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Strain Hardening From Elastic–Perfectly Plastic to Perfectly Elastic Flattening Single Asperity Contact

For elastic contact, an exact analytical solution for the stresses and strains within two contacting bodies has been known since the 1880s. Despite this, there is no similar solution for elastic–plastic contact due to the integral nature of plastic deformations, and the few models that do exist develop approximate solutions for the elastic–perfectly plastic material model. In this work, the full transition from elastic–perfectly plastic to elastic materials in contact is studied using a bilinear material model in a finite element environment for a frictionless dry flattening contact. Even though the contact is considered flattening, elastic deformations are allowed to happen on the flat. The real contact radius is found to converge to the elastic contact limit at a tangent modulus of elasticity around 20%. For the contact force, the results show a different trend in which there is a continual variation in forces across the entire range of material models studied. A new formulation has been developed based on the finite element results to predict the deformations, real contact area, and contact force. A second approach has been introduced to calculate the contact force based on the approximation of the Hertzian solution for the elastic deformations on the flat. The proposed formulation is verified for five different materials sets. [DOI: 10.1115/1.4041537]

1 Introduction

Contact mechanics is one of the most common problems in mechanical engineering and tribology, with a variety of applications in collision mechanics [1–3], joint structures [4,5], electrical contact [6], thermal contact [7], solid mechanics [8–11], seals and bearings [12], biomechanical systems [13,14], turbines [15], and additive manufacturing [16,17]. The studies in contact mechanics can be categorized into: single asperity spherical, elliptical, cylindrical and flat contacts, with single asperity spherical contact being the most employed one [18–22]. Many analytical and numerical studies have been performed to simulate and predict the contact properties of a single asperity spherical contact, such as contact radius, contact force, average pressure, and stress distributions [18]. However, because of the complexities of such problems no closed-form solution has been provided for elastic–plastic contact. The majority of the recent works use numerical methods, such as finite element or finite difference, to simulate or model the elastic–plastic behavior of contact; however, even by using the numerical methods, the studies are limited to the elastic–perfectly plastic materials, and the effect of strain hardening has been neglected in almost all the previous contact models [18].

In general, the single asperity contact models can be divided into flattening and indentation models [23], with the exception of Ghaednia et al. [24] and Olsson and Larsson [25] works. The flattening models consider the contact of a deformable sphere with a rigid flat, while indentation models consider the reverse case. It is obvious that both of the cases are unrealistic since the elastic deformations on the harder object are inevitable. Ghaednia et al. [24] studied this effect by introducing the yield strength ratio in the modeling of elastic–plastic contact. They formulated the whole transition from flattening to indentation based on the variation of the yield strength ratio. Olsson and Larsson [25] studied the effect of radius of curvature in elastic–plastic contact. Their model is partly based on dimensionless quantities emerging from

Brinell hardness tests and is developed for granular material simulation.

1.1 Elastic–Perfectly Plastic Contact. The occurrence of elastic–perfectly plastic contact is divided into three phases: elastic, elastic–plastic, and fully plastic regimes. For the elastic regime, a closed form solution was derived by Hertz [22,26]. This solution approximates the curvature of two dissimilar spheres with polynomials to solve for the elastic stress distribution in the objects. Even though the Hertzian contact theory is used in most contact models, it is important to understand that the solution is still an approximation. In the Hertzian theory, the contact radius is approximated as

$$a = \sqrt{R\Delta} \quad (1)$$

Here, a is the real contact radius, R is the equivalent radius of curvature, and Δ is the relative normal displacement of the spheres' centers during contact. Equation (1) is a first-order approximation of the exact solution that is accurate for $\Delta/R \ll 1$, and yields to around 5% error for contact with $\Delta/R = 0.05$. The elastic–plastic regime is defined to start at the initiation of yield. The onset of yield is solved using the von Mises criterion for the maximum amplitude of stress which occurs at some depth below the surface [22]. This criteria is used to solve for the deformation necessary to initiate yield

$$\delta_y = \frac{R}{F(\nu)} \left(\frac{\pi S_{ys}}{2E} \right)^2 \quad (2)$$

where S_y is the yield strength of the softer object, ν is the Poisson's ratio of the more compliant material, and E is the effective modulus of elasticity

$$\frac{1}{E} = \frac{1 - \nu_s^2}{E_s} + \frac{1 - \nu_f^2}{E_f} \quad (3)$$

ν_s , ν_f , E_s , and E_f are the Poisson's ratio and modulus of elasticity of the sphere and the flat, respectively. The function $F(\nu)$ is solved from [22]

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$$F(\nu) = \max_{z \geq 0} \left[-(1+\nu) \left(1 - \frac{z}{a} \tan^{-1} \left(\frac{a}{z} \right) \right) + \frac{3}{2} \frac{1}{1 + \left[\frac{z}{a} \right]^2} \right]^2 \quad (4)$$

However, since the equation cannot be solved analytically, Jackson and Green [27] proposed

$$\delta_c = \left(\frac{\pi C S_{ys}}{2E} \right)^2 R \quad (5)$$

such that C is a fit to the numerical solution of Eq. (4) and can be calculated as

$$C = 1.295 e^{0.736 \nu_s} \quad (6)$$

Such a fit was also proposed by Chang [28]; however, Chang's derivation assumes a fixed value for hardness, which recent studies proved not to be accurate.

There is no closed form analytical solution for the elastic-plastic regime. Most researchers utilize empirical formulations based on FE models. To have a material properties independent formulation, all of the parameters are normalized in empirical formulation. It has to be noted that geometrical values such as radius of a sphere contacting a flat is addressed in empirical formulations but the shape, such as the difference between a spherical and a conical indenter is not.

One of the first FEA models of indentation contact was published by Hardy et al. [29]. Following this, researchers introduced their modeling approach in which a low-order model was calibrated using the results from FEA studies [30,31]. The first elastic-plastic empirical model was proposed by Chang et al. [32] often known as the CEB model. The model is based on the volume conservation of the deformed sphere during large plastic deformation. In their model, the average normal pressure for the fully plastic regime is considered constant, and the contact radius and contact force formulations contain discontinuities at the transition from the elastic to plastic regimes.

Subsequent models, such as Kogut and Etsion's [33], Zhao et al. [34], Thornton and coworkers [35], Wu et al. [36] Jackson and Green [27], Shankar and Mayuram [36,37], and Lin and Lin [38] were developed to predict the contact parameters of an elastic-perfectly plastic flattening contact. Several of these models have been compared with each other and to experimental results of normal and oblique collision tests in Ghaednia et al. [3] and Brake [39,40]. Both Brake's and Ghaednia's studies show very good matches between experiments and quasi-static numerical simulations using the aforementioned contact models in terms of the coefficient of restitution; however, recently, Ghaednia and Marghitu [1] showed that in the case of permanent deformations after collision, numerical simulation predictions are an order of magnitude off compared to the experimental results. They proposed a semi-analytical semi-experimental method to account for this issue.

As detailed in Ref. [40], the elastic-plastic regime must be smooth and continuous at all parts, including the transition from a purely elastic stress state to the fully plastic flow. During the fully plastic regime, the average normal pressure was assumed to be constant by Tabor [41]. Based off of experimental evidence, Tabor proposed the value 2.84 for the ratio between the average normal pressure (hardness), and the yield strength. To exclude the effect of strain hardening, Tabor used work-hardened metals in his Brinell hardness tests. Analytical support of Tabor's conclusion was first provided by Ishlinsky [42], then later by Kogut and Komvopoulos [43], who showed that the hardness value could be empirically related to the material properties E and S_{ys} . However, Jackson and Green [27] showed that the hardness, which describes the plastic response, is a function of geometry (contact radius) in

addition to the material properties. Jackson et al. [44] also analytically derived the dependence of hardness to the contact radius, using the slip-line pattern proposed by Ishlinsky [42] for perfectly plastic materials.

One challenge with experimental studies of elastic-plastic behavior is that the transition from elastic to fully developed plastic flow occurs in less than a micrometer of deflection for typical engineering materials and applications [40,45,46]. Thus, the fully plastic contact has been studied extensively compared to the elastic-plastic regime in order to match the theoretical predictions to Brinell [47] hardness tests. The majority of these studies, however, are for indentation contact, which is not the topic of this work. For a detailed review of the fully plastic regime for single asperity contact studies, the reviews by Bhushan [19] Adam and Nosonovsky [21] and Ghaednia et al. [18] are recommended.

1.2 Strain Hardening. One of the main simplifying assumptions in most analytical, numerical, and experimental studies is the effect of strain hardening. There are only a few studies that consider the effect of strain hardening, starting with the modeling of Meyer's hardness test [48] in which the strain hardening appears in the equation relating the contact force to the contact radius

$$F_p = p_0 \pi \frac{a^n}{a_p^{n-2}} \quad (7)$$

Here, a_p' is the contact radius at which the fully plastic regime starts, n is the Meyer hardness exponent [41,49], and is related to the strain-hardening exponent, n_e , by $n_e = n - 2$. For elastic-perfectly plastic materials, $n = 2$, and for perfectly elastic materials, $n = 3$. The strain-hardening exponent, n , was recently used by Brake [40] to analytically describe the elastic-plastic regime of indentation contact considering the effect of strain hardening.

Zhao et al. [34] studied the loading and unloading of a flattening contact for an elastic-plastic sphere considering power-law strain hardening in their FE model. Their results showed a decrease in the contact area and an increase in the contact force compared to an elastic-perfectly plastic contact at the same deformation. The empirical formulation developed by Zhao et al. is an improved version of the equations proposed by Brizmer et al. [50]. The power-law strain-hardening effect was also studied by Mesarovic and Fleck [51]; however, only one strain-hardening exponent was looked at, and no quantification or generalized prediction method was presented.

Another strain-hardening material model that has been used in contact models is the bilinear isotropic behavior in the stress-strain curve. For these models, a tangent modulus of elasticity, E_t , is defined for the material in terms of a hardening parameter

$$HP = \frac{E_t}{E_s - E_t} \quad (8)$$

From the formulation, HP shows the ratio between the tangent modulus and the part of elasticity that has been transformed to plasticity. For most metals, HP varies from 0 to 0.5 resulting in an upper limit of $E_t = E/3$ for the tangent modulus. This model is simpler to formulate compared to power law models, is less time consuming for the FE models to calculate the contact, and better matches the true stress-true strain curve of the metallic engineering materials.

There are a few contact studies that considered this material model in their formulation [18]. In Brizmer et al. [50], the effect of bilinear strain hardening is mentioned briefly relative to the stick-slip condition in contact, and the results were presented as observations rather than as a constitutive model. Kogut and Etsion [33] stated that the effect of bilinear strain hardening was less

than 4.5% for tangent modulus less than $E_t = 0.05E_s$; however, the comparison was performed for deformations smaller than $\delta/\delta_c = 20$. Shankar and Mayuram [36] (SM model) studied the effect of bilinear strain hardening on the flattening contact. They improved Kogut and Etsion's model [33] for three different tangent modulus, $E_t = 0.025 E_s$, $E_t = 0.05 E_s$, and $E_t = 0.1 E_s$. They showed a decrease in contact radius and an increase in force compared to elastic-perfectly plastic materials. Finally, Sharma and Jackson [52] studied the line contact of a deformable cylinder and a rigid flat and provided the empirical relation between the contact width and the contact force for small E_t/E_s ratios (i.e., typical engineering materials). They used an effective yield strength to account for hardening. In Ref. [52], they also used an unorthodox normalization by dividing the Hertz elastic solution rather than the critical values at the initiation of yielding. This appears to be effective at collapsing the results for different cases onto nearly one universal curve.

From the literature, there are two major gaps in the field that need to be addressed. First, the effect of strain hardening from $E_t = 0$ to $E_t = E_s$ that has not been studied yet and will be the main focus of this work. In what follows, the range of E_t/E_s from 0 to 1 is studied in order to capture the physical behavior of the material with a better accuracy. Second, in most of the previous models with exception of Ghaednia et al. [24], and Rodriguez et al. [45], one of the objects has been implicitly assumed to be rigid. This is far from reality since elastic deformations on the tougher objects are inevitable. In this work, the flat has been considered fully elastic instead of being rigid to account for the effect of deformations on the flat on the contact area and the contact force.

2 Finite Element Analysis

2.1 Model. The frictionless axisymmetric contact of a hemisphere with a half space is modeled in an FE environment using ANSYS. The same meshing as used in Ghaednia et al. [2,24] is

used, shown in Fig. 1, and the detailed specification of the modeling is:

- eight-node PLANE183 elements have been used;
- the mesh is finer at the contact tip, and overall includes 9784 elements;
- the sphere is modeled as a bilinear material, and the flat is considered to be fully elastic;
- frictionless contact elements used for all contacting surfaces;
- all nodes on the axis of symmetry are constrained in the x direction;
- the nodes on the bottom line of the flat are fixed in both the x and y directions;
- a uniform displacement (Δ) has been applied on top of the sphere in 25 substeps from $\Delta/R = 0$ to $\Delta/R = 0.05$;

The mesh has been verified with convergence of less than 1% for both small and large deformations in case of contact area, contact force, and the smoothness of the contact pressure profile. The displacement of the node at the tip of the contact is used to calculate the maximum deformation of the sphere and the sum of the reaction forces on all of the nodes on the top of the sphere is calculated as the contact force. To calculate the real contact radius, the contact status of all of the nodes on the top line of the flat are checked, and the last node in contact is found. The real contact radius is the initial position plus the displacement of the selected node. The mesh convergence is done for elastic-perfectly plastic case to have less than 1% deviation for all the results.

2.2 Initial Observations. Figures 2–6 show the FE simulation results of the contact parameters for a contact with $R = 0.001$ m, $E_s = 200$ GPa, $E_f = 200$ GPa, $S_{ys} = 300$ MPa, $\nu_s = 0.33$, $\nu_f = 0.30$, and $E_{ts} = 4$ GPa. Figure 2 depicts the von Mises stress distribution, it can be seen that due to the strain hardening of the sphere at some parts inside the sphere the stresses exceed the yield strength and the flat reaches higher stresses. The

Fig. 1 Meshing of the FE model

Fig. 2 Von Mises stress distribution contour for a flattening contact, of a bilinear sphere and an elastic sphere

Fig. 4 Contact radius versus Δ/R and E_t/E_s , Hertz and JG model show upper and lower limits, respectively

Fig. 3 Deformation ratio of the sphere for, $0 \leq E_t/E_s \leq 1$ and $0 < \Delta/R \leq 0.05$ using FE analysis

Fig. 5 Average contact pressure versus Δ/R and E_t/E_s . Hertz and JG models show upper and lower limits, respectively.

265 applied displacement for the stress distribution shown in Fig. 2 is
 266 $\Delta/R = 0.05$. It can be seen that the maximum stress occurs on the
 267 flat, as expected, and the sphere experiences a very high amount
 268 of plastic deformation.
 269 Figure 3 shows the behavior of the sphere deformation ratio, δ_s^* ,
 270 of the sphere

$$\delta_s^* = \frac{\delta_s}{\Delta} \quad (9)$$

271 as the tangent modulus of elasticity, E_t/E_s , and the applied dis-
 272 placement, Δ/R , change. The deformation ratio has limits of $\delta_{se}^* =$
 273 $E(1 - \nu_s^2)/E_s$ for very small deformations (the elastic regime) and
 274 for $E_t/E = 1$ (elastic contact). The sphere deformation ratio, δ_s^* ,
 275 shows convergence as the applied displacement, Δ/R and tangent
 276 modulus, E_t/E_s increase.

277 Figure 4 shows the real contact radius, a , for 13 different tan-
 278 gent modulus, E_t/E_s and for $0 \leq \Delta/R \leq 0.05$. The dashed line
 279 shows the elastic contact limit, which is calculated from the
 280 Hertzian theory [26], and the continuous thick line shows the JG
 281 model [27] for elastic-perfectly plastic contact. The contact radius
 282 for all of the cases fall between the two limits. The gradient of the
 283 contact radius with respect to E_t/E_s is larger around the
 284 elastic-perfectly plastic limit ($E_t/E_s = 0$). The contact radius con-
 verges to elastic limit for $E_t/E_s \geq 0.2$. The small difference
 between the FE modeling of $E_t/E_s = 1$ and Hertzian theory is due

Fig. 6 Contact Force versus Δ/R and E_t/E_s . Hertz and JG model show upper and lower limits, respectively.

to the large deformations that have been compared in Fig. 4. For large deformations, the Hertzian theory results in inaccuracies due to the first-order approximation that was used in Ref. [26] to calculate the contact radius, Eq. (1). The small noises that can be seen for the results in Fig. 4 are due to the fact that mesh convergence was done for elastic-perfectly plastic; however, they are all within 2% error.

In Fig. 5, the dependence of the average normal pressure on the applied displacement and the tangent modulus is shown. For the elastic-perfectly plastic materials and within the fully plastic contact regime, the normal average pressure is called hardness. Here, however, since strain hardening is considered, there is no perfectly plastic regime. Therefore, the subsequent discussion focuses on the average normal pressure instead of the initiation of the plastic regime.

The thick continuous line in Fig. 5 shows the hardness predicted by the JG model for elastic-perfectly plastic materials. In the JG model, the hardness is a function of a/R . This is one of the main contributions of the JG model. For the elastic-perfectly plastic materials, the hardness decreases as the contact radius increases. The dashed line shows the average normal pressure for a fully elastic contact, calculated from the Hertzian theory. Similarly to before (i.e., Fig. 4), the discrepancy between the FE model of fully elastic contact and the Hertzian theory is the first-order approximation of the real contact radius, which results in 3–5% difference between the predictions and FE calculations. The FE results show a monotonic increase from the elastic-perfectly plastic contact to the fully elastic contact as E_t/E_s increases.

The contact force, shown in Fig. 6, is bounded between the two limits: purely elastic and elastic-perfectly plastic contact. The contact force increases faster around the elastic-perfectly plastic limit and reaches to the Hertzian contact around the fully elastic limit. Contrary to the contact radius results, the contact force shows a continuous increase from the elastic-perfectly plastic limit to the elastic limit. Again the difference between the Hertzian theory and the FE results at large deformations is due to the first-order approximation of the contact curvature in the Hertzian theory.

3 Model Development

The modeling proposed in this section tries to provide predictive formulas for the contact radius, contact force, and the deformations on both the sphere and the flat. Since the flat is considered perfectly elastic, the plastic deformations will occur only in the sphere. For each of these variables, the prediction should fall between the upper ($E_t = E_s$) and lower ($E_t = 0$) limits of the tangent modulus, which are the fully elastic and elastic-perfectly plastic contact. For the fully elastic contact, the Hertzian [26] theory will be used; for the elastic-perfectly plastic contact, Jackson and Green's model [27] is employed.

3.1 Deformation. Let the deformation ratio of the sphere be as defined in Eq. (9). This ratio is assumed to be a function of E_t/E_s and Δ/R , while satisfying the boundaries. Note that other variables, such as Poisson's ratio and the effective modulus of elasticity, are convoluted in the elastic and elastic-perfectly plastic limits. With these assumptions, let the sphere deformation ratio be a function of the applied displacement and the tangent modulus of elasticity

$$\delta_s^* = f\left(\frac{\Delta}{R}, \frac{E_t}{E_s}\right) \quad (10)$$

where both Δ/R and E_t/E_s vary from 0 to 1. The deformation ratio of the sphere, δ_s^* , should satisfy the following governing conditions (GCs) (see Fig. 3 for visualization).

The elastic limit of the deformation ratio for small deformations

- (i) $\lim_{\Delta/R \rightarrow 0} \delta_s^* = \delta_{se}^* = E(1 - \nu_s^2)/E_s$, $0 \leq E_t/E_s \leq 1$
The elastic limit of the deformation ratio for all applied deformations
- (ii) $\lim_{E_t/E_s \rightarrow 1} \delta_s^* = \delta_{se}^* = E(1 - \nu_s^2)/E_s$, $0 < \Delta/R \leq 1$
The δ_{se}^* in GCs (I, II) is the elastic limit of the deformation ratio that can be found from the Hertzian theory [24]. GC (III) sets the large deformation limit of the elastic-perfectly plastic materials to δ_{sp}^*
- (iii) $\lim_{E_t/E_s \rightarrow 0} \delta_s^* = \delta_{sp}^*$, $\Delta/R = 1$
where the large deformation limit was found by considering the sphere to be fully elastic with modulus of elasticity $E_s = E_r$. Therefore, the deformation ratio at large deformation limit, δ_{sp}^* , can be approximated as

$$\delta_{sp}^* = \frac{E_t(1 - \nu_s^2)}{E_t}, \quad \frac{1}{E_t} = \frac{1 - \nu_s^2}{E_t} + \frac{1 - \nu_f^2}{E_f} \quad (11)$$

here E_t^* is the contact's reduced modulus of elasticity if the sphere was fully plastic with modulus equal to E_r . The comparison between Eq. (11) and the finite element results for deformation ratio, δ_s^* , is shown in Fig. 7. Here, the finite element results are for applied displacement $\Delta/R = 0.05$ that is much smaller than the maximum displacement $\Delta/R = 1$; however, as shown in Fig. 3, the deformation ratio converges to the large deformation limit very quickly. Equation (11) includes a singularity for elastic-perfectly plastic contact, $E_t = 0$; however, the limit of Eq. (11) yields to $\delta_s^* = 1$. Therefore, the predictions are reasonable.

GC (IV) shows that for large deformations, δ_s^* converges to the large deformation limit and sets the gradient of δ_s^* with respect to applied displacement to 0 for large Δ/R . GC (V) shows the convergence of the deformation ratio for elastic contact and sets the gradient of δ_s^* with respect to tangent modulus to 0 for large E_t/E_s .

- (iv) $\lim_{\Delta/R \rightarrow 1} \partial f / \partial (\Delta/R) = 0$, $0 \leq E_t/E_s \leq 1$
- (v) $\lim_{E_t/E_s \rightarrow 1} \partial f / \partial (E_t/E_s) = 0$, $0 < \Delta/R \leq 1$
- GCs (VI, VII) set constraints on the first partial derivation of the deformation ratio for both the Δ/R and E_t/E_s dimensions
- (vi) $\lim_{E_t/E_s \rightarrow 0} \partial f / \partial (\Delta/R) \neq 0$, $0 < \Delta/R \leq 1$
- (vii) $\lim_{\Delta/R \rightarrow 0} \partial f / \partial (E_t/E_s) \neq 0$, $0 \leq E_t/E_s \leq 1$

To simplify the modeling, a separation of variables for f is used

$$\frac{\delta_s}{\Delta} = f\left(\frac{\Delta}{R}, \frac{E_t}{E_s}\right) = \phi\left(\frac{\Delta}{R}\right)\psi\left(\frac{E_t}{E_s}\right) + D \quad (12)$$

Fig. 7 Comparison of deformation ratio's large deformation limit, δ_{sp}^* , between FE results at $\Delta/R = 0.05$ and Eq. (11)

Application of GC (I) yields to

$$\delta_s^* = \phi\left(\frac{\Delta}{R} = 0\right)\psi\left(\frac{E_t}{E_s}\right) + D = \delta_{se}^* \quad (13)$$

therefore, as ψ cannot be a constant

$$\phi\left(\frac{\Delta}{R} = 0\right) = 0, \quad D = \delta_{se}^* \quad (14)$$

and the same holds for GC (II)

$$\psi\left(\frac{E_t}{E_s} = 1\right) = 0 \quad (15)$$

Using GC (III), and Eq. (14)

$$\delta_s^* = \phi\left(\frac{\Delta}{R} = 1\right)\psi\left(\frac{E_t}{E_s} = 0\right) + \delta_{se}^* = \delta_{sp}^* \quad (16)$$

From GC (IV)

$$\lim_{\frac{\Delta}{R} \rightarrow 1} \frac{\partial f}{\partial \left(\frac{\Delta}{R}\right)} = \lim_{\frac{\Delta}{R} \rightarrow 1} \left[\phi' \left(\frac{\Delta}{R} \right) \psi \left(\frac{E_t}{E_s} \right) \right] = 0 \quad (17)$$

and since $\psi(E_t/E_s) \neq 0$, therefore

$$\lim_{\frac{\Delta}{R} \rightarrow 1} \phi' \left(\frac{\Delta}{R} \right) = 0 \quad (18)$$

which is the same as GC (IV). Similarly application of GC (V) to Eq. (12) results in

$$\lim_{\frac{E_t}{E_s} \rightarrow 1} \psi' \left(\frac{E_t}{E_s} \right) = 0 \quad (19)$$

Further, from GCs (VI, VII), it can be derived that

$$\lim_{\frac{E_t}{E_s} \rightarrow 0} \psi' \left(\frac{E_t}{E_s} \right) \neq 0, \quad \lim_{\frac{\Delta}{R} \rightarrow 1} \phi' \left(\frac{\Delta}{R} \right) \neq 0 \quad (20)$$

Thus, from Eqs. (12) to (20) the following is proposed:

$$\begin{aligned} \phi\left(\frac{\Delta}{R}\right) &= 1 - e^{\beta \frac{\Delta}{R}} \\ \psi\left(\frac{E_t}{E_s}\right) &= \lambda(\delta_{sp}^* - \delta_{se}^*) \left(1 - \left[\frac{E_t}{E_s}\right]^\alpha\right) \\ \delta_s^* &= \delta_{se}^* + \lambda(\delta_{sp}^* - \delta_{se}^*) \left(1 - \left[\frac{E_t}{E_s}\right]^\alpha\right) (1 - e^{\beta \frac{\Delta}{R}}) \end{aligned} \quad (21)$$

where λ , α , and β are the fitting variables and have been calculated from fitting to FEM results for $E_s = 200$ GPa, $E_f = 200$ GPa, $S_{ys} = 300$ MPa, $\nu_s = 0.33$, $\nu_f = 0.3$, and for $E_t/E_s = 0, 0.01, 0.02, 0.03, 0.04, 0.06, 0.1, 0.2, 0.3, 0.4, 0.6, 0.9, 1$ and for 25 increments of Δ/R from $\Delta/R = 0$ to $\Delta/R = 0.05$

$$\lambda = 0.8, \quad \alpha = \pi/2, \quad \beta = -\frac{E_s}{2.7S_{ys}} \quad (22)$$

These are the only three parameters in this work that are calculated from fitting to the FE results, and they are satisfying the transition between the two limits. It has to be considered that according to GC (III), λ should be equal to 1; however, since the upper limit of $\Delta/R = 1$ is extremely large compare to real metal

contacts (before this limit, materials will reach fracture), the coefficient λ was added to help the fittings with this issue. The fittings have been performed for the results presented in Fig. 3. The absolute mean error of the fittings is 4.2% with the maximum error 9%. Therefore, the presented formulation is considered to be stable for the same range as the two limits, Hertz theory and the JG model for elastic-perfectly plastic contact.

3.2 Contact Radius. The contact radius is divided into two phases: elastic and elastic-plastic. For the elastic phase, the Hertzian solution, Eq. (1), is used. For the elastic-plastic phase ($\delta > 1.9\delta_c$), a new formulation is proposed. The lower limit of the contact radius follows Hertzian theory, Eq. (1). For the upper limit (elastic-perfectly plastic) of the contact radius, a_p , Jackson and Green's equation has been used

$$a_p = a_{JG} = R \left(\frac{\pi C S_{ys}}{2E} \right) \sqrt{\left(\frac{\Delta}{\delta_c} \left[\frac{\Delta}{1.9} \right]^B \right)} \quad (23)$$

where C and δ_c can be found from Eqs. (5) and (6), and from Ref. [27]

$$B = 0.14e^{23\frac{S_{ys}}{E}} \quad (24)$$

For the contact radius, the initial FE results showed convergence toward $E_t/E_s \rightarrow 1$. Thus, the following is proposed for the transition from fully elastic to elastic-perfectly plastic contact

$$a = a_e + (a_p - a_e)e^{\Upsilon\left(\frac{E_t}{E_s}, \frac{\Delta}{R}\right)} \quad (25)$$

where Υ is found from fitting to FEM results

$$\Upsilon\left(\frac{E_t}{E_s}, \frac{\Delta}{R}\right) = \frac{\sqrt{\frac{E_s}{S_{ys}}\left(\frac{E_t}{E_s}\right)^{(1-\frac{\Delta}{R})}}}{\frac{1}{2}\left(4 - 3e^{-2\sqrt{\frac{E_s E_t}{S_{ys} E_s}}}\right)\left(1 - \frac{E_t}{E_s}\right)} \quad (26)$$

The results shown in Fig. 4 are used for the fitting of the constants in Eq. (26) resulting in absolute mean error and maximum errors 4% and 10%, respectively.

3.3 Contact Force. For the contact force, two approaches can be taken: (1) derivation of a transition from fully elastic to elastic-perfectly plastic with fitting the FEM results, similar to the contact radius approach, and (2) using the elastic deformations on the flat Eq. (21) and the contact radius from Eq. (25) to calculate the contact force from the Hertzian theory. Using the second approach, the radius of the curvature is updated to satisfy the contact radius for the flat. Here, equations for both approaches will be derived, compared, and discussed in the next sections.

3.3.1 Contact Force:Fitting. Similar to the contact radius fittings, Jackson and Green's [27] model is used for the elastic-perfectly plastic limit

$$F_p = F_c \left(e^{-0.25[\Delta/\delta_c]^{5/12}} \left(\frac{\Delta}{\delta_c} \right)^{\frac{3}{2}} + \frac{4H}{CS_{ys}} (1 - e^{\frac{1}{25}[\Delta/\delta_c]^{5/9}}) \frac{\Delta}{\delta_c} \right) \quad (27)$$

where H/S_{ys} is from JG model [53]

$$\frac{H}{S_{ys}} = 2.84 - 0.92 \left(1 - \cos \left(\pi \frac{a}{R} \right) \right) \quad (28)$$

and F_c is the critical force at the initiation of yield that is used in JG model [27]

Table 1 Material sets used in the comparisons

Mat. sets	Mat1	Mat2	Mat3	Mat4	Mat5
E_s (GPa)	200	200	71	71	200
E_f (GPa)	200	200	200	200	200
E_f/E_s (%)	4	2	0.5	10	2
S_{ys} (MPa)	500	750	200	500	300
ν_s	0.33	0.3	0.33	0.33	0.33
ν_f	0.33	0.3	0.33	0.3	0.3

$$F_c = \frac{4}{3} \left(\frac{R}{E} \right)^2 \left(\frac{C}{2} \pi S_{ys} \right)^3 \quad (29)$$

and C is from Eq. (6). Then, using a mathematical template to transition between elastic and elastic-perfectly plastic contact

$$F = F_p + (F_e - F_p)(1 - e^{-3.3E_f/E_s}) \quad (30)$$

where F_e is the elastic force calculated from Hertzian theory

$$F_e = \frac{4}{3} \pi R^{1/2} E \Delta^{3/2} \quad (31)$$

The results from Fig. 6 are used to find the constant in the exponent's power with absolute mean and maximum errors 6% and 10%, respectively.

3.3.2 Contact Force: Elastic Force on the Flat. Since the contact radius is a shared parameter between the flat and the sphere, it could be approximated as

$$\sqrt{R' \delta_f} = a \quad (32)$$

Fig. 8 Comparison of deformation ratio, δ_s^* , contact radius, a , and contact force, F , between the FE modeling and proposed formulation for material set 1

Table 2 Normalized deformations and mean absolute errors

Mat. sets	Mat1	Mat2	Mat3	Mat4	Mat5
Δ/δ_c	366	163	635	101	1018
e_{δ_s}	0.020	0.017	0.030	0.012	0.053
e_a	0.041	0.045	0.035	0.050	0.034
e_{F1}	0.043	0.027	0.015	0.040	0.055
e_{F2}	0.160	0.200	0.300	0.120	0.160

where δ_f is indirectly found from Eq. (21), and R' is assumed to be the updated radius of curvature that accounts for the elastic-plastic deformations on the sphere, and a is calculated from Eq. (25). The updated radius of curvature, R' , can be calculated from Eq. (32). By assuming that the deformation curve is approximately the same shape as for an elastic contact, the elastic contact force is calculated from the flat as

$$F_{ef} = \frac{4}{3} R' \frac{E_f}{1 - \nu_f^2} \delta_f^{\frac{3}{2}} \quad (33)$$

here, $\delta_f = \Delta - \delta_s$.

3.4 Final Formulation. Contact starts at purely elastic regime for small deformations $\Delta < 1.9\delta_c$. At this regime, the Hertzian solution can be used to calculate the contact parameters, Eq. (1) for contact radius and Eq. (31) for the contact force. The sphere deformation ratio for this phase is constant $\delta_s^* = \delta_{se}^*$. For larger deformations $\Delta > 1.9\delta_c$, the contact is in elastic-plastic regime, and the contact parameters can be found from Eq. (21) for the deformation ratio, Eq. (25) for the real contact radius and Eq. (30) for the contact force.

4 Results and Discussion

To verify the accuracy of the formulation, five different material sets have been analyzed. Table 1 shows the material properties of the studied sets. For each set, the deformation ratio, real contact radius, and the contact force have been compared for the proposed formulation and the FE models. The contact force has also been studied for the two approaches that were discussed. The tangent modulus of elasticity has been kept below $0.1 E_s$ for the comparisons since the majority of the engineering materials are within this range; however, the formulation quantifies for the whole range from elastic-perfectly plastic to fully elastic contact.

For the contact radius and contact force, the results have also been compared with the model of Shankar and Mayuram [36]. However, since in their work they provide fits for three discrete tangent modulus, the closest tangent modulus to each set of the properties is used. Therefore, the differences between the FE results and the SM model for cases 1, 2, 3, 5 are reasonable.

Figure 8 shows the comparison of contact parameters for material set 1. Although the deformation results show a good match between the predictions and the FE results, for large applied displacement there are small deviations between the results. The mean absolute error between predictions, Eq. (21), and the FE results is $e_{\delta_s} = 0.020$ with maximum error, $e_{\delta_{sm}} = 0.066$. The predictions for contact radius show a good match with the FEM results with average mean error, $e_a = 0.041$ and maximum $e_{am} = 0.060$. The SM model for $E_f/E_s = 0.05$ is used for comparison, which shows larger contact radius compare to the FE results. Note that since the SM formulation used here was for $E_f/E_s = 0.05$, smaller predictions are expected for the FE results compared to the SM model. The reason for this large difference is that the elastic deformation of the flat is allowed in the present model, which is more realistic. The first contact force approach explained in Secs. 3.3.1, shows a good agreement with the FE results. The mean absolute error for first and second approaches

Fig. 9 Comparison of deformation ratio, δ_s^* , contact radius, a , and contact force, F , between the FE modeling and proposed formulation for material set 2

Fig. 10 Comparison of deformation ratio, δ_s^* , contact radius, a , and contact force, F , between the FE modeling and proposed formulation for material set 3

are $e_{F1} = 0.043$ and $e_{F2} = 0.160$, respectively (Table 2), and the maximum error for the first approach is $e_{F1m} = 0.056$. The second approach shows smaller results specially around $\Delta/R = 0.025$. The SM model also shows a good agreement.

The same comparison as Fig. 8 has been shown in Fig. 9 for the second material set. Compared to set 1, set 2 requires a larger contact displacement before the onset of plasticity since the S_{ys} is larger for this set, resulting in a greater contact force than for set 1. The predictions match the FE results for both sphere deformation ratio, δ_s^* , and contact radius, a , with absolute errors $e_{\delta_s} = 0.017$ and $e_a = 0.045$ and maximum errors $e_{\delta_s,m} = 0.062$ and $e_{a,m} = 0.041$, respectively. The first approach for the force shows a good agreement of the FE results with mean absolute error as, $e_{F1} = 0.027$ and maximum error $e_{F1m} = 0.043$. The second approach shows large errors, $e_{F2} = 0.200$, which is not an acceptable match.

Figure 10 compares the FE model and analytical predictions for material set 3. For this material set, the tangent modulus is small ($E_t/E_s = 0.005$). The errors for deformation ratio and contact radius are $e_{\delta_s} = 0.031$ and $e_a = 0.035$ and maximum errors $e_{\delta_s,m} = 0.067$ and $e_{a,m} = 0.125$, respectively. The first force approach shows very small absolute mean error, $e_{F1} = 0.015$ with maximum error $e_{F1m} = 0.063$, while the second approach shows larger errors $e_{F2} = 0.30$. The SM model shows much larger values for both contact radius and contact force.

Material set 4 represents the same material properties as set 3 with the exception of a significantly larger $E_t/E_s = 0.1$. The comparison for deformation ratio, contact area, and contact force representing this set is shown in Fig. 11. The mean absolute errors for the deformation ratio and the contact radius of the analytical model are $e_{\delta_s} = 0.012$ and $e_a = 0.050$ and maximum errors $e_{\delta_s,m} = 0.024$ and $e_{a,m} = 0.034$, respectively. The first and second contact force approaches show absolute mean error $e_{F1} = 0.040$ and $e_{F2} = 0.12$, respectively, while the maximum error for the first approach is $e_{F1m} = 0.045$. The SM model shows similar agreement

Fig. 11 Comparison of deformation ratio, δ_s^* , contact radius, a , and contact force, F , between the FE modeling and proposed formulation for material set 4

Fig. 12 Comparison of deformation ratio, δ_s^* , contact radius, a , and contact force, F , between the FE modeling and proposed formulation for material set 5

as the present model for both contact radius and contact force, with slightly larger contact radius.

The predictions for material set 5 are shown in Fig. 12. For set 5, the tangent modulus is $E_t/E_s = 0.02$, which is the same as set 2. The difference between sets 5 and 2 is the yield strength of the sphere, which is smaller than set 2 (300 MPa versus 750 MPa). The predictions for both contact radius and deformation ratio show good match with the FE results with $e_{\delta_s} = 0.053$, and $e_a = 0.0345$ and maximum errors $e_{\delta_{sm}} = 0.135$ and $e_{am} = 0.092$, respectively. The SM model shows much larger values for the contact radius. The first contact force approach shows a good agreement to the FE results with, absolute mean error $e_{F1} = 0.055$ and maximum error $e_{F1m} = 0.071$ while second approach shows larger difference with $e_{F2} = 0.16$. The SM model also shows a good agreement with the numerical model.

All of the comparisons shown in Figs. 8–12 are for a large range of deformations, which shows the stability of the formulation for a high range of contact loads and its applicability to typical engineering materials. A summary of all the comparisons can be seen in Table 2. A good measure of the plastic deformation is the normalized interference, Δ/δ_c that has been used in almost all models. For the JG model, the critical interference in which the yield starts is calculated from Eq. (5). The maximum normalized deformation analyzed for the sets shown in Table 1 are: set 1, $\max(\Delta/\delta_c) = 366$, set 2, $\max(\Delta/\delta_c) = 163$, set 3, $\max(\Delta/\delta_c) = 635$, set 4, $\max(\Delta/\delta_c) = 101$, and for set 5, $\max(\Delta/\delta_c) = 1018$.

The predictions for sphere deformation ratio and the real contact radius are in agreement with the FE results for all of the cases. The mean absolute error is less than 6% in all of the cases. It can be seen that in case of the deformation predictions, the maximum errors happen for cases 3 and 5, which represent smaller E_t/E_s . The reason for this trend can be the effect of E_t/E_s on the accuracy of Eq. (11). To find the large deformation limit Eq. (11), the

purely elastic regime of the material has been neglected. The applicability of this assumption decreases for smaller E_t/E_s .

For the contact force predictions, the first approach (empirical formulation) shows a very good agreement with the FE results (mean absolute error is less than 5% for all the cases). No trend can be seen between the errors and any of the material properties parameters. The second approach (using the elastic deformations on the flat) does not show a very good agreement with the FE results for all the material sets. For material sets 1, 4, 5 the approach predicts a reasonable approximation but still over 10%. This shows even though the contact radius is updated for this approach, the pressure distribution over the contact area has a large effect on the predictions of this approach.

5 Conclusion

In this work, a new formulation for single asperity frictionless flattening contact of a sphere and a flat considering the effect of bilinear strain hardening was presented. The two main focuses of this record were first, investigating the whole range of material plasticity from elastic–perfectly plastic behavior to perfectly elastic material and second, allowing elastic deformations on the flat for a flattening contact. A new approach for predicting contact force based on the elastic force on the flat has been studied that shows potential for developing simpler formulations of the contact force.

The formulation of the deformation ratio, and contact radius were developed based on FE results. For all of the parameters there were two limiting cases, the perfectly elastic contact and elastic–perfectly plastic contact. The prediction of the sphere deformation was based on seven governing conditions and separation of variables to account for the effect of bilinear strain hardening, E_t/E_s and the applied displacement, Δ/R . The formulation of the real contact area was based on the FE results and showed that including the elastic deformations on the flat affects the real contact radius significantly, but not the force.

For the contact force two approaches were introduced: the empirical fits to the FE results that are conventionally used in the contact mechanics field and calculating the elastic forces from the elastic deformation on the flat. In the second approach, the radius of the curvature was modified to account for the elastic–plastic deformations on the sphere. Although in our study this method did not present acceptable results, it shows a potential for future investigations. The model compared to results from simulations for five different material sets, and the predictions were verified for all sets with a mean absolute error less than 6% for deformation, real contact radius, and the contact force. It has to be noted that since we provide the transition from elastic–perfectly plastic to purely elastic material models, the accuracy of our formulation and its limits are highly dependent on the contact models we used for lower and upper limits (Hertzian theory and JG model). Therefore, our formulation limit is bounded to the limits of JG model which is very close to $a/R = 0.5$.

In this work, we are considering the flat to be perfectly elastic; however, due to the hardening of the sphere, the stresses in the flat increase as the deformations increase. The stresses for special cases can reach to very large values depending on the deformations and the tangent modulus of the sphere that in reality the flat would reach plastic deformations. These phenomena should not happen in majority of the regular engineering applications, such as the rolling element bearings as the yield strength of the bearing is extremely large; however, must be considered. Predicting this issue, requires analyzing of extremely large number of material combinations for different yields strength, elastic modulus, tangential modulus, and Poisson's ratio of both sphere and the flat. In case the yield strength of the materials is very close to each other, the effect of strain hardening will decrease and our previous study [24] can be used.

It was shown in this work that the effect of strain hardening for flattening contact even for very small tangent moduli as little as

$E_f/E_s = 1\%$ cannot be neglected contrary to the findings of Kogut and Etsion [33]. This effect is anticipated to be larger for indentation contact, since the strain hardening might affect the pile-up or sink-in which significantly affects the real contact radius and the pressure distribution in contact region. In the future work, authors will study the bilinear strain-hardening effect for elastic-plastic indentation contact.

Nomenclature

a = real contact radius
 a_e = real contact radius for purely elastic materials
 a_p = real contact radius for elastic-perfectly plastic materials
 a_{IG} = real contact radius presented by Jackson and Green [27]
 a'_p = real contact radius at which the contact reaches the fully plastic regime
 C = coefficient defined by Green [54] to account for the effect of Poisson's ratio in the initiation of yield
 E = effective modulus of elasticity
 E_f = modulus of elasticity of the flat
 E_s = modulus of elasticity of the sphere
 E_t = tangent modulus of elasticity
 E^* = effective modulus of elasticity for a sphere with $E = E_t$
 F = contact force
 F_c = critical contact force at which the elastic-plastic regime effectively starts
 F_e = contact force for purely elastic materials
 F_{ef} = elastic contact force on flat
 F_p = contact force for elastic-perfectly plastic materials
 H = hardness
 HP = hardness parameter
 n = Meyer's hardness exponent
 n_e = strain-hardening exponent
 p_0 = average normal pressure during the fully plastic regime
 R = equivalent radius of curvature
 R' = modified radius of curvature
 S_y = yield strength
 S_{ys} = yield strength of the sphere
 ν = Poisson's ratio
 ν_f = Poisson's ratio of the flat
 ν_s = Poisson's ratio of the sphere
 z = distance depth on the axis of the symmetry from the contact tip
 α, β, λ = fitting parameters
 Δ = total relative normal displacement of the objects during contact
 δ = deformation on one of the objects
 δ_c = critical deformation at which the elastic-plastic regime effectively starts
 δ_f = deformation of the flat
 δ_s = deformation of the sphere
 δ_y = deformation at which yield initiates
 δ_s^* = deformation ratio of the sphere
 δ_s^{*e} = sphere deformation ratio limit for elastic contact
 δ_{sp}^{*e} = sphere deformation ratio limit for larger deformations
 ϕ = function accounting for the effect of applied displacement on the sphere deformation ratio
 ψ = function accounting for the effect of bilinear strain hardening on the sphere deformation ratio
 Υ = function accounting for the effect of bilinear strain hardening on the real contact radius

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