Ste. = 0.0552, R = 3.272 cm, $F_0 = 0.068$

θ	η^*	θ	η^*
0.101	0.405	1.723	0.858
0.304	0.415	1.925	0.965
0.507	0.435	2.128	0.986
0.709	0.466	2.331	0.990
0.912	0.512	2.534	0.992
1.115	0.573	2.736	0.993
1.317	0.652	2.939	0.993
1.520	0.748	3.1416	0.993

Table 4.2.a. Interface position at various stages of melting
 Ste. = 0.255, R = 3.272 cm, $F_0 = 0.008$

θ	η^*	θ	η^*
0.101	0.757	1.723	0.956
0.304	0.764	1.925	0.976
0.507	0.777	2.128	0.985
0.709	0.796	2.331	0.988
0.912	0.822	2.534	0.990
1.115	0.852	2.736	0.991
1.317	0.886	2.939	0.991
1.520	0.922	3.1416	0.991

Table 4.2.b. Interface position at various stages of melting
 Ste. = 0.255, R = 3.272 cm, $F_0 = 0.014$

θ	η^*	θ	η^*
0.101	0.579	1.723	0.928
0.304	0.588	1.925	0.976
0.507	0.607	2.128	0.984
0.709	0.637	2.331	0.987
0.912	0.677	2.534	0.989
1.115	0.727	2.736	0.991
1.317	0.788	2.939	0.991
1.520	0.856	3.1416	0.991

Table 4.2.c. Interface position at various stages of melting
 Ste. = 0.255, R = 3.272 cm, $F_0 = 0.022$

θ	η^*	θ	η^*
0.101	0.350	1.723	0.831
0.304	0.359	1.925	0.971
0.507	0.379	2.128	0.983
0.709	0.410	2.331	0.987
0.912	0.455	2.534	0.989
1.115	0.518	2.736	0.990
1.317	0.602	2.939	0.991
1.520	0.708	3.1416	0.991

Table 4.2.d. Interface position at various stages of melting
 Ste. = 0.255, R = 3.272 cm, $F_0 = 0.030$

θ	η^*	θ	η^*
0.101	0.132	1.723	0.635
0.304	0.137	1.925	0.861
0.507	0.147	2.138	0.985
0.709	0.165	2.331	0.986
0.912	0.194	2.534	0.988
1.115	0.240	2.736	0.990
1.317	0.316	2.939	0.990
1.520	0.443	3.1416	0.990

Table 4.3.a. Interface position at various stages of melting
 Ste. = 0.500, R = 3.272 cm, $F_0 = 0.004$

θ	η^*	θ	η^*
0.101	0.784	1.723	0.960
0.304	0.790	1.925	0.977
0.507	0.802	2.128	0.984
0.709	0.820	2.331	0.987
0.912	0.843	2.534	0.989
1.115	0.870	2.736	0.990
1.317	0.901	2.939	0.990
1.520	0.932	3.1416	0.991

Table 4.3.b. Interface position at various stages of melting
 Ste. = 0.500, R = 3.272 cm, $F_0 = 0.008$

θ	η^*	θ	η^*
0.101	0.572	1.723	0.933
0.304	0.581	1.925	0.977
0.507	0.601	2.128	0.984
0.709	0.631	2.331	0.987
0.912	0.671	2.534	0.989
1.115	0.723	2.736	0.990
1.317	0.786	2.939	0.990
1.520	0.857	3.1416	0.990

Table 4.3.c. Interface position at various stages of melting
 Ste. = 0.500, R = 3.272 cm, $F_0 = 0.012$

θ	η^*	θ	η^*
0.101	0.366	1.723	0.842
0.304	0.376	1.925	0.978
0.507	0.396	2.128	0.983
0.707	0.427	2.331	0.986
0.912	0.473	2.534	0.988
1.115	0.536	2.736	0.989
1.317	0.620	2.939	0.990
1.520	0.724	3.1416	0.990

Table 4.3.d. Interface position at various stages of melting
 Ste. = 0.500, R = 3.272 cm, $F_0 = 0.015$

θ	η^*	θ	η^*
0.101	0.218	1.723	0.727
0.304	0.225	1.925	0.948
0.507	0.240	2.128	0.983
0.709	0.265	2.331	0.986
0.912	0.304	2.534	0.988
1.115	0.363	2.736	0.989
1.317	0.448	2.939	0.989
1.520	0.566	3.1416	0.990

Table 4.4.a. Interface position at various stages of melting
 $Ste. = 0.769$, $R = 3.272$ cm, $F_o = 0.001$

θ	η^*	θ	η^*
0.101	0.929	1.723	0.973
0.304	0.930	1.925	0.978
0.507	0.934	2.128	0.983
0.709	0.939	2.331	0.986
0.912	0.945	2.534	0.988
1.115	0.952	2.734	0.989
1.317	0.959	2.939	0.990
1.520	0.966	3.1416	0.990

Table 4.4.b. Interface position at various stages of melting
 $Ste. = 0.769$, $R = 3.272$ cm, $F_o = 0.004$

θ	η^*	θ	η^*
0.101	0.698	1.723	0.957
0.304	0.706	1.925	0.977
0.507	0.722	2.128	0.983
0.709	0.746	2.331	0.986
0.912	0.778	2.534	0.988
1.115	0.817	2.736	0.989
1.317	0.861	2.939	0.990
1.520	0.909	3.1416	0.990

Table 4.4.c. Interface position at various stages of melting
 $Ste. = 0.769$, $R = 3.272$ cm, $F_o = 0.006$

θ	η^*	θ	η^*
0.101	0.548	1.723	0.930
0.304	0.558	1.925	0.975
0.507	0.517	2.128	0.983
0.709	0.608	2.331	0.986
0.912	0.651	2.534	0.988
1.115	0.705	2.736	0.989
1.317	0.771	2.939	0.989
1.520	0.847	3.1416	0.990

Table 4.4.d. Interface position at various stages of melting
 $Ste. = 0.769$, $R = 3.272$ cm, $F_o = 0.008$

θ	η^*	θ	η^*
0.101	0.401	1.723	0.861
0.302	0.411	1.925	0.979
0.507	0.432	2.128	0.983
0.709	0.464	2.331	0.985
0.912	0.510	2.534	0.987
1.115	0.573	2.734	0.988
1.317	0.654	2.939	0.989
1.520	0.754	3.1416	0.989

Table 4.5. Temperature as a function of radial position for
 $\theta = 0, \pi/2$, and $F_0 = 0.002$

Ste. = 0.769						
η	0.10	0.20	0.40	0.60	0.80	0.90
$\theta = 0, \psi$	0.01	0.02	0.11	0.31	0.62	0.81
$\theta = \pi/2, \psi$	0.04	0.09	0.24	0.45	0.71	0.85
Ste. = 0.500						
η	0.10	0.20	0.40	0.60	0.80	0.90
$\theta = 0, \psi$	0.03	0.07	0.21	0.42	0.69	0.85
$\theta = \pi/2, \psi$	0.06	0.13	0.30	0.51	0.75	0.87
Ste. = 0.255						
η	0.10	0.20	0.40	0.60	0.80	0.90
$\theta = 0, \psi$	0.06	0.14	0.32	0.54	0.77	0.89
$\theta = \pi/2, \psi$	0.09	0.18	0.38	0.59	0.80	0.90
Ste. = 0.100						
η	0.10	0.20	0.40	0.60	0.80	0.90
$\theta = 0, \psi$	0.09	0.18	0.38	0.59	0.80	0.90
$\theta = \pi/2, \psi$	0.10	0.20	0.40	0.61	0.81	0.91

Table 4.6. Temperature as a function of radial direction for
 $\theta = 0, \pi/2$, and $F_0 = 0.008$.

Ste. = 0.769						
η	0.10	0.20	0.40	0.60	0.80	0.90
$\theta = 0, \psi$	0.00	0.00	0.01	0.10	0.43	0.70
$\theta = \pi/2, \psi$	0.00	0.00	0.00	0.05	0.39	0.69
Ste. = 0.500						
η	0.10	0.20	0.40	0.60	0.80	0.90
$\theta = 0, \psi$	0.00	0.01	0.05	0.21	0.55	0.77
$\theta = \pi/2, \psi$	0.01	0.01	0.06	0.22	0.56	0.78
Ste. = 0.255						
η	0.10	0.20	0.40	0.60	0.80	0.90
$\theta = 0, \psi$	0.02	0.06	0.19	0.40	0.69	0.84
$\theta = \pi/2, \psi$	0.05	0.11	0.26	0.47	0.73	0.86
Ste. = 0.100						
η	0.10	0.20	0.40	0.60	0.80	0.90
$\theta = 0, \psi$	0.07	0.15	0.33	0.54	0.77	0.88
$\theta = \pi/2, \psi$	0.09	0.18	0.37	0.57	0.78	0.89

Table 4.7.a. Temperature as a function of radial position at $\theta = 0, \pi/4, \pi/2, 3\pi/4$. For $F_o = 0.008$. $Ste = 0.255$.

η	0.1	0.2	0.4	0.6	0.8	0.9
$\theta = 0, \Psi$	0.02	0.06	0.19	0.40	0.69	0.84
$\theta = \pi/4, \Psi$	0.03	0.07	0.20	0.42	0.70	0.85
$\theta = \pi/2, \Psi$	0.05	0.11	0.26	0.47	0.73	0.86
$\theta = 3\pi/4, \Psi$	0.10	0.20	0.40	0.60	0.80	0.90

Table 4.7.b. Temperature as a function of radial position at $\theta = 0, \pi/4, \pi/2, 3\pi/4$. For $F_o = 0.014$. $Ste = 0.255$.

η	0.1	0.2	0.4	0.6	0.8	0.9
$\theta = 0, \Psi$	0.01	0.03	0.13	0.33	0.64	0.82
$\theta = \pi/4, \Psi$	0.01	0.03	0.13	0.34	0.65	0.82
$\theta = \pi/2, \Psi$	0.02	0.05	0.14	0.34	0.65	0.82
$\theta = 3\pi/4, \Psi$	0.10	0.20	0.40	0.60	0.80	0.90

Table 4.8.a. Temperature as a function of radial position at $\theta = 0, \pi/4, \pi/2, 3\pi/4$. For $F_o = 0.022$. $Ste = 0.255$.

η	0.1	0.2	0.4	0.6	0.8	0.9
$\theta = 0, \Psi$	0.00	0.01	0.08	0.27	0.61	0.80
$\theta = \pi/4, \Psi$	0.00	0.01	0.08	0.26	0.60	0.80
$\theta = \pi/2, \Psi$	0.01	0.01	0.06	0.21	0.55	0.77
$\theta = 3\pi/4, \Psi$	0.10	0.20	0.39	0.60	0.80	0.90

Table 4.8.b. Temperature as a function of radial position at $\theta = 0, \pi/4, \pi/2, 3\pi/4$. For $F_o = 0.030$. $Ste = 0.255$.

η	0.1	0.2	0.4	0.6	0.8	0.9
$\theta = 0, \Psi$	0.00	0.01	0.06	0.23	0.58	0.79
$\theta = \pi/4, \Psi$	0.00	0.00	0.04	0.20	0.55	0.78
$\theta = \pi/2, \Psi$	0.00	0.00	0.02	0.11	0.45	0.72
$\theta = 3\pi/4, \Psi$	0.10	0.20	0.43	0.61	0.80	0.90

Table 4.9. Heat flux at shell surface as a function of θ for
 $F_o = 0.002, 0.024, \text{ and } 0.050$. Ste. = 0.10, R = 3.272 cm.

θ	$F_o = 0.002$	$F_o = 0.024$	$F_o = 0.050$
0.00	$0.144 \times 10^7 \frac{\text{erg}}{\text{cm}^2 \cdot \text{s}}$	$0.179 \times 10^6 \frac{\text{erg}}{\text{cm}^2 \cdot \text{s}}$	$0.948 \times 10^5 \frac{\text{erg}}{\text{cm}^2 \cdot \text{s}}$
0.405	0.149×10^7	0.187×10^6	0.983×10^5
0.811	0.164×10^7	0.222×10^6	0.112×10^6
1.216	0.194×10^7	0.325×10^6	0.152×10^6
1.621	0.245×10^7	0.740×10^6	0.306×10^6
2.027	0.323×10^7	0.317×10^7	0.287×10^7
2.432	0.421×10^7	0.533×10^7	0.497×10^7
2.838	0.504×10^7	0.622×10^7	0.632×10^7
3.142	0.527×10^7	0.693×10^7	0.656×10^7

Table 4.10. Heat flux at interface as a function of θ for
 $F_o = 0.002, 0.024, 0.050$. Ste. = 0.10, R = 3.272 cm.

θ	$F_o = 0.002$	$F_o = 0.024$	$F_o = 0.050$
0.00	$0.133 \times 10^7 \frac{\text{erg}}{\text{cm}^2 \cdot \text{s}}$	$0.448 \times 10^5 \frac{\text{erg}}{\text{cm}^2 \cdot \text{s}}$	$0.913 \times 10^4 \frac{\text{erg}}{\text{cm}^2 \cdot \text{s}}$
0.405	0.14×10^7	0.496×10^5	0.927×10^4
0.811	0.159×10^7	0.842×10^6	0.101×10^5
1.216	0.195×10^7	0.112×10^6	0.136×10^5
1.621	0.254×10^7	0.307×10^6	0.318×10^5
2.027	0.346×10^7	0.286×10^7	0.268×10^7
2.432	0.458×10^7	0.533×10^7	0.496×10^7
2.838	0.552×10^7	0.673×10^7	0.635×10^7
3.142	0.578×10^7	0.702×10^7	0.666×10^7

Figure 4.11. Pressure as a function of θ for $F_o = 0.002$, 0.024 , and 0.050 . $Ste. = 0.10$, $R = 3.272$ cm.

θ	$F_o = 0.002$	$F_o = 0.024$	$F_o = 0.050$
0.00	0.00 $\frac{\text{dyne}}{\text{cm}^2}$	0.00 $\frac{\text{dyne}}{\text{cm}^2}$	0.00 $\frac{\text{dyne}}{\text{cm}^2}$
0.405	0.588×10^0	0.00	0.00
0.811	0.285×10^1	0.00	0.00
1.216	0.865×10^1	0.00	0.00
1.621	0.229×10^2	0.00	0.00
2.027	0.573×10^2	0.649×10^1	0.157×10^1
2.432	0.139×10^3	0.789×10^2	0.518×10^2
2.838	0.399×10^3	0.349×10^3	0.247×10^3
3.142	0.1235×10^4	0.113×10^4	0.797×10^4

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APPENDIX A

A.1 Calculating dV_p , α , β , γ

Let η^* refer to the coordinates of the interface at the end of the current time step. η^* is a discrete function of θ . We wish to calculate dV_p for a given value of θ , namely, θ_p .

Let us define the mesh descriptor J , where $J + 1$ is the number of unknowns in the θ direction. Then

$$\Delta\theta = \pi/J \quad (A.1.1.a)$$

$$\theta_j = (j-2)\Delta\theta \quad (A.1.1.b)$$

$$IJ = J + 3 \quad (A.1.1.c)$$

The j mesh in the untransformed plane is shown in Figure (A.1.1)

Let

$$dV_p = \frac{2\pi R^3}{3}(B-C) \quad (A.1.2)$$

where $2\pi R^3 B/3$ is the volume of fluid present at the end of phase one within the region defined by requiring that $\theta_p \leq \theta \leq \pi$, $r^* \leq r \leq R$, and $0 \leq \phi \leq 2\pi$. $2\pi R^3 C/3$ is the volume present at the end of phase two within the same region.

The interface position at the end of phase two is simply η^* . We define η_j^* as the interface coordinate at θ_j , and we will use these coordinates to calculate C . The interface position at the end of

phase one is approximated by shifting the interface as established by η^* vertically upward by an amount equal to the distance that the solid fell during the time step, which is $-V_D \Delta t$. We define η_j^* as the coordinate of the shifted interface at θ_j .

Both B and C are calculated in increments.

$$B = \sum_{j=p}^{IJ-2} B_{j+1} \quad (A.1.3)$$

$$C = \sum_{j=p}^{IJ-2} C_{j+1} \quad (A.1.4)$$

B and C are calculated by identical procedures, so it will suffice to illustrate the calculation of C. C_{j+1} refers to the volume bounded by θ_j and θ_{j+1} . See Figure (A.1.2).

We will assume that the interface can be locally approximated by a quadratic through the points η_{j-1}^* , η_j^* , η_{j+1}^* . We can find the coefficients easily enough.

$$\eta^* = a + b\theta + c\theta^2 \quad (A.1.5)$$

where

$$a = \frac{[\eta_{j-1}^*(\theta_j \theta_{j+1}^2 - \theta_{j+1} \theta_j^2) - \eta_j^*(\theta_{j-1} \theta_{j+1}^2 - \theta_{j+1} \theta_{j-1}^2) + \eta_{j+1}^*(\theta_{j-1} \theta_j^2 - \theta_j \theta_{j-1}^2)]}{DETI}$$

$$b = \frac{[(\eta_j^* \theta_{j+1}^2 - \eta_{j+1}^* \theta_j^2) - (\eta_{j-1}^* \theta_{j+1}^2 - \eta_{j+1}^* \theta_{j-1}^2) + (\eta_{j-1}^* \theta_j^2 - \eta_j^* \theta_{j-1}^2)]}{DETI}$$

$$c = \frac{[(\theta_j \eta_{j+1}^* - \theta_{j+1} \eta_j^*) - (\theta_{j-1} \eta_{j+1}^* - \theta_{j+1} \eta_{j-1}^*) + (\theta_{j-1} \eta_j^* - \eta_{j-1}^* \theta_j)]}{DETI}$$

$$DETI = (\theta_j \theta_{j+1}^2 - \theta_{j+1} \theta_j^2) - (\theta_{j-1} \theta_{j+1}^2 - \theta_{j+1} \theta_{j-1}^2) + (\theta_{j-1} \theta_j^2 - \theta_j \theta_{j-1}^2)$$

Figure (A.1.1) Sample Mesh in θ Direction For Case of $J = 12$.

Figure (A.1.2) Defining C_{j+1}

Then, in accordance with equations (2.4.3) and (A.1.2),

$$C_{j+1} = 3 \int_{\theta}^{\theta_{j+1}} \int_{a+b\theta+c\theta^2}^1 (\eta')^2 \sin\theta \, d\eta' d\theta \quad (\text{A.1.6})$$

or

$$C_{j+1} = \int_{\theta_j}^{\theta_{j+1}} (1 - (a + b\theta + c\theta^2)^3) \sin\theta \, d\theta \quad (\text{A.1.7})$$

After defining the constants c_0 through c_6 , as

$$c_0 = a^3 \quad (\text{A.1.8.a})$$

$$c_1 = 3a^2b \quad (\text{A.1.8.b})$$

$$c_2 = 3ab^2 + 3a^2c \quad (\text{A.1.8.c})$$

$$c_3 = b^3 + 6abc \quad (\text{A.1.8.d})$$

$$c_4 = 3b^2c + 3ac^2 \quad (\text{A.1.8.e})$$

$$c_5 = 3bc^2 \quad (\text{A.1.8.f})$$

$$c_6 = c^3 \quad (\text{A.1.8.g})$$

we can complete the integration. The result is

$$C_{j+1} = (c_0 - 1)(\cos\theta_{j+1} - \cos\theta_j) -$$

$$c_1(\sin\theta_{j+1} - \sin\theta_j - \theta_{j+1}\cos\theta_{j+1} + \theta_j\cos\theta_j) -$$

$$c_2(2\theta_{j+1}^2\sin\theta_{j+1} - 2\theta_j^2\sin\theta_j - (\theta_{j+1}^2 - 2)\cos\theta_{j+1} + (\theta_j^2 - 2)\cos\theta_j) -$$

$$c_3((3\theta_{j+1}^2 - 6)\sin\theta_{j+1} - (3\theta_j^2 - 6)\sin\theta_j) +$$

$$c_4(-\theta_{j+1}^4\cos\theta_{j+1} + \theta_j^4\cos\theta_j + (12\theta_{j+1}^2 - 24)\cos\theta_{j+1} - (12\theta_j^2 - 24)\cos\theta_j) -$$

$$c_4((4\theta_{j+1}^3 - 24\theta_{j+1})\sin\theta_{j+1} - (4\theta_j^3 - 24\theta_j)\sin\theta_j) -$$

$$\begin{aligned}
& c_5(-\theta_{j+1}^5 \cos \theta_{j+1} + \theta_j^5 \cos \theta_j + 5\theta_{j+1}^4 \sin \theta_{j+1} - 5\theta_j^4 \sin \theta_j) - \\
& c_5((120 - 60\theta_{j+1}^2) \sin \theta_{j+1} - (120 - 60\theta_j^2) \sin \theta_j + (20\theta_{j+1}^3 - 120\theta_{j+1}) \cos \theta_{j+1}) + \\
& c_5((20\theta_j^3 - 120\theta_j) \cos \theta_j) - \\
& c_6(-\theta_{j+1}^6 \cos \theta_{j+1} + \theta_j^6 \cos \theta_j + 6\theta_{j+1}^5 \sin \theta_{j+1} - 6\theta_j^5 \sin \theta_j) - \\
& c_6(30\theta_{j+1}^4 \cos \theta_{j+1} - 30\theta_j^4 \cos \theta_j + (720 - 360\theta_{j+1}^2) \cos \theta_{j+1}) + \\
& c_6((720 - 360\theta_j^2) \cos \theta_j + (720\theta_{j+1} - 120\theta_{j+1}^3) \sin \theta_{j+1}) + \\
& c_6((720\theta_j - 120\theta_j^3) \sin \theta_j)
\end{aligned} \tag{A.1.9}$$

Equation (2.4.5) can be recast into the following form.

$$(B-C)_p = -3 \sin \theta_p (\Delta F_o) (\text{RePr}) \left[\frac{\alpha}{2} (1 - \eta_p^{*2}) + \frac{\beta}{3} (1 - \eta_p^{*3}) + \frac{\gamma}{4} (1 - \eta_p^{*4}) \right] \tag{A.1.10}$$

This expression can be used in conjunction with the boundary conditions on the velocity to calculate α , β , and γ . Using Cramer's Rule, we get

$$\alpha = \frac{-(B-C)_p}{3\Delta F_o \text{RePr} \sin \theta_p} \left(\frac{\eta_p^* - \eta_p^{*2}}{\text{DETC}} \right) \tag{A.1.11.a}$$

$$\beta = \frac{(B-C)_p}{3\Delta F_o \text{RePr} \sin \theta_p} \left(\frac{1 - \eta_p^{*2}}{\text{DETC}} \right) \tag{A.1.11.b}$$

$$\gamma = \frac{-(B-C)_p}{3\Delta F_o \text{RePr} \sin \theta_p} \left(\frac{1 - \eta_p^*}{\text{DETC}} \right) \tag{A.1.11.c}$$

where

$$\text{DETC} = (-1 + 3\eta_p^* - 2\eta_p^{*2} - 2\eta_p^{*3} + 3\eta_p^{*4} - \eta_p^{*5})/12 \tag{A.1.11.d}$$

A.2 Calculating the Momentum Flux and A_3

We have from equation (2.25) that

$$V_\theta = V_D(\alpha + \beta\eta' + \gamma(\eta')^2) \quad (\text{A.2.1})$$

then

$$\int_A \rho_L V_\theta^2 dA = V_D^2 \rho_L R^2 \int_0^{2\pi} \int_{\eta^*}^1 (\alpha + \beta\eta' + \gamma(\eta')^2)^2 \eta' \sin\theta d\eta' d\phi$$

or

$$\begin{aligned} \int_A \rho_L V_\theta^2 dA = 2\pi R^2 V_D^2 \rho_L \sin\theta & \left[\frac{\alpha^2}{2}(1-\eta^{*2}) + \frac{\beta^2 + 2\alpha\gamma}{4}(1-\eta^{*4}) + \right. \\ & \left. \frac{\gamma^2}{6}(1-\eta^{*6}) + \frac{2\alpha\beta}{3}(1-\eta^{*3}) + \frac{2\beta\gamma}{5}(1-\eta^{*5}) \right] \end{aligned} \quad (\text{A.2.2})$$

Area A_3 can be calculated on the same pattern as C_{j+1} . The local interface is approximated by equation (A.1.5). Then the expression for A_3 becomes

$$\begin{aligned} A_{3_{j+1}} &= \int_0^{2\pi} \int_{\theta_j}^{\theta_{j+1}} r^2 \sin\theta d\theta d\phi \\ A_{3_{j+1}} &= 2\pi R^2 \left[C_{c2} \cos\theta_{j+1} - C_{c1} \cos\theta_j + C_{s2} \sin\theta_{j+1} - C_{s1} \sin\theta_j \right] \end{aligned} \quad (\text{A.2.3})$$

where

$$C_c = \left[-a^2 - 2ab\theta + (2-\theta^2)(b^2 + 2ac) + 2bc(6\theta - \theta^3) - c^2\theta^4 + c^2(12\theta^2 - 24) \right] \quad (\text{A.2.4.a})$$

$$C_s = \left[2ab + (b^2 + 2ac)(2\theta) + 2bc(3\theta^2 - 6) + c^2(4\theta^3 - 24\theta) \right] \quad (\text{A.2.4.b})$$

The subscript 2 indicates the expression in brackets should be evaluated at $\theta = \theta_{j+1}$, while the subscript 1 indicates the expression in brackets should be evaluated at $\theta = \theta_j$.

A.3 Calculating the Vertical Components of the Shear and Pressure Forces.

The magnitude of the shear force at the solid, T_3 , and the area A_3 , are both documented in a manner analogous to convention adopted for the volumes B and C. That is, T_{3j+1} is the shear force acting on the sector bounded by θ_j and θ_{j+1} , and that sector's surface area is A_{3j+1} . The pressure, P_j , is the pressure calculated at $\theta = \theta_j$. The total shear and pressure force is calculated by summing the appropriate components for all the sectors between $\theta = 0$ and $\theta = \pi$. Refer to Figure (A.3.1) for the shear force calculation nomenclature.

First we convert η_j^* and η_{j+1}^* to rectangular coordinates. Then we observe that

$$T_{3j+1}^\perp = T_{3j+1} \cos \alpha \quad (\text{A.3.1})$$

and that

$$\cos \alpha = (y_j - y_{j+1}) / ((x_j + x_{j+1})^2 + (y_j + y_{j+1})^2)^{1/2} \quad (\text{A.3.2})$$

a). Positive Orientation of Shear Force exerted on a solid.

b) Geometry used to find Vertical Components of T_3 .

Figure A.3.1

Finally, the total shear in the $\hat{k}$ direction is

$$\text{SHEAR} = \sum_{j=2}^{IJ-2} T_{3_{j+1}} (y_j - y_{j+1}) / ((x_j + x_{j+1})^2 + (y_j + y_{j+1})^2)^{1/2} \quad (\text{A.3.3})$$

The pressure force is calculated in an entirely analogous fashion. Our final expression is

$$\text{DYPRES} = \sum_{j=2}^{IJ-2} \frac{1}{2} (P_j + P_{j+1}) (A_{3_{j+1}}) (x_j - x_{j+1}) / ((x_j + x_{j+1})^2 + (y_j + y_{j+1})^2)^{1/2} \quad (\text{A.3.4})$$

APPENDIX B

FLOW CHART

Suppose we have just completed time step n and are about to calculate the next. First we assume that the interface heat flux calculated at time n remains constant throughout the time step, and we use this to calculate a first approximation to η_{n+1}^* . The boundary conditions used to do this are the reflection condition at $\theta = 0$, and the assumption that $\eta_{n+1}^*(\theta = \pi) = \eta_n^*(\theta = \pi)$. (The reflection condition at $\theta = \pi$ is used to calculate VD.). Knowing η_{n+1}^* and η_n^* we can solve for the velocities and ultimately decide whether the solid is in equilibrium, or whether the gap at the bottom must be bigger or smaller. If the gap requires modification, we alter $\eta_{n+1}^*(\theta = \pi)$ accordingly, and recalculate VD and η_{n+1}^* , and eventually the new velocities, shear forces, etc. This loop is continued until the solid is balanced. We then return to the energy equation and calculate ψ_{n+1} , by using the average between η_{n+1}^* and η_n^* where ever η^* is required in the energy equation coefficients. Then the heat flux is recalculated using an average value of ψ_{n+1} and ψ_n , and the balancing routine reinitiated. This process continues until successive calculations of ψ_{n+1} yield nearly identical results. At this point the time step is said to have converged, and we prepare to calculate ψ_{n+2} .

APPENDIX C

EXPERIMENTAL RESULTS

1. Experimental Procedure

Glass spheres of the type used to make light bulbs were donated by General Electric's Lamp Glass Products Department. The two sizes of bulbs most frequently used had radii of 3.272 cm and 2.789 cm, and an average wall thickness of 0.06 in. A water bath was constructed to maintain an approximate constant wall temperature boundary condition for the bulb. The bath was contained in a 19.5 x 10.2 x 11.5 in deep glass tank (10 gallon) insulated on the sides and bottom with 2 in. of styrofoam, and heated by a Fisher Model 73 Immersion Circulator, controlled by a mercury contact switch. The heater's circulating pump was rated at 15 liters/min., and was supplemented by a 0.7 Amp. Teel Model 1P805 sump pump. Uniformity of temperature within the bath was tested by setting the heater to 104°F. Measurements taken at numerous locations and depths indicated that the bath's temperature was 103.5 ± 0.5 °F.

The bulb was supported from a wooden top as illustrated in Figure (C.1.1). The bulb could be viewed through a hole cut in the insulation, and photographs were made using a Fujica 35 mm camera and TRI-X film. The only significant quantitative measurement that could be readily made was the position of the interface at $\theta = 0$. A Mitutoyo height gage was employed for this purpose. The gage's scale ranged from 0 to 25 cm and a vernier allowed measurements to within 0.02 mm. The

gage was supported between two steel angles which spanned the top of the tank. The experimental setup is illustrated in Figure's (C.1.2) and (C.1.3).

A bulb containing solid n-octadecane wax was placed in the tank and held at its saturation temperature of 28°C for about 1.5 hours, then removed and placed in a holding bath while the main bath's temperature was brought to the testing temperature. Then the bulb was reinserted into the bath and the height gage's wooden probe was lowered through the neck of the bulb until it touched the interface. Time and gage position were noted to serve as reference points. Periodic measurements were then made to determine the change in height of the interface.

2. Experimental Results

Photographs of the interface at various stages of melting are shown in Figures (C.2.1) through (C.2.4). Note that the bottom remains spherical. The dark, odd shaped patches seen near the bottom of the bulb are flattened air bubbles working their way to the free surface. Tkachev [18] reported that trapped air bubbles had no measurable effect on the liquid to solid heat transfer between ice and water, and the same is assumed to apply here. The fact that the bubbles are flattened indicates that the interface is indeed very near the glass and we are not viewing a parallax phenomena.

A plot of η^* ($\theta = 0$) for $\text{Ste} = 0.0552$ is compared to data obtained at a test temperature of 34°C (roughly $\text{Ste} = 0.05$) in Figure (C.2.5). Figure (C.2.6) shows a similar plot taken from $\text{Ste} = 0.10$

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results and data obtained at a test temperature of 41°C (roughly $\text{Ste} = 0.10$). It is apparent that the numerical results are reasonable, at least for small Stefan numbers. The numerically generated quantities are collected in Table (C.2.1) while the experimental data is collected in Tables (C.2.2) and (C.2.3). Physical property data used for the wax is collected in Table (C.2.4).

Figure C.1.1. Detailed drawing of the press board top and bulb support. The bulb is cantilevered via a test tube holder from a 0.4 in. diameter stainless steel bar. The bar is bolted to an aluminum block which can slide back and forth within the two channels.

Figure C.1.2. Measuring r^* ($\theta = 0$). The height gage is supported by two steel angles.

Figure C.1.3. Experimental apparatus showing heater, height gage and viewing port.

Figure C.2.1. Interface after 7.5 minutes. $R = 3.272$ cm, $T_w = 34^\circ\text{C}$.
Run #8, TRI-X film exposed at f-16, 1/15 sec.

Figure C.2.2. Interface after 11.5 minutes. $R = 3.272$ cm, $T_w = 34^{\circ}\text{C}$.
Run #8, TRI-X film exposed at f-8, $1/8$ sec.

Figure C.2.3. Interface after 15 minutes. $R = 3.272$ cm, $T_w = 34^{\circ}\text{C}$.
Run #8, TRI-X film exposed at f-11, 1/4 sec.

Figure C.2.4. Interface after 18.5 minutes. $R = 3.272$ cm, $T_w = 34^{\circ}\text{C}$.
Run #8, TRI-X film exposed at f-11, 1/4 sec.

Figure C.2.5. Comparison of experimental data with numerical results for η^* ($\theta = 0$) for $Ste \approx 0.05$, $R = 3.272$ cm.

Figure C.2.6. Comparison of experimental data with numerical results for η^* ($\theta = 0$) for $Ste \approx 0.10$ and $R = 3.272$ cm.

Table C.2.1. Numerically generated value of η^* ($\theta = 0$) for $Ste = 0.0552$ and $Ste = 0.10$, as a function of time. $R = 3.272$ cm.

Ste = 0.055		Ste = 0.10	
F_o	$\eta^*(\theta=0)$	F_o	$\eta^*(\theta=0)$
0.004	0.96	0.004	0.94
0.012	0.88	0.008	0.87
0.020	0.81	0.012	0.82
0.028	0.74	0.016	0.76
0.036	0.67	0.020	0.70
0.044	0.60	0.024	0.64
0.052	0.53	0.028	0.59
0.060	0.47	0.034	0.50
0.068	0.40	0.040	0.42
0.072	0.37	0.046	0.34
		0.050	0.29

Table C.2.2. Experimentally determined values of η^* ($\theta = 0$) as a function of time for $T_w = 34^\circ\text{C}$ and $R = 3.272$ cm.

Run #3		Run #4		Run #5	
F_o	$\eta^*(\theta=0)$	F_o	$\eta^*(\theta=0)$	F_o	$\eta^*(\theta=0)$
0.00	1.00	0.00	1.00	0.00	1.00
0.009	0.93	0.002	0.98	0.014	0.92
0.013	0.89	0.006	0.92	0.031	0.79
0.018	0.84	0.009	0.90	0.053	0.64
0.023	0.80	0.016	0.84	0.068	0.53
0.033	0.74	0.023	0.78	0.089	0.40
0.038	0.70	0.040	0.66	0.103	0.32
0.049	0.63	0.047	0.63	0.119	0.23
0.054	0.58	0.053	0.56		
0.064	0.52	0.060	0.50		
0.073	0.47	0.069	0.44		
0.079	0.43	0.081	0.36		
0.086	0.38	0.096	0.27		
0.092	0.35	0.106	0.21		
0.097	0.32	0.114	0.16		

Table C.2.3. Experimentally determined values of $\eta^*(\theta = 0)$ as a function of time for $T_w = 41^\circ\text{C}$ and $R = 3.272\text{ cm}$.

Run #22		Run #23		Run #30	
F_o	$\eta^*(\theta=0)$	F_o	$\eta^*(\theta=0)$	F_o	$\eta^*(\theta=0)$
0.00	1.00	0.00	1.00	0.00	1.00
0.008	0.89	0.010	0.90	0.012	0.85
0.016	0.77	0.019	0.75	0.021	0.74
0.025	0.65	0.026	0.64	0.031	0.61
0.037	0.49	0.033	0.55	0.039	0.51
0.048	0.35	0.040	0.47	0.049	0.40
0.056	0.26			0.060	0.29
0.067	0.12			0.068	0.21

Table C.2.4. Property data for n-octadecane wax. References are noted in brackets.

Liquid density [19]

T °C	ρ_L g/cm ³	T °C	ρ_L g/cm ³
28.17	0.7768	70	0.7481
30.00	0.7755	80	0.7415
40.00	0.7684	90	0.7351
50.00	0.7615	100	0.7287
60.00	0.7547	110	0.7224

Viscosity [19]

T °C	μ centipoise	T °C	μ centipoise
30	3.879	60	2.081
35	3.448	70	1.755
40	3.084	80	1.501
45	2.744	90	1.299
50	2.508	100	1.137

Thermal Conductivity [20]

T °C	k w/m °C	T °C	k w/m °C
40	0.1493	120	0.1346
60	0.1456	140	0.1309
80	0.1419	160	0.1273
100	0.1382	180	0.1237

Heat Capacity [21]

$$C_p = (0.008213)T - 0.14237 \quad \text{w-sec/gm K}$$

Latent Heat of Fusion [19]

$$14.81 \quad \text{Kcal/mole}$$

Melting Temperature [19]

$$28.18 \quad ^\circ\text{C}$$

APPENDIX D

RELATIVE IMPORTANCE OF FREE CONVECTION

Mack and Hardee [22] solved for the flow field and temperature distribution between concentric spheres held at two different temperatures by expressing the solution as a power series in the Rayleigh number and solving for the first three terms. A recent paper by Powe, Warrington, and Scanlan [23] summarize experimental results through 1979. They present qualitative descriptions of the various flow regimes possible between eccentric spheres and vertical cylinders within the spherical enclosure. An earlier paper by Scanlan, Bishop, and Powe [24] presents results for the heat transfer between concentric spheres. They account for the presence of free convection by replacing the thermal conductivity by an effective conductivity within the simple concentric shell conduction equation. They recommend the following correlation equation:

$$k_{\text{eff}}/k = 0.228(Ra^*)^{0.226} \quad (D.1)$$

where k_{eff} is the effective conductivity and Ra^* is defined by $Ra^* = g\beta (r_o - r_i)^4 \Delta T / \nu \alpha r_i$. Here r_o has the same meaning as R , and r_i is the fixed inner shell radius. β is the thermal expansion coefficient and g is the acceleration due to gravity. This equation is valid for $1.2 \times 10^2 < Ra^* < 1.1 \times 10^9$.

The region of thermal instability in the melting problem occurs in the lower hemisphere, where the gap is small and nearly constant

throughout the portion of the melting time considered in this paper. This implies that the volumetric flow caused by the buoyancy forces should also remain fairly constant. Given this, free convection will be most prominent a heat transfer mode in the early stages of melting when the gap is everywhere small and the volumetric flux will translate into appreciable velocities. It is for this reason that the experimental results for concentric spheres is felt to be applicable.

β is calculated according to the approximate formula given by Eckert and Drake [25] as

$$\rho_w - \rho_{l,sat} = \beta \rho_w (T_{sat} - T_w) \quad (D.2)$$

where ρ_w is the liquid density evaluated at the wall temperature. Thermal properties are evaluated at the volumetric mean temperature, as recommended in [24]. The relation is

$$T_m = \left[(r_{avg}^3 - r_i^3) T_{sat} + (r_o^3 - r_{avg}^3) T_w \right] / (r_o^3 - r_i^3) \quad (D.3)$$

where $r_{avg} = (r_o + r_i)/2$.

Table (D.1) presents the results for the Stefan numbers used in this paper for the cases of $r_i = 0.90r_o$ and $r_i = 0.95r_o$. Recall that in the region near $\theta = \pi$, $r^* \approx 0.99R$ for all of the Stefan numbers considered. The results indicate that the size of the gap is a very important parameter, and one that is not clearly defined in the present context. Free convection appears to be negligible for small Stefan numbers under the conditions set for this problem. In future work which is not restricted by requiring that $r^* > 0.0$,

it is evident that free convection will be significant, particularly for moderate values of the Stefan number, since then the observation that the gap is small will no longer hold true.

Table D.1. Ra^* and k_{eff}/k for $r_i = 0.90r_o$ and $r_i = 0.95r_o$ for various Stefan Numbers.

$r_i = 0.90r_o$			$r_i = 0.95r_o$		
Ste	Ra^*	k_{eff}/k	Ste	Ra^*	k_{eff}/k
0.055	165	≈ 1	0.055	10	≈ 1
0.100	694	≈ 1	0.100	41	≈ 1
0.255	1630	1.2	0.255	100	≈ 1
0.500	4450	1.5	0.500	265	≈ 1
0.769	8559	1.8	0.769	507	≈ 1