CHARACTERIZATION OF SPHERICAL AND PSEUDO-SPHERICAL SETS OF POINTS

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A semi-metric space is defined to be a set of undefined elements, called "points", such that for any two elements p, q of the set there corresponds a not negative real number pq , called the "distance" between the points p and q , which satisfies the following postulates:

$$(1). \quad pq = qp;$$

$$(2). \quad pq = 0 \text{ if and only if the points } p \text{ and } q \text{ are identical.}$$

Two pairs of points $p, q; p', q'$ such that $pq = p'q'$ are said to be congruent. A mapping of a set S upon a set S' is called a congruent mapping if to each pair of points of S there corresponds a congruent pair of points of S' . Two sets S and S' are said to be congruent provided there exists a congruent mapping of one upon the other. The nature of the investigations presented here justifies their division into two sections, a description of which follows.

Section I. The Circle. A set is said to be d-cyclic if it is congruent to a subset of a circle of metric diameter d , the distance between any two points of the circle being defined to be the length of the shorter arc joining them; while a set is pseudo d-cyclic if each three points of the set is d-cyclic, but the set itself is not d-cyclic. D-cyclic and pseudo d-cyclic sets of points are characterized by means of distance relations expressible in determinantal form. The principal theorem concerning pseudo d-cyclic sets shows that such sets are equilateral provided they contain more than four points, and no four of the points form a convex tripod*. This theorem is equivalent to a new

* Four points are said to form a convex tripod if one of the points lies between each of the three pairs of points contained in the remaining three points. The point q is said to lie between the points p and r if $pq + qr = pr$.

theorem on n th-order determinants of a certain class. Using synthetic methods, L. M. Blumenthal has proved most of the theorems contained in this section*. Finally, a method of construction of the three kinds of pseudo d -cyclic quadruples is obtained.

Section II. The Sphere $S_{2,r}$. The two-dimensional spherical space $S_{2,r}$ consists of the points of the surface of a three-dimensional sphere of radius r in a Euclidean space of three dimensions, with the distance between two points of the surface defined to be the length of the shorter arc of a great circle through the two points. The $S_{2,r}$ is characterized among general semi-metric spaces by means of four principal theorems. A set of points which is congruent to a subset of the $S_{2,r}$ is said to be r -spheric. A set such that each four points of the set is r -spheric while the set itself is not r -spheric is said to be pseudo r -spheric. A method of constructing pseudo r -spheric quintuples is obtained and some of their properties are derived. A theorem concerning pseudo r -spheric sets similar to the "equilateral" theorem proved for pseudo d -cyclic sets has not yet been proved.

I. The Circle

Let p_i and p_j be any two points of a semi-metric space. We denote by α_{ij} the angle $\frac{p_i p_j}{r}$ radians, where r is the radius (Euclidean) of the circle of metric diameter d ; i.e. $d = \pi r$.

* These theorems are contained in two papers published in the American Journal of Mathematics; "A Complete Characterization of Proper Pseudo d -Cyclic Sets of Points", Vol. 54 (1932), pp. 387-396; and "Concerning Regular Pseudo d -Cyclic Sets", Vol. 54, pp. 729-738.

If $p_1, p_2, \dots, p_n$ are n points of a semi-metric space, we denote by $\Delta(p_1, p_2, \dots, p_n)$ the axisymmetric determinant

$$\begin{vmatrix} 1 & \cos \alpha_{12} & \dots & \cos \alpha_{1n} \\ \cos \alpha_{21} & 1 & \dots & \cos \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \cos \alpha_{n1} & \cos \alpha_{n2} & \dots & 1 \end{vmatrix}$$

Theorem 1. Three points p_1, p_2, p_3 of a semi-metric space are d-cyclic if and only if $0 < \alpha_{ij} \leq \pi$ and $\Delta(p_1, p_2, p_3) = 0$. ($i, j = 1, 2, 3$).

Evaluating the determinant $\Delta(p_1, p_2, p_3)$, we have

$$\begin{aligned} \Delta(p_1, p_2, p_3) &= 1 - \cos^2 \alpha_{12} - \cos^2 \alpha_{13} - \cos^2 \alpha_{23} + 2 \cos \alpha_{12} \cos \alpha_{13} \cos \alpha_{23} \\ &= -[\cos \alpha_{23} - \cos(\alpha_{12} + \alpha_{13})][\cos \alpha_{23} - \cos(\alpha_{12} - \alpha_{13})] \\ &= 4 \sin \frac{\alpha_{12} + \alpha_{23} + \alpha_{13}}{2} \sin \frac{\alpha_{12} + \alpha_{23} - \alpha_{13}}{2} \sin \frac{\alpha_{12} - \alpha_{23} + \alpha_{13}}{2} \sin \frac{-\alpha_{12} + \alpha_{23} + \alpha_{13}}{2} \end{aligned} \quad (1)$$

We shall prove first that if $0 < \alpha_{ij} \leq \pi$ and $\Delta(p_1, p_2, p_3) = 0$, the points p_1, p_2, p_3 are d-cyclic. If $\Delta(p_1, p_2, p_3) = 0$, then by (1) one of the following conditions must subsist:

$$\begin{aligned} &[\alpha_{12} + \alpha_{23} + \alpha_{13} = 2k\pi]; [\alpha_{12} + \alpha_{23} - \alpha_{13} = 2k\pi]; \\ &[\alpha_{12} - \alpha_{23} + \alpha_{13} = 2k\pi]; [\alpha_{12} + \alpha_{23} + \alpha_{13} = 2k\pi]. \end{aligned}$$

where k is an integer. Since $0 < \alpha_{ij} \leq \pi$, the only possible value for k in the first of these equations is 1, while in the three remaining equations the only possible value for k is 0. Hence, the four equations are equivalent to the following:

$$p_1 p_2 + p_1 p_3 + p_2 p_3 = 2d; p_1 p_2 + p_2 p_3 = p_1 p_3; p_1 p_2 + p_1 p_3 = p_2 p_3; p_1 p_3 + p_2 p_3 = p_1 p_2. \quad (2)$$

But, when one of these conditions exists, the points p_1, p_2, p_3 are d-cyclic* and the sufficiency of the theorem is proved.

* L. M. Blumenthal, "A Complete Characterization of Proper Pseudo d-Cyclic Sets of Points", American Journal of Mathematics, Vol. 54, p. 388

If p_1, p_2, p_3 are d -cyclic, then $0 < p_i p_j \leq d$ and one of the conditions (2) subsists*. Hence, one of the factors in (1) vanishes. Thus, we have $0 < \alpha_{ij} \leq \pi$ and $\Delta(p_1, p_2, p_3) = 0$, which establishes the necessity of the theorem.

From (1) we have immediately the following corollary:

Corollary. The determinant $\Delta(p_1, p_2, p_3)$ is zero if and only if one of the relations

$\alpha_{12} + \alpha_{23} + \alpha_{13} = 2\pi$; $\alpha_{12} + \alpha_{23} = \alpha_{13}$; $\alpha_{12} + \alpha_{13} = \alpha_{23}$; $\alpha_{23} + \alpha_{13} = \alpha_{12}$;
is satisfied.

Definition. Three points of a semi-metric space are said to be circular provided the points are congruent to three points of some circle.

Theorem 2. A necessary and sufficient condition that three points p_1, p_2, p_3 of a semi-metric space be circular is that the points satisfy the triangular inequality.

For if p_1, p_2, p_3 are congruent to three points of a circle, then the points either are linear† (in which case the triangular inequality is obviously satisfied) or else they satisfy the relation

$$p_1 p_2 + p_2 p_3 + p_1 p_3 = 2d \quad (1)$$

where d is the metric diameter of the circle. Suppose p_1, p_2, p_3 do not satisfy the triangular inequality. We may assume the labeling so that

$$p_1 p_2 > p_2 p_3 + p_1 p_3 \quad (2)$$

Adding (1) and (2) we have $p_1 p_2 > d$, which is impossible, since d is the metric diameter of the circle containing the three points congruent to p_1, p_2, p_3 . Hence, the necessity of the condition is established.

Suppose now that p_1, p_2, p_3 satisfy the triangular inequality.

* Loc. cit.

† Three points are said to be linear if one is between the other two.

If the triple is linear, then p_1, p_2, p_3 are congruent to three points of a circle of metric diameter d , where d is greater than or equal to the largest of the numbers $p_1, p_2, p_2 p_3, p_1 p_3$. If the triple is not linear, we may write

$$p_1 p_2 + p_2 p_3 + p_1 p_3 = 2d'.$$

Then the points are congruent to three points of a circle of metric diameter d' . To show this we need only note that each of the distances $p_1 p_2, p_2 p_3, p_1 p_3$ is less than d' . For suppose one of these distances, say $p_1 p_2$, is not less than d' . Then

$$p_1 p_2 + p_2 p_3 + p_1 p_3 \leq 2p_1 p_2 \quad \text{that is}$$

$$p_2 p_3 + p_1 p_3 \leq p_1 p_2.$$

But this is impossible since the points p_1, p_2, p_3 are not linear and the triangular inequality holds.

Using this result one may define a metric space as a semi-metric space each three points of which is circular.

Theorem 2. Four points p_1, p_2, p_3, p_4 of a semi-metric space are d-cyclic if and only if each three of the points is d-cyclic and
 $\Delta(p_1, p_2, p_3, p_4) = 0$.

To prove the necessity of the conditions we suppose that p_1, p_2, p_3, p_4 are congruent to four points p_1', p_2', p_3', p_4' of the circle. Then each three of the points is congruent to three points of the circle, and we have only to show that $\Delta(p_1, p_2, p_3, p_4) = 0$. We take as the origin of a two-dimensional rectangular coordinate system the center of the circle through p_1', p_2', p_3', p_4' . Denoting by A_i the cosine of the angle between the x-axis and the line joining the origin with p_i' , and by B_i the cosine of the angle between the y-axis and this line, we may write

$$\cos \alpha_{ij} = A_i A_j + B_i B_j ;$$

$$1 = A_i^2 + B_i^2 .$$

Employing these two relations we may express $\Delta(p_1, p_2, p_3, p_4)$ in the form

$$\Delta(p_1, p_2, p_3, p_4) = \begin{vmatrix} A_1^2 + B_1^2 & A_1 A_2 + B_1 B_2 & A_1 A_3 + B_1 B_3 & A_1 A_4 + B_1 B_4 \\ A_2 A_1 + B_2 B_1 & A_2^2 + B_2^2 & A_2 A_3 + B_2 B_3 & A_2 A_4 + B_2 B_4 \\ A_3 A_1 + B_3 B_1 & A_3 A_2 + B_3 B_2 & A_3^2 + B_3^2 & A_3 A_4 + B_3 B_4 \\ A_4 A_1 + B_4 B_1 & A_4 A_2 + B_4 B_2 & A_4 A_3 + B_4 B_3 & A_4^2 + B_4^2 \end{vmatrix}$$

$$= \begin{vmatrix} A_1 & B_1 & 0 & 0 \\ A_2 & B_2 & 0 & 0 \\ A_3 & B_3 & 0 & 0 \\ A_4 & B_4 & 0 & 0 \end{vmatrix} \chi \begin{vmatrix} A_1 & A_2 & A_3 & A_4 \\ B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

In proving the sufficiency of the conditions, we suppose that p_1, p_2, p_3, p_4 are such that each three of the points is d-cyclic and $\Delta(p_1, p_2, p_3, p_4) = 0$, and we show that there exist four points, say p'_1, p'_2, p'_3, p'_4 of the circle such that $p_1, p_2, p_3, p_4 \approx p'_1, p'_2, p'_3, p'_4$. At least one of the angles α_{ij} is not equal to π , for if we suppose that each angle has the value π , we find that $\Delta(p_1, p_2, p_3, p_4)$ equals -16 , which contradicts the hypothesis that $\Delta(p_1, p_2, p_3, p_4) = 0$. We may assume the labeling so that $\alpha_{12} \neq \pi$. Hence $p_1, p_2 \neq d$. By hypothesis there exist three points, say p'_1, p'_2, p'_3 , and three points, say $\bar{p}_1, \bar{p}_2, \bar{p}_4$, of the circle of metric diameter d such that

$$p_1, p_2, p_3 \approx p'_1, p'_2, p'_3 \quad \text{and}$$

$$p_1, p_2, p_4 \approx \bar{p}_1, \bar{p}_2, \bar{p}_4 .$$

Thus $p'_1 p'_2 = p_1 p_2 = \bar{p}_1 \bar{p}_2 \neq d$.

Now we may make a congruent transformation of the circle into

itself so that $\bar{p}_1$ goes into the point p_1' , and $\bar{p}_2$ goes into p_2' . This transformation sends $\bar{p}_4$ into some point, say p_4' , such that $p_1'p_4' = \bar{p}_1\bar{p}_4 = p_1p_4$ and $p_2'p_4' = \bar{p}_2\bar{p}_4 = p_2p_4$. Since p_4' has its distances from two points, p_1' and p_2' , determined and since these two points are not diametral, the point p_4' is uniquely determined. We now have

$$p_1, p_2, p_3 \approx p_1', p_2', p_3',$$

$$p_1, p_2, p_4 \approx p_1', p_2', p_4'.$$

In order to show that the quadruple p_1, p_2, p_3, p_4 is congruent to the d-cyclic quadruple p_1', p_2', p_3', p_4' , we have only to show that $p_3p_4 = p_3'p_4'$.

We define the expression $\Delta(x)$ as follows:

$$\Delta(x) = \begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{13} & \cos \alpha_{14} \\ \cos \alpha_{21} & 1 & \cos \alpha_{23} & \cos \alpha_{24} \\ \cos \alpha_{31} & \cos \alpha_{32} & 1 & \cos x \\ \cos \alpha_{41} & \cos \alpha_{42} & \cos x & 1 \end{vmatrix}$$

$\Delta(x)$ does not vanish identically since the coefficient of $\cos^2 x$, namely $-(1 - \cos^2 \alpha_{12})$, does not vanish. By the necessity of the conditions of the theorem, a root of $\Delta(x) = 0$ is $\frac{p_3'p_4'}{r}$, since p_1', p_2', p_3', p_4' are d-cyclic. By hypothesis, a root of $\Delta(x) = 0$ is $\frac{p_3p_4}{r}$. Since $\Delta(x) = 0$ is a quadratic equation in $\cos x$, $\Delta(x) = 0$ has only two roots in the interval $0 < x \leq \pi$. We shall show that $\frac{p_3p_4}{r}$ and $\frac{p_3'p_4'}{r}$ are double roots. We may expand $\Delta(x)$ in the form *

$$\Delta(x) = \frac{\Delta(p_1, p_2, p_3) \Delta(p_1, p_2, p_4) - \begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{13} \\ \cos \alpha_{21} & 1 & \cos \alpha_{23} \\ \cos \alpha_{41} & \cos \alpha_{42} & \cos x \end{vmatrix}^2}{1 - \cos^2 \alpha_{12}}$$

* E. B. Stouffer, American Mathematical Monthly, Vol. 39 (1932), p. 165.

By Theorem 1, $\Delta(p_1, p_2, p_4)$ and $\Delta(p_1, p_2, p_3)$ have the value 0. Thus the equation $\Delta(x) = 0$ may be written in the form

$$\begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{13} \\ \cos \alpha_{21} & 1 & \cos \alpha_{23} \\ \cos \alpha_{41} & \cos \alpha_{42} & \cos x \end{vmatrix}^2 = 0.$$

Hence $\frac{p_3 p_4}{n}$ and $\frac{p_3 p_1}{n}$ are double roots of the equation $\Delta(x) = 0$.

Therefore $p_3 p_4 = p_3 p_1$ and the theorem is proved.

Theorems 1 and 3 characterize d-cyclic triples and quadruples.

It has been shown that the circle has the congruence order four.* That is, a set is d-cyclic if each quadruple contained in the set is d-cyclic, Hence the necessary and sufficient condition that a set of points be d-cyclic is that each quadruple of the set satisfy the conditions stated in Theorem 3.[†]

We have defined a pseudo d-cyclic set of points as a set such that each three of the points is d-cyclic while the set itself is not d-cyclic. The remainder of this section is devoted to the characterization of such sets. First we shall characterize the pseudo d-cyclic quadruples. The three types of pseudo d-cyclic quadruples are the convex tripod, the pseudo-linear pseudo d-cyclic quadruple, and the proper pseudo d-cyclic quadruple, containing exactly three, exactly four, and [†]no linear triples respectively.

Lemma. If p_1, p_2, p_3, p_4 form a convex tripod such that each triple is d-cyclic, then the sum of opposite distances equals d and no two

* Karl Menger, "New Foundations of Euclidean Geometry", American Journal of Mathematics, Vol. 53 (1931), p. 725.

† It is obvious that if $p_1, p_2, \dots, p_n$ ($n > 4$) are d-cyclic, then $\Delta(p_1, p_2, \dots, p_n) = 0$. Cf. Bocher, Introduction to Higher Algebra, p. 57.
[†]L. M. Blumenthal, Loc. cit.

two points are diametral.

From the definition of the convex tripod, we may assume the labeling so that

$$\begin{aligned} p_2 p_4 &= p_2 p_1 + p_1 p_4; \\ p_3 p_4 &= p_3 p_1 + p_1 p_4; \\ p_2 p_3 &= p_2 p_1 + p_1 p_3; \\ p_2 p_3 + p_3 p_1 + p_1 p_4 &= 2d; \text{ where } p_1 p_3 \leq d. \end{aligned}$$

From these relations we obtain immediately

$$\left. \begin{aligned} p_1 p_3 + p_1 p_2 + p_1 p_4 &= d; \\ p_1 p_3 &= d - p_2 p_3; \\ p_1 p_2 &= d - p_3 p_4; \\ p_1 p_4 &= d - p_2 p_3. \end{aligned} \right\} \quad (1)$$

The relations (1) show that the sum of opposite distances equals d , and hence that no two of the points are diametral, since the points are distinct and consequently none of the distances is zero, which is what we wished to prove.

From (1) we have

$$\cos d_{24} = -\cos d_{13}; \cos d_{34} = -\cos d_{12}; \cos d_{14} = -\cos d_{23}; \cos d_{23} = \cos(\alpha_{12} + \alpha_{13})$$

Applying these relations, we may write

$$\Delta(p_1, p_2, p_3, p_4) = \begin{vmatrix} 1 & \cos d_{12} & \cos d_{13} & \cos d_{14} \\ \cos d_{12} & 1 & \cos d_{23} & \cos d_{13} \\ \cos d_{13} & \cos d_{23} & 1 & \cos d_{12} \\ \cos d_{14} & \cos d_{13} & \cos d_{12} & 1 \end{vmatrix} \quad (I)$$

If p_1, p_2, p_3, p_4 form a pseudo-linear pseudo d -cyclic quadruple, then opposite distances are equal*, and no two points are diametral†.

We may assume the labeling so that

* Menger, Loc. cit., p. 744.

† L. M. Blumenthal, Loc. cit., p. 730.

$$p_2 p_3 = p_1 p_2 + p_4 p_3 ; p_1 p_4 = p_1 p_2 + p_2 p_4 ;$$

$$p_1 p_4 = p_1 p_3 + p_3 p_4 ; p_2 p_3 = p_2 p_4 + p_3 p_4 ;$$

Then $p_1 p_2 = p_3 p_4 ; p_1 p_3 = p_2 p_4 ; p_1 p_4 = p_2 p_3$. Hence the following relations exist;

$$\cos \alpha_{24} = \cos \alpha_{13} ; \cos \alpha_{34} = \cos \alpha_{12} ; \cos \alpha_{14} = \cos \alpha_{23} ; \cos \alpha_{23} = \cos (\alpha_{12} + \alpha_{13}) \dots (2)$$

Making use of these relations we may again write $\Delta(p_1, p_2, p_3, p_4)$ in the form (I).

If p_1, p_2, p_3, p_4 form a proper pseudo d-cyclic quadruple, then "opposite distances are equal, no one of the distances equals the sum of the other two, the sum of three of the distances, no two of which are opposite, is equal to the length of the circle" and no two of the points are diametral*. Thus $\alpha_{12} + \alpha_{23} + \alpha_{13} = 2\pi$, and hence $\cos \alpha_{23} = \cos (\alpha_{12} + \alpha_{13})$. We note that again in this case the relations (2) hold, and $\Delta(p_1, p_2, p_3, p_4)$ may be written in the form (I).

Summarizing our investigations of the three kinds of pseudo d-cyclic quadruples, we find that if p_1, p_2, p_3, p_4 form a pseudo d-cyclic quadruple of any of the three kinds, we may express $\Delta(p_1, p_2, p_3, p_4)$ in the form (I), where $\cos \alpha_{23} = \cos (\alpha_{12} + \alpha_{13})$ and where none of the angles appearing in the determinant has the value π . Expanding this determinant, we have

$$\begin{aligned} \Delta(p_1, p_2, p_3, p_4) &= \left[\cos \alpha_{12} - \cos^3 \alpha_{12} - 2 \cos \alpha_{12} \cos \alpha_{23} + \cos \alpha_{12} \cos^2 \alpha_{13} \right. \\ &\quad \left. + \cos \alpha_{12} \cos^2 \alpha_{23} \right]^2 \\ &= \left[\cos \alpha_{12} (1 - \cos^2 \alpha_{12} - \cos^2 \alpha_{13} + \cos^2 \alpha_{12} \cos^2 \alpha_{13} + \sin^2 \alpha_{12} \sin^2 \alpha_{13}) \right. \\ &\quad \left. + 2 \sin \alpha_{12} \sin \alpha_{13} \cos \alpha_{13} (1 - \cos^2 \alpha_{12}) \right]^2 \end{aligned}$$

* L. M. Blumenthal, "A Complete Characterization of d-Cyclic and Pseudo d-Cyclic Sets of Points", American Journal of Mathematics, Vol. 54, p. 390.

$$\begin{aligned}\Delta(p_1, p_2, p_3, p_4) &= -4 \sin^2 \alpha_{12} \sin^2 \alpha_{13} [\sin \alpha_{13} \cos \alpha_{12} + \sin \alpha_{12} \cos \alpha_{13}]^2 \\ &= -4 \sin^2 \alpha_{12} \sin^2 \alpha_{13} \sin^2 (\alpha_{12} + \alpha_{13}) \\ &= -4 \sin^2 \alpha_{12} \sin^2 \alpha_{13} \sin^2 \alpha_{23}\end{aligned}\quad (3)$$

This value for $\Delta(p_1, p_2, p_3, p_4)$ is definitely negative, since no two of the points are diametral. Hence we have the following theorem:

Theorem 4. The determinant of a pseudo d-cyclic quadruple is negative.

Applying the relation (3) and making use of theorems 1 and 3, we have immediately the following theorem and corollaries.

Theorem 5. If all four third-order principal minors of the determinant $\Delta(p_1, p_2, p_3, p_4)$ vanish, and $0 < \alpha_{ij} < \pi$ ($i, j = 1, 2, 3, 4$), then $\Delta(p_1, p_2, p_3, p_4)$ either vanishes or has the value $-4 \sin^2 \alpha_{12} \sin^2 \alpha_{13} \sin^2 \alpha_{23}$, where $\alpha_{12} < \pi$, and $\cos^2 \alpha_{12} = \cos^2 \alpha_{34}$, $\cos^2 \alpha_{23} = \cos^2 \alpha_{14}$, $\cos^2 \alpha_{13} = \cos^2 \alpha_{24}$.

Corollary 1. The points p_1, p_2, p_3, p_4 form a pseudo d-cyclic convex tripod if and only if $\Delta(p_1, p_2, p_3, p_4) \neq 0$, each third-order principal minor vanishes, and $\cos \alpha_{12} = -\cos \alpha_{34}$, $\cos \alpha_{23} = -\cos \alpha_{14}$, $\cos \alpha_{13} = -\cos \alpha_{24}$.

Corollary 2. The points p_1, p_2, p_3, p_4 form a pseudo-linear pseudo d-cyclic quadruple if and only if $\Delta(p_1, p_2, p_3, p_4) \neq 0$, each third-order principal minor vanishes, $\alpha_{12} = \alpha_{34}$, $\alpha_{23} = \alpha_{14}$, $\alpha_{13} = \alpha_{24}$, and all triples are linear.

Corollary 3. The points p_1, p_2, p_3, p_4 form a proper pseudo d-cyclic quadruple if and only if $\Delta(p_1, p_2, p_3, p_4) \neq 0$, each third-order principal minor vanishes, $\alpha_{12} = \alpha_{34}$, $\alpha_{23} = \alpha_{14}$, $\alpha_{13} = \alpha_{24}$, and no triples are linear.

We shall now describe methods of constructing the three forms of pseudo d-cyclic quadruples.

The pseudo d-cyclic convex tripod and the proper pseudo d-cyclic quadruple. Let p'_1, p'_2, p'_3 be three non-linear points of the circle of

Fig. 1.

metric diameter d . Reflect p'_3 in the diameter through p'_1 , obtaining the point p^*_3 . Reflect p'_3 in the diameter through p'_2 , obtaining p^{**}_3 . Let p'_4 and $\bar{p}'_4$ be the two points of the circle equidistant from p^*_3 and p^{**}_3 .

We shall let p'_4 be the midpoint of the shorter arc joining p^*_3 and p^{**}_3 , and $\bar{p}'_4$ the point diametral to p'_4 (see Fig.1). Define the quadruple p, p_2, p_3, p_4 such that $p, p_2 = p'_1 p'_2$; $p, p_3 = p'_1 p'_3$; $p_2 p_3 = p'_2 p'_3$; $p, p_4 = p'_1 p'_4$; $p_2 p_4 = p'_2 p'_4$; $p_3 p_4 = p^*_3 p'_4 = p^{**}_3 p'_4$. The triples p, p_2, p_4 ; p, p_3, p_4 ; p_2, p_3, p_4 are d-cyclic and linear with p_4 between p and p_2 , p and p_3, p_2 and p_3 . The triple p, p_2, p_3 is not linear, but is d-cyclic. Hence, p, p_2, p_3, p_4 form a pseudo d-cyclic convex tripod.

If p, p_2, p_3, p_4 are defined such that $p, p_2 = p'_1 p'_2$, $p, p_3 = p'_1 p'_3$, $p_2 p_3 = p'_2 p'_3$, $p, p_4 = p'_1 \bar{p}'_4$, $p_2 p_4 = p'_2 \bar{p}'_4$, $p_3 p_4 = p^*_3 \bar{p}'_4 = p^{**}_3 \bar{p}'_4$, then it is seen immediately that each triple is d-cyclic and not linear and that opposite distances are equal. In this case p, p_2, p_3, p_4 form a proper pseudo d-cyclic quadruple. Provided that p'_1, p'_2, p'_3 are not linear, it is true for any arrangement of the points that the convex tripod and the proper pseudo d-cyclic quadruple can be constructed by the methods described above. One of the points p'_4 and $\bar{p}'_4$ —which one depending upon the positions of p'_1, p'_2, p'_3 — will give the convex tripod and the other will give the proper pseudo d-cyclic quadruple. By choosing the points p'_1, p'_2, p'_3 such that the triple is equilateral and by applying the methods described above, a proper pseudo d-cyclic quadruple which is equilateral may be constructed.

The pseudo-linear pseudo d-cyclic quadruple. If the points p_1', p_2', p_3' are chosen so that the triple is linear and no two of the points are diametral and if p_4' and $\bar{p}_4'$ are determined as in the previous case and the quadruple p_1, p_2, p_3, p_4 defined as above, then in either case each triple is d-cyclic and linear, opposite distances are equal, and the four points form a pseudo-linear pseudo d-cyclic quadruple.

Pseudo d-cyclic sets no four points of which form a convex tripod are called regular. In order to establish the theorem characterizing sets of this type which contain more than four points we shall prove two lemmas concerning the fifth-order determinant

$\Delta(p_1, p_2, p_3, p_4, p_5)$. Consider the determinant $\Delta(p_1, p_2, p_3, p_4, p_5)$ subject to the following hypotheses:

- (a). $0 < \alpha_{ij} \leq \pi$
- (b). Each third-order principal minor vanishes.
- (c). At least one fourth-order principal minor does not vanish.
- (d). If $\Delta(p_i, p_j, p_k, p_\ell)$ is any non-vanishing fourth-order principal minor, it is possible so to arrange the labeling of the elements of the minor that $\cos \alpha_{ij} + \cos \alpha_{k\ell} \neq 0$.

Lemma 1. $\Delta(p_1, p_2, p_3, p_4, p_5)$ does not contain exactly one non-vanishing fourth-order principal minor.

We shall assume that $\Delta(p_1, p_2, p_3, p_4) \neq 0$. Then by theorem 5, $\Delta(p_1, p_2, p_3, p_4)$ is negative and none of the six angles contained in this minor has the value π . By the three corollaries to theorem 5 we have

$$\cos \alpha_{12} = \cos \alpha_{34}; \cos \alpha_{13} = \cos \alpha_{24}; \cos \alpha_{23} = \cos \alpha_{14}. \quad (1)$$

We suppose that the four remaining fourth-order principal minors

$$\Delta(p_1, p_2, p_3, p_5), \Delta(p_1, p_2, p_4, p_5), \Delta(p_1, p_3, p_4, p_5), \text{ and } \Delta(p_2, p_3, p_4, p_5)$$

all vanish and show that this assumption leads to a contradiction.

Expanding each of these four minors, we obtain the following relations.

$$\begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{13} \\ \cos \alpha_{12} & 1 & \cos \alpha_{23} \\ \cos \alpha_{15} & \cos \alpha_{25} & \cos \alpha_{35} \end{vmatrix} = 0; \quad \begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{14} \\ \cos \alpha_{12} & 1 & \cos \alpha_{24} \\ \cos \alpha_{15} & \cos \alpha_{25} & \cos \alpha_{45} \end{vmatrix} = 0;$$

$$\begin{vmatrix} 1 & \cos \alpha_{13} & \cos \alpha_{14} \\ \cos \alpha_{13} & 1 & \cos \alpha_{34} \\ \cos \alpha_{15} & \cos \alpha_{35} & \cos \alpha_{45} \end{vmatrix} = 0; \quad \begin{vmatrix} 1 & \cos \alpha_{23} & \cos \alpha_{24} \\ \cos \alpha_{23} & 1 & \cos \alpha_{34} \\ \cos \alpha_{25} & \cos \alpha_{35} & \cos \alpha_{45} \end{vmatrix} = 0.$$

Writing $a = \cos \alpha_{12} = \cos \alpha_{34}$; $b = \cos \alpha_{13} = \cos \alpha_{24}$; $c = \cos \alpha_{23} = \cos \alpha_{14}$,

and expanding the four determinants above, we have

$$(a-c)\cos \alpha_{15} + (ab-c)\cos \alpha_{25} + (1-a^2)\cos \alpha_{35} = 0$$

$$(ab-c)\cos \alpha_{15} + (a-c)\cos \alpha_{25} + (1-a^2)\cos \alpha_{45} = 0$$

$$(ab-c)\cos \alpha_{15} + (bc-a)\cos \alpha_{35} + (1-b^2)\cos \alpha_{45} = 0$$

$$(a-c)\cos \alpha_{45} + (bc-a)\cos \alpha_{35} + (1-c^2)\cos \alpha_{45} = 0$$

Now $\cos \alpha_{15}$, $\cos \alpha_{25}$, $\cos \alpha_{35}$, $\cos \alpha_{45}$ are not all zero. In fact, we

get a contradiction by supposing that two of them, say $\cos \alpha_{15}$ and

$\cos \alpha_{25}$, are zero. For if so, then $\alpha_{15} = \alpha_{25} = \frac{\pi}{2}$. Therefore, since

$\Delta(p_1, p_2, p_5) = 0$, α_{12} must have the value 0 or π , which is impossible.

The four equations cannot be satisfied, then, unless

$$\begin{vmatrix} ac-b & ab-c & 1-a^2 & 0 \\ ab-c & ac-b & 0 & 1-a^2 \\ ab-c & 0 & bc-a & 1-b^2 \\ 0 & ac-b & bc-a & 1-c^2 \end{vmatrix} = 0$$

That is

$$\begin{vmatrix} \cos \alpha_{12} \cos \alpha_{23} - \cos \alpha_{13} & \cos \alpha_{12} \cos \alpha_{13} - \cos \alpha_{23} & \sin^2 \alpha_{12} \\ \cos \alpha_{12} \cos \alpha_{13} - \cos \alpha_{23} & \cos \alpha_{12} \cos \alpha_{23} - \cos \alpha_{13} & 0 & \sin^2 \alpha_{12} \\ \cos \alpha_{12} \cos \alpha_{13} - \cos \alpha_{23} & 0 & \cos \alpha_{13} \cos \alpha_{23} - \cos \alpha_{12} & \sin^2 \alpha_{13} \\ 0 & \cos \alpha_{12} \cos \alpha_{23} - \cos \alpha_{13} & \cos \alpha_{13} \cos \alpha_{23} - \cos \alpha_{12} & \sin^2 \alpha_{23} \end{vmatrix} = 0. \quad (2)$$

In the proof of theorem 1, we showed that α_{12} , α_{13} , α_{23} are such that

either $\alpha_{12} + \alpha_{13} + \alpha_{23} = 2\pi$ or one angle equals the sum of the other two. Thus, two cases present themselves.

Case 1. $\alpha_{12} + \alpha_{13} + \alpha_{23} = 2\pi$ (3)

In this case equation (2) becomes

$$\begin{vmatrix} \sin \alpha_{12} \sin \alpha_{23} & \sin \alpha_{12} \sin \alpha_{13} & \sin^2 \alpha_{12} & 0 \\ \sin \alpha_{12} \sin \alpha_{13} & \sin \alpha_{12} \sin \alpha_{23} & 0 & \sin^2 \alpha_{12} \\ \sin \alpha_{12} \sin \alpha_{13} & 0 & \sin \alpha_{13} \sin \alpha_{23} & \sin^2 \alpha_{13} \\ 0 & \sin \alpha_{12} \sin \alpha_{23} & \sin \alpha_{13} \sin \alpha_{23} & \sin^2 \alpha_{23} \end{vmatrix} = 0$$

Factoring $\sin \alpha_{12}$ out of the first two rows, $\sin \alpha_{13}$ out of the third row, and $\sin \alpha_{23}$ out of the fourth row, and interchanging the first and fourth columns and the second and third columns, we have

$$\begin{vmatrix} 0 & \sin \alpha_{12} & \sin \alpha_{13} & \sin \alpha_{23} \\ \sin \alpha_{12} & 0 & \sin \alpha_{23} & \sin \alpha_{13} \\ \sin \alpha_{13} & \sin \alpha_{23} & 0 & \sin \alpha_{12} \\ \sin \alpha_{23} & \sin \alpha_{13} & \sin \alpha_{12} & 0 \end{vmatrix} = 0$$

Evaluation of this determinant gives the relation

$$\begin{aligned} & (\sin \alpha_{12} + \sin \alpha_{23} + \sin \alpha_{13}) (\sin \alpha_{12} + \sin \alpha_{23} - \sin \alpha_{13}) \\ & \times (\sin \alpha_{12} - \sin \alpha_{23} + \sin \alpha_{13}) (-\sin \alpha_{12} + \sin \alpha_{23} + \sin \alpha_{13}) = 0. \end{aligned} \quad (4)$$

The first factor cannot vanish. Hence one of the remaining factors vanishes. Suppose the second factor vanishes. Then

$$\sin \alpha_{12} + \sin \alpha_{23} - \sin \alpha_{13} = 0$$

From (3) $\sin \alpha_{13} + \sin \alpha_{12} \cos \alpha_{23} + \cos \alpha_{12} \sin \alpha_{23} = 0$

Adding these two equations we have

$$\sin \alpha_{12} (1 + \cos \alpha_{23}) + \sin \alpha_{23} (1 + \cos \alpha_{12}) = 0.$$

This is impossible since both terms are definitely positive. Similarly we may show it is impossible that any of the remaining factors vanish.

Case 2. One of the angles $\alpha_{12}, \alpha_{13}, \alpha_{23}$ equals the sum of the other two. We may assume that

$$\alpha_{13} = \alpha_{12} + \alpha_{23}. \quad (5)$$

In this case, equation (2) becomes

$$\begin{vmatrix} \sin \alpha_{12} \sin \alpha_{23} & -\sin \alpha_{12} \sin \alpha_{13} & \sin^2 \alpha_{12} & 0 \\ -\sin \alpha_{12} \sin \alpha_{13} & \sin \alpha_{12} \sin \alpha_{23} & 0 & \sin^2 \alpha_{12} \\ \sin \alpha_{12} \sin \alpha_{13} & 0 & -\sin \alpha_{13} \sin \alpha_{23} & \sin^2 \alpha_{13} \\ 0 & \sin \alpha_{12} \sin \alpha_{23} & -\sin \alpha_{13} \sin \alpha_{23} & \sin^2 \alpha_{23} \end{vmatrix} = 0$$

Factoring $\sin \alpha_{12}$ out of the first two rows, $\sin \alpha_{13}$ out of the third row, and $\sin \alpha_{23}$ out of the fourth row, and interchanging the first and fourth columns and the second and third columns, we have

$$\begin{vmatrix} 0 & \sin \alpha_{12} & -\sin \alpha_{13} & \sin \alpha_{23} \\ \sin \alpha_{12} & 0 & \sin \alpha_{23} & -\sin \alpha_{13} \\ \sin \alpha_{13} & -\sin \alpha_{23} & 0 & -\sin \alpha_{12} \\ \sin \alpha_{23} & -\sin \alpha_{13} & \sin \alpha_{12} & 0 \end{vmatrix} = 0$$

This determinant is the same as the corresponding determinant considered in Case 1 except for the factor -1, as may be seen by multiplying the second and fourth columns and the first row by -1. Hence, again we have the equation (4). Assume first that the second factor of (4) vanishes.

$$\text{Then} \quad \sin \alpha_{12} + \sin \alpha_{23} - \sin \alpha_{13} = 0.$$

$$\text{From (5)} \quad \sin \alpha_{12} \cos \alpha_{23} + \sin \alpha_{23} \cos \alpha_{12} - \sin \alpha_{13} = 0$$

Subtracting these two equations we have

$$\sin \alpha_{12} (1 - \cos \alpha_{23}) + \sin \alpha_{23} (1 - \cos \alpha_{12}) = 0.$$

This is impossible, since both terms are definitely positive. Assuming the third factor of (4) vanishes we have

$$\sin \alpha_{13} + \sin \alpha_{12} - \sin \alpha_{23} = 0,$$

$$\text{From (5)} \quad \sin \alpha_{13} \cos \alpha_{12} - \sin \alpha_{12} \cos \alpha_{13} - \sin \alpha_{23} = 0.$$

Subtracting these two equations, we obtain the relation

$$\sin \alpha_{12} (1 + \cos \alpha_{13}) + \sin \alpha_{13} (1 - \cos \alpha_{12}) = 0.$$

which is also impossible. Similarly we may show that it is impossible for the fourth factor of (4) to vanish. Hence the proof of the lemma is completed.

Lemma 2. No fourth-order principal minor of $\Delta(p_1, p_2, p_3, p_4, p_5)$ vanishes.

By the preceding lemma, there are at least two non-vanishing fourth-order principal minors. We may assume, then, that $\Delta(p_1, p_2, p_3, p_4)$ and $\Delta(p_1, p_2, p_3, p_5)$ do not vanish. Then none of the nine angles contained in these two minors has the value π , and

$$\left. \begin{aligned} \cos \alpha_{12} &= \cos \alpha_{34} = \cos \alpha_{35} \\ \cos \alpha_{13} &= \cos \alpha_{24} = \cos \alpha_{25} \\ \cos \alpha_{23} &= \cos \alpha_{14} = \cos \alpha_{15} \end{aligned} \right\} \quad (1)$$

by the corollaries to theorem 5.

Suppose that any other fourth-order principal minor, say $\Delta(p_1, p_2, p_4, p_5)$ vanishes. Then from the expansion of $\Delta(p_1, p_2, p_4, p_5)$ we have

$$\begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{14} \\ \cos \alpha_{12} & 1 & \cos \alpha_{24} \\ \cos \alpha_{15} & \cos \alpha_{25} & \cos \alpha_{45} \end{vmatrix} = 0$$

Applying (1), we may write this equation in the form

$$\begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{14} \\ \cos \alpha_{12} & 1 & \cos \alpha_{24} \\ \cos \alpha_{14} & \cos \alpha_{24} & \cos \alpha_{45} \end{vmatrix} = 0.$$

Consider the function $\varphi(x)$ defined as follows:

$$\varphi(x) = \begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{14} \\ \cos \alpha_{12} & 1 & \cos \alpha_{24} \\ \cos \alpha_{14} & \cos \alpha_{24} & x \end{vmatrix}.$$

Since $\Delta(p_1, p_2, p_4) = 0$, one root of $\varphi(x) = 0$ is $x = 1$. Moreover, since $\varphi(x) = 0$ is a linear equation and since the coefficient of x , namely $(1 - \cos^2 \alpha_{12})$ does not vanish, 1 is the only root of $\varphi(x) = 0$. Therefore $\cos \alpha_{45} = 1$, and hence $\alpha_{45} = 0$, which is impossible. Hence the lemma is proved.

Hence, under the conditions imposed, no fourth-order principal minor of the determinant $\Delta(p_1, p_2, p_3, p_4, p_5)$ vanishes. By theorem 3 each fourth-order principal minor is negative and, by the corollaries to theorem 5, opposite angles in each quadruple are equal. Making use of this fact, we have immediately the following theorem.

Theorem 6. Each element of the determinant $\Delta(p_1, p_2, p_3, p_4, p_5)$ outside the principal diagonal has the same value, and this value is $-\frac{1}{2}$.

The first part of this theorem is obtained by equating opposite angles in each quadruple. Since each triple is d-cyclic, in order that each of the three angles in each triple have the same value, this value must be $\frac{2\pi}{3}$. Hence $\cos \alpha_{ij} = -\frac{1}{2}$. The value of this determinant is found to be $-(\frac{3}{2})^4$.

Stating theorem 6 as a theorem on sets of points, we have the

Theorem 7. A regular pseudo d-cyclic quintuple is equilateral*.

By applying mathematical induction we obtain a more general theorem

Theorem 8. A regular pseudo d-cyclic set containing more than four points is equilateral.

We have shown that the theorem holds if the set consists of exactly five points. We shall assume the theorem holds for sets containing k points ($k > 4$) and show that it follows that the theorem is true

* This theorem as well as the following one are proved by L. M. Blumenthal in the paper "Concerning Regular Pseudo d-Cyclic Sets", American Journal of Mathematics, Vol. 54, pp. 729-738. The proof given above for theorem 7 is quite different from the proof given by Blumenthal.

for sets containing $k+1$ points.

Let $p_1, p_2, \dots, p_k, p_{k+1}$ be a regular pseudo d-cyclic set containing exactly $k+1$ points. At least one k -tuple contained in these $k+1$ points is a pseudo d-cyclic set, for otherwise each k points and hence each four points is d-cyclic. But since the circle has the congruence order four, if each four points is d-cyclic, the $k+1$ points are d-cyclic, which contradicts the hypothesis that the set is pseudo d-cyclic. We may assume the labeling, then, so that $p_1, p_2, \dots, p_k$ are pseudo d-cyclic.

At least one other k -tuple is pseudo d-cyclic. For if we suppose that all k -tuples except $p_1, p_2, \dots, p_k$ are d-cyclic, then each quadruple contained in them is d-cyclic. But all the quadruples contained in $p_1, p_2, \dots, p_k$ are contained in the k remaining k -tuples. Hence, from our supposition it follows that the set $p_1, p_2, \dots, p_k$ is d-cyclic, which is impossible. Thus, we may assume that $p_2, p_3, \dots, p_{k+1}$ is a pseudo d-cyclic k -tuple. Since we assume the theorem true for pseudo d-cyclic sets containing k points, the sets $p_1, p_2, \dots, p_k$ and $p_2, p_3, \dots, p_k, p_{k+1}$ are equilateral and each distance contained in the two sets equals $\frac{2d}{3}$. All the distances, then, except p_1, p_{k+1} , have been shown to have the value $\frac{2d}{3}$. To show that p_1, p_{k+1} equals $\frac{2d}{3}$, we consider the triple p_1, p_2, p_{k+1} . This triple is d-cyclic and

$$p_1 p_2 = p_2 p_{k+1} = \frac{2d}{3}$$

Evidently the triple p_1, p_2, p_{k+1} cannot be linear. Hence

$$p_1 p_2 + p_2 p_{k+1} + p_1 p_{k+1} = 2d$$

Therefore $p_1 p_{k+1} = \frac{2d}{3}$, and the theorem is proved.

Changing back to our determinantal notation, we have the following theorem as the analogue to theorem 8.

Theorem 9. If $\Delta(p_1, p_2, \dots, p_n)$, $n > 4$, is such that

- (a). $0 < \alpha_{ij} \leq \pi$.
- (b). Each third-order principal minor vanishes.
- (c). At least one fourth-order principal minor does not

vanish.

(d). If $\Delta(p_i, p_j, p_k, p_l)$ is any non-vanishing fourth-order principal minor, it is possible so to arrange the labeling of the elements of the minor that

$$\cos \alpha_{ij} + \cos \alpha_{kl} \neq 0,$$

then each angle contained in the determinant has the value $\frac{2\pi}{3}$, and the determinant has the value $-\frac{1}{2} \left(\frac{3}{2}\right)^{n-1} (n-3)$.

In order to obtain the value of $\Delta(p_1, p_2, \dots, p_n)$ given above, we evaluate the determinant of the n -th order defined as follows.

$$\begin{aligned} A_n &= \begin{vmatrix} 1 & a & a & a & \dots & a \\ a & 1 & a & a & \dots & a \\ a & a & 1 & a & \dots & a \\ a & a & a & 1 & \dots & a \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & a & a & \dots & 1 \end{vmatrix} = \begin{vmatrix} 1-a & a-1 & 0 & 0 & \dots & 0 \\ a & 1 & a & a & \dots & a \\ 0 & a-1 & 1-a & 0 & \dots & 0 \\ 0 & a-1 & 0 & 1-a & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a-1 & 0 & 0 & \dots & 1-a \end{vmatrix} \\ &= (1-a) \begin{vmatrix} 1 & a & a & \dots & a \\ a-1 & 1-a & 0 & \dots & 0 \\ a-1 & 0 & 1-a & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a-1 & 0 & 0 & \dots & 1-a \end{vmatrix} - a \begin{vmatrix} a-1 & 0 & 0 & \dots & 0 \\ a-1 & 1-a & 0 & \dots & 0 \\ a-1 & 0 & 1-a & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a-1 & 0 & 0 & \dots & 1-a \end{vmatrix} \\ &= (1-a) A_{n-1} + a(1-a)^{n-1} \end{aligned}$$

(1)

Since $A_1 = 1$, we have

$$A_1 = 1$$

$$A_2 = (1 - a)(1 + a)$$

$$A_3 = (1 - a)^2(1 + 2a)$$

which suggest the formula

$$A_n = (1 - a)^{n-1} [1 + (n - 1) a] \quad (2).$$

If we assume the formula (2) holds for $n = k$, then by applying (1) we find that the formula holds for $n = k + 1$. Hence, by mathematical induction, the formula (2) is correct. If we substitute $-\frac{1}{2}$ for a in (2) we obtain the value $-\frac{1}{2} \left(\frac{3}{2}\right)^{n-1} (n-1)$ for A_n which is what we wished to show.

II. The Sphere, $S_{n,r}$.

We shall characterize r -spheric and pseudo r -spheric sets by means of theorems similar to the theorems characterizing d -cyclic and pseudo d -cyclic sets. It is evident that d -cyclic sets are included among the r -spheric sets. Throughout this section d equals r .

We define the angle α_{ij} and the determinant $\Delta(p_1, p_2, \dots, p_n)$ as in Section I. We shall denote by o the center of the sphere of radius r and by $v(p_1, p_2, \dots, p_n, o)$ the volume of the simplex determined by the $n+1$ points $p_1, p_2, \dots, p_n, o$.

Lemma 1. $\Delta(p_1, p_2, \dots, p_n) = \left(\frac{n!}{n^n}\right)^2 v^2(p_1, p_2, \dots, p_n, o).$

Denoting the Euclidean distance between p_i and p_j by $\overline{ij}$ we have

$$\overline{ij}^2 = 2r^2 - 2r^2 \cos \alpha_{ij} \quad (1)$$

But
$$v^2(p_1, p_2, \dots, p_n, o) = \frac{(-1)^{n+1}}{(n!)^2 2^n} D(p_1, p_2, \dots, p_n, o)^* \quad (2)$$

* Menger, Loc. cit., p. 738.

$$\text{where } D(p_1, p_2, \dots, p_n, 0) = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & \overline{12}^2 & \dots & \overline{1n}^2 & \lambda^2 \\ 1 & \overline{21}^2 & 0 & \dots & \overline{2n}^2 & \lambda^2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \overline{n1}^2 & \overline{n2}^2 & \dots & 0 & \lambda^2 \\ 1 & \lambda^2 & \lambda^2 & \dots & \lambda^2 & 0 \end{vmatrix}$$

Subtracting the last row of the determinant $D(p_1, p_2, \dots, p_n, 0)$ from each preceding row except the first, and subtracting the last column from each preceding column, we have

$$D(p_1, p_2, \dots, p_n, 0) = \begin{vmatrix} -2\lambda^2 & \overline{12}^2 - 2\lambda^2 & \dots & \overline{1n}^2 - 2\lambda^2 \\ \overline{21}^2 - 2\lambda^2 & -2\lambda^2 & \dots & \overline{2n}^2 - 2\lambda^2 \\ \vdots & \vdots & & \vdots \\ \overline{n1}^2 - 2\lambda^2 & \overline{n2}^2 - 2\lambda^2 & \dots & -2\lambda^2 \end{vmatrix}$$

Applying (1) this equation may be written in the form

$$D(p_1, p_2, \dots, p_n, 0) = (-1)^{n+1} (2\lambda^2)^n \Delta(p_1, p_2, \dots, p_n).$$

Substituting this value for $D(p_1, p_2, \dots, p_n, 0)$ in (2) we obtain the desired relation

$$\Delta(p_1, p_2, \dots, p_n) = \left(\frac{n!}{\lambda^n}\right)^2 v^2(p_1, p_2, \dots, p_n, 0).$$

As a corollary to this lemma we have the following lemma.

Lemma 2. If $p_1, p_2, \dots, p_n$ ($n > 3$) are r -spheric, then

$$\underline{\Delta(p_1, p_2, \dots, p_n) = 0}.$$

If n equals 3, then $v^2(p_1, p_2, p_3, 0)$ has the value zero if $p_1, p_2, p_3, 0$ lie in a plane and is positive if $p_1, p_2, p_3, 0$ do not lie in a plane. Since $p_1, p_2, p_3, 0$ lie in a plane if and only if p_1, p_2, p_3 lie on a great circle of the sphere, we have the following lemma.

Lemma 3. If p_1, p_2, p_3 are r -spheric and not d -cyclic then

$$\underline{\Delta(p_1, p_2, p_3) \text{ is positive.}}$$

Theorem 10. Three points p_1, p_2, p_3 of a semi-metric space are r -spheric if and only if $0 < \alpha_i \leq \pi$ ($i=1,2,3$) and $\Delta(p_1, p_2, p_3) \geq 0$.

The necessity of the conditions follows immediately from Lemma 3 and theorem 1. To prove the sufficiency of the conditions we consider two cases.

Case 1. $\Delta(p_1, p_2, p_3) = 0$. In this case the three points are d -cyclic, by theorem 1, and hence are congruent to three points of a great circle of the sphere.

Case 2. $\Delta(p_1, p_2, p_3) > 0$. We shall show that the three angles $\alpha_{12}, \alpha_{13}, \alpha_{23}$ satisfy the triangular inequality and that their sum is less than 2π . We have

$$\Delta(p_1, p_2, p_3) = 4 \sin \frac{\alpha_{12} + \alpha_{13} + \alpha_{23}}{2} \sin \frac{\alpha_{12} + \alpha_{23} - \alpha_{13}}{2} \sin \frac{\alpha_{12} - \alpha_{13} + \alpha_{23}}{2} \sin \frac{-\alpha_{12} + \alpha_{13} + \alpha_{23}}{2} \quad (1)$$

Each of the three angles cannot have the value π , for if so then

$\Delta(p_1, p_2, p_3)$ has the value -4 , which is impossible. Hence

$$0 < \alpha_{12} + \alpha_{23} + \alpha_{13} < 3\pi.$$

We shall show first that

$$0 < \alpha_{12} + \alpha_{23} + \alpha_{13} < 2\pi$$

Obviously $\alpha_{12} + \alpha_{23} + \alpha_{13} \neq 2\pi$ for if so, then by (1) $\Delta(p_1, p_2, p_3)$ equals zero which is impossible. We suppose, then, that

$$2\pi < \alpha_{12} + \alpha_{23} + \alpha_{13} < 3\pi \quad (2)$$

and show that this supposition leads to a contradiction. We shall

denote by A, B , and C the angles $\frac{\alpha_{12} + \alpha_{23} - \alpha_{13}}{2}$, $\frac{\alpha_{12} - \alpha_{23} + \alpha_{13}}{2}$, and $\frac{-\alpha_{12} + \alpha_{23} + \alpha_{13}}{2}$

respectively. Then $|A|$, $|B|$, and $|C|$ are all less than π . Now, no

one of the angles A, B, C is negative; for suppose that any one of

them, say A , is negative. Then $\alpha_{12} + \alpha_{23} < \alpha_{13} \leq \pi$. Hence $\alpha_{12} + \alpha_{23} + \alpha_{13} < 2\pi$

which contradicts (2). Therefore $\sin A, \sin B, \sin C$ are all positive.

From (2), $\sin \frac{\alpha_{12} + \alpha_{23} + \alpha_{13}}{2}$ is negative. Substituting in (1), then,

we find that $\Delta(p_1, p_2, p_3)$ is negative and we have a contradiction.

Hence

$$0 < \alpha_{12} + \alpha_{23} + \alpha_{13} < 2\pi$$

Therefore $\sin \frac{\alpha_{12} + \alpha_{23} + \alpha_{13}}{2}$ is positive. Since $\Delta(p_1, p_2, p_3)$ is positive, we see from (1) that $\sin A \sin B \sin C$ must be positive. Hence, either all three of the factors $\sin A$, $\sin B$, and $\sin C$ are positive, or two of them are negative and one is positive. We show that the latter condition cannot arise. Suppose that two of the factors, say $\sin A$ and $\sin B$, are negative. Then since $|A|$ and $|B|$ are less than π , A and B themselves must be negative. Hence, we have

$$\alpha_{12} + \alpha_{23} - \alpha_{13} < 0 \quad \text{and}$$

$$\alpha_{12} - \alpha_{23} + \alpha_{13} < 0$$

Therefore, $\alpha_{12} < 0$

which is impossible. Hence, $\sin A$, $\sin B$, $\sin C$ are all positive and consequently A , B , C are all positive. Thus,

$$\alpha_{12} + \alpha_{23} > \alpha_{13}$$

$$\alpha_{12} + \alpha_{13} > \alpha_{23}$$

$$\alpha_{23} + \alpha_{13} > \alpha_{12}$$

These relations show that α_{12} , α_{23} , and α_{13} satisfy the triangular inequality, and we have shown above that the sum of the three angles does not exceed 2π . Then there exist three points of the sphere of radius r congruent to p_1, p_2, p_3 .

Theorem 11. Four points p_1, p_2, p_3, p_4 of a semi-metric space are r -spheric if and only if each three of the points is r -spheric and $\Delta(p_1, p_2, p_3, p_4) = 0$.

If the four points are r -spheric then each three is r -spheric and, by lemma 2, $\Delta(p_1, p_2, p_3, p_4)$ equals zero, and the necessity of the conditions is proved.

To prove the sufficiency we suppose $\Delta(p_1, p_2, p_3, p_4) = 0$ and that each three points is r -spheric, and we show there exist four points of the sphere congruent to p_1, p_2, p_3, p_4 . The theorem is immediate if each three points is d -cyclic for then, by theorem 3, the four points are congruent to four points of a great circle of the sphere. We shall suppose, then, that at least one triple, say p_1, p_2, p_3 , is not d -cyclic. Then no two of the points p_1, p_2, p_3 have the distance πr ; i. e. α_{12} , α_{23} , and α_{13} are all less than π . By hypothesis there exist three points, say p'_1, p'_2, p'_3 , of the sphere and three points, say $\bar{p}_1, \bar{p}_2, \bar{p}_4$, of the sphere such that

$$p_1, p_2, p_3 \approx p'_1, p'_2, p'_3 \quad \text{and}$$

$$p_1, p_2, p_4 \approx \bar{p}_1, \bar{p}_2, \bar{p}_4.$$

Then

$$\bar{p}_1 \bar{p}_2 = p'_1 p'_2 = p_1 p_2.$$

Now we may make a congruent transformation of the sphere into itself so that $\bar{p}_1$ goes into p'_1 , and $\bar{p}_2$ goes into p'_2 . This transformation sends $\bar{p}_4$ into some point of the sphere, say p_4^* , such that $p'_1 p_4^* = p_1 p_4$ and $p'_2 p_4^* = p_2 p_4$. Since p_4^* has its distances from the two points p'_1 and p'_2 determined and since these two points are not diametral, there are at most two such points p_4^* . Two cases present themselves.

Case 1. p_1, p_2, p_4 are not d -cyclic. In this case there are exactly two possible positions on the sphere for p_4^* . We shall denote these points by p_4^{I} and p_4^{II} . Then these two points are reflections of each other in the plane through p'_1 and p'_2 and the center of the sphere. Since p_1, p_2, p_3 are not d -cyclic, the point p'_3 is not on this plane and hence $p'_3 p_4^{\text{I}} \neq p'_3 p_4^{\text{II}}$. We now have

$$p_1, p_2, p_3 \approx p'_1, p'_2, p'_3$$

$$p_1, p_2, p_4 \approx p'_1, p'_2, p_4^{\text{I}} \approx p'_1, p'_2, p_4^{\text{II}}.$$

Hence, in order to prove the theorem we have only to show that $p_3 p_4 = p_3' p_4'$ or $p_3 p_4 = p_3' p_4''$. In order to do this we define the expression

$\Delta(x)$ as follows:

$$\Delta(x) = \begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{13} & \cos \alpha_{14} \\ \cos \alpha_{21} & 1 & \cos \alpha_{23} & \cos \alpha_{24} \\ \cos \alpha_{31} & \cos \alpha_{32} & 1 & \cos x \\ \cos \alpha_{41} & \cos \alpha_{42} & \cos x & 1 \end{vmatrix}$$

$\Delta(x)$ is not identically zero for the coefficient of $\cos^2 x$, namely $-(1 - \cos^2 \alpha_{12})$ does not vanish. By the necessity of the conditions of the theorem, $\Delta(x) = 0$ has the two unequal roots $\frac{p_3' p_4'}{r}$ and $\frac{p_3' p_4''}{r}$. By hypothesis a root of $\Delta(x) = 0$ is $\frac{p_3 p_4}{r}$. Since $\Delta(x) = 0$ is a quadratic equation in $\cos x$, the equation has only two roots in the interval $0 < x \leq \pi$. Hence either

$$p_3 p_4 = p_3' p_4' \quad \text{or}$$

$$p_3 p_4 = p_3' p_4''.$$

In either case the theorem is proved, for either

$$p_1, p_2, p_3, p_4 \approx p_1', p_2', p_3', p_4' \quad \text{or}$$

$$p_1, p_2, p_3, p_4 \approx p_1', p_2', p_3', p_4''.$$

Case 2. p_1, p_2, p_4 are d-cyclic. In this case the point p_4^* is unique. We shall denote this point by p_4' and shall show that

$$p_1, p_2, p_3, p_4 \approx p_1', p_2', p_3', p_4'.$$

We have

$$p_1, p_2, p_3 \approx p_1', p_2', p_3' \quad \text{and}$$

$$p_1, p_2, p_4 \approx p_1', p_2', p_4'.$$

Thus we need only to show that $p_3 p_4 = p_3' p_4'$. To do this we define the expression $\Delta(x)$ as in the preceding case and we show that $\frac{p_3 p_4}{r}$

and $\frac{p_3' p_4'}{r}$ are double roots of $\Delta(x) = 0$. By the necessity of the conditions of the theorem a root of $\Delta(x) = 0$ is $\frac{p_3' p_4'}{r}$. By hypothesis a root of $\Delta(x) = 0$ is $\frac{p_3 p_4}{r}$. Expanding $\Delta(x)$ we obtain

$$\Delta(x) = \frac{\begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{13} \\ \cos \alpha_{21} & 1 & \cos \alpha_{23} \\ \cos \alpha_{31} & \cos \alpha_{32} & \cos x \end{vmatrix}}{1 - \cos^2 \alpha_{12}} \quad (1)$$

$\Delta(x)$ is not identically zero for the same reason as in the preceding case, and $\Delta(x) = 0$ has only two roots in the interval $0 < x \leq \pi$. From (1) we see that $\frac{p_3' p_4'}{r}$ and $\frac{p_3 p_4}{r}$ are double roots of $\Delta(x) = 0$. Hence $p_3 p_4 = p_3' p_4'$. Therefore

$$p_1, p_2, p_3, p_4 \approx p_1', p_2', p_3', p_4'$$

and the proof is completed.

Theorem 12. Five points p_1, p_2, p_3, p_4, p_5 of a semi-metric space are r-spheric if and only if each four of the points is r-spheric and $\Delta(p_1, p_2, p_3, p_4, p_5) = 0$.

If the five points are r-spheric then obviously each four of the points is r-spheric. By lemma 2, $\Delta(p_1, p_2, p_3, p_4, p_5) = 0$. Hence, the necessity of the conditions is established.

By theorem 11 each fourth-order principal minor vanishes. If each three points is d-cyclic, then by theorem 3 each four points is d-cyclic. The sufficiency of the conditions is immediate in this case since the circle has the congruence order four. In this case the five points are congruent to five points of a great circle of the sphere.

Now we need only to establish the sufficiency of the conditions for the case in which at least one triple is r-spheric and not d-cyclic. We may assume the labeling so that p_1, p_2, p_3 are not d-cyclic. Then $\Delta(p_1, p_2, p_3)$ is positive. By hypothesis there exist four points, say

p'_1, p'_2, p'_3, p'_4 of the sphere and four points, say $\bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_5$, of the sphere such that

$$p_1, p_2, p_3, p_4 \approx p'_1, p'_2, p'_3, p'_4 \quad \text{and}$$

$$p_1, p_2, p_3, p_5 \approx \bar{p}_1, \bar{p}_2, \bar{p}_3, \bar{p}_5.$$

We may make a congruent transformation of the sphere into itself so that $\bar{p}_1$ goes into p'_1 , $\bar{p}_2$ goes into p'_2 , $\bar{p}_3$ goes into p'_3 , and $\bar{p}_5$ goes into some point, say p'_5 . The point p'_5 has its distances from the three points p'_1, p'_2, p'_3 fixed and equal respectively to p_1, p_2, p_3 and p_3, p_5 . Now p'_1, p'_2, p'_3 are not on a great circle of the sphere because p'_1, p'_2, p'_3 are congruent to p_1, p_2, p_3 and this triple is not d-cyclic. Hence the point p'_5 is unique. We wish to show that

$$p_1, p_2, p_3, p_4, p_5 \approx p'_1, p'_2, p'_3, p'_4, p'_5.$$

We have

$$p_1, p_2, p_3, p_4 \approx p'_1, p'_2, p'_3, p'_4 \quad \text{and}$$

$$p_1, p_2, p_3, p_5 \approx p'_1, p'_2, p'_3, p'_5.$$

Hence we need only to show that $p_4 p_5 = p'_4 p'_5$.

In order to do this we define the expression $\Delta(x)$ as follows:

$$\Delta(x) = \begin{vmatrix} 1 & \cos \alpha_{12} & \cos \alpha_{13} & \cos \alpha_{14} & \cos \alpha_{15} \\ \cos \alpha_{21} & 1 & \cos \alpha_{23} & \cos \alpha_{24} & \cos \alpha_{25} \\ \cos \alpha_{31} & \cos \alpha_{32} & 1 & \cos \alpha_{34} & \cos \alpha_{35} \\ \cos \alpha_{41} & \cos \alpha_{42} & \cos \alpha_{43} & 1 & \cos x \\ \cos \alpha_{51} & \cos \alpha_{52} & \cos \alpha_{53} & \cos x & 1 \end{vmatrix}$$

$\Delta(x)$ does not vanish identically, since the coefficient of $\cos^2 x$, namely $-\Delta(p_1, p_2, p_3)$ is not zero. By the necessity of the conditions of the theorem a root of $\Delta(x) = 0$ is $\angle p'_1 p'_2 p'_3$. By hypothesis a root of $\Delta(x) = 0$ is $\angle p_4 p_5$. Since $\Delta(x) = 0$ is a quadratic equation in $\cos x$ the equation has only two roots in the interval $0 < x \leq \pi$. Expanding $\Delta(x)$ we have

$$\Delta(x) = \frac{- \begin{vmatrix} 1 & \cos d_{12} & \cos d_{13} & \cos d_{14} \\ \cos d_{21} & 1 & \cos d_{23} & \cos d_{24} \\ \cos d_{31} & \cos d_{32} & 1 & \cos d_{34} \\ \cos d_{41} & \cos d_{42} & \cos d_{43} & \cos x \end{vmatrix}^2}{\Delta(p_1, p_2, p_3)}$$

Hence $\frac{p_1' p_2'}{r}$ and $\frac{p_1' p_3'}{r}$ are double roots of $\Delta(x) = 0$. Therefore

$$p_4 p_5 = p_4' p_5' \text{ and hence}$$

$$p_1, p_2, p_3, p_4, p_5 \approx p_1', p_2', p_3', p_4', p_5'$$

which is what we wished to show.

Theorems 10, 11, and 12 characterize r-spheric triples, quadruples, and quintuples. Since the sphere has the congruence order five*, it follows that the necessary and sufficient condition that a set of points be r-spheric is that each quintuple of the set satisfy the conditions stated in theorem 12.[†] The remainder of this section consists of a treatment pseudo r-spheric sets containing five points.

Theorem 13. If five points p_1, p_2, p_3, p_4, p_5 of a semi-metric space are pseudo r-spheric then $\Delta(p_1, p_2, p_3, p_4, p_5)$ is negative.

From the definition of the pseudo r-spheric quintuple and theorem 11 each fourth-order principal minor of $\Delta(p_1, p_2, p_3, p_4, p_5)$ vanishes. Hence, at least one third-order principal minor does not vanish; that is, at least one triple is not d-cyclic, for otherwise each four points is d-cyclic by theorem 3. But if each four points is d-cyclic, the five points are d-cyclic, since the circle has the congruence order four, and hence are r-spheric. Thus, we may

* Menger, Loc. cit., p. 725.

† It is obvious that if $p_1, p_2, \dots, p_n$ ($n > 5$) are r-spheric, then $\Delta(p_1, p_2, \dots, p_n) = 0$. Cf. Bocher, Loc. cit.

assume the labeling so that p_1, p_2, p_3 are r-spheric and are not d-cyclic.

Hence $\Delta(p_1, p_2, p_3) > 0$.

Evaluating $\Delta(p_1, p_2, p_3, p_4, p_5)$ we obtain

$$\Delta(p_1, p_2, p_3, p_4, p_5) = \frac{\begin{vmatrix} 1 & \cos d_{12} & \cos d_{13} & \cos d_{14} \\ \cos d_{21} & 1 & \cos d_{23} & \cos d_{24} \\ \cos d_{31} & \cos d_{32} & 1 & \cos d_{34} \\ \cos d_{41} & \cos d_{42} & \cos d_{43} & \cos d_{44} \end{vmatrix}}{\Delta(p_1, p_2, p_3)}$$

Therefore $\Delta(p_1, p_2, p_3, p_4, p_5) \leq 0$.

But if $\Delta(p_1, p_2, p_3, p_4, p_5)$ equals zero, the five points are r-spheric by the preceding theorem. Hence

$$\Delta(p_1, p_2, p_3, p_4, p_5) < 0.$$

Theorem 14. None of the triples contained in a pseudo r-spheric quintuple is d-cyclic and each two points are isogonal conjugates with respect to the other three points.

We shall let the pseudo r-spheric quintuple be p_1, p_2, p_3, p_4, p_5 . As was shown in the proof of theorem 13, at least one triple contained in these points is not d-cyclic. We may assume the labeling so that p_1, p_2, p_3 is such a triple. There exist three points p'_1, p'_2, p'_3 of the sphere such that

$$p_1, p_2, p_3 \approx p'_1, p'_2, p'_3.$$

Then p'_1, p'_2, p'_3 are not on a great circle. There exist four points $p''_1, p''_2, p''_3, p''_4$ of the sphere such that

$$p_1, p_2, p_3, p_4 \approx p''_1, p''_2, p''_3, p''_4$$

Hence $p_1, p_2, p_3 \approx p''_1, p''_2, p''_3 \approx p'_1, p'_2, p'_3$.

We may make a congruent transformation of the sphere into itself so that p''_1 goes into p'_1 , p''_2 goes into p'_2 , p''_3 goes into p'_3 , and p''_4

goes into some point, say p_4^I . Similarly, there exist four points, say $p_1^{II}, p_2^{II}, p_3^{II}, p_5^{II}$ of the sphere such that

$$p_1, p_2, p_3, p_5 \approx p_1^{II}, p_2^{II}, p_3^{II}, p_5^{II}.$$

We may make a congruent transformation of the sphere into itself so that p_1^{II} goes into p_1^I , p_2^{II} goes into p_2^I , p_3^{II} goes into p_3^I , and p_5^{II} goes into some point, say p_5^I . The points p_4^I and p_5^I so determined are unique since their distances from the three points p_1^I, p_2^I, p_3^I are fixed and since these three points are not on a great circle of the sphere. Thus we have

$$p_1, p_2, p_3, p_4 \approx p_1^I, p_2^I, p_3^I, p_4^I \quad (1)$$

$$p_1, p_2, p_3, p_5 \approx p_1^I, p_2^I, p_3^I, p_5^I \quad (2)$$

Hence
$$p_4 p_5 \neq p_4^I p_5^I \quad (3)$$

for otherwise p_1, p_2, p_3, p_4, p_5 are congruent to $p_1^I, p_2^I, p_3^I, p_4^I, p_5^I$ and thus are r -spheric, which is impossible.

There exist four points, say $\bar{p}_2, \bar{p}_3, \bar{p}_4, \bar{p}_5$, of the sphere such that

$$p_2, p_3, p_4, p_5 \approx \bar{p}_2, \bar{p}_3, \bar{p}_4, \bar{p}_5.$$

Then
$$p_2, p_3, p_5 \approx \bar{p}_2, \bar{p}_3, \bar{p}_5 \approx p_2^I, p_3^I, p_5^I.$$

We may make a congruent transformation of the sphere into itself sending $\bar{p}_2$ into p_2^I , $\bar{p}_3$ into p_3^I , and $\bar{p}_5$ into some point, say $p_5^{\perp}$.

Hence we have

$$p_2, p_3, p_4, p_5 \approx p_2^I, p_3^I, p_4^I, p_5^{\perp}. \quad (4)$$

From (1), (2), (3), and (4)

$$p_2^I p_5^{\perp} = p_2 p_5 = p_2^I p_5^I$$

$$p_3^I p_5^{\perp} = p_3 p_5 = p_3^I p_5^I$$

$$p_4^I p_5^{\perp} = p_4 p_5 \neq p_4^I p_5^I.$$

Therefore p_5^I and $p_5^{\perp}$ are distinct and $p_5^{\perp}$ is the image of p_5^I in the plane through p_2^I, p_3^I , and the center of the sphere. Treating in a similar

manner the two remaining quadruples; namely, p_1, p_3, p_4, p_5 and p_1, p_2, p_4, p_5 we obtain the relations

$$p_1, p_3, p_4, p_5 \approx p_1', p_3', p_4', p_5'^2 \quad (4)$$

$$p_1, p_2, p_4, p_5 \approx p_1', p_2', p_4', p_5'^3 \quad (5)$$

where $p_5'^2$ is the image of p_5' in the plane through p_1' , p_3' , and the center of the sphere, and $p_5'^3$ is the image of p_5' in the plane through p_1' , p_2' , and the center of the sphere.

Fig. 2

From (2), (3), (4), and (5) we have

$$p_4 p_5 = p_4' p_5'^1 = p_4' p_5'^2 = p_4' p_5'^3 \neq p_4' p_5'.$$

Thus p_4' is equidistant from the points $p_5'^1$, $p_5'^2$, and $p_5'^3$, which are, respectively, the images of p_5' in the plane through p_1' , p_3' , and the center of the sphere; p_1' , p_3' , and the center of the sphere; and p_1' , p_2' , and the center of the sphere. Therefore p_4' is the isogonal conjugate of p_5' with respect to p_1' , p_2' , p_3' .

The above analysis itself may be used to show that p_1, p_2, p_5 are not d-cyclic, for if so, then p_1', p_2', p_5' are on a great circle and the points p_4' and p_3' must be identical, which is impossible. Likewise p_1, p_3, p_5 and p_2, p_3, p_5 are not d-cyclic. Now we may use p_1, p_2, p_5 in place of p_1, p_2, p_3 in the analysis above. In this manner we find p_4' and p_3' are isogonal conjugates with respect to p_1', p_2', p_5' . Using p_1, p_3, p_5 in place of p_1, p_2, p_3 we find that p_4' and p_2' are isogonal conjugates with respect to p_1', p_3', p_5' and so on. Finally, we find that each two

points are isogonal conjugates with respect to the other three points and that no three points are d-cyclic, for if three of the points are d-cyclic then the isogonal conjugate construction breaks down, two of the points being identical, which is impossible.