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Syzygies for translational surfaces

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ABSTRACT

A translational surface is a rational tensor product surface generated from two rational space curves by translating one curve along the other curve. Translational surfaces are invariant under rigid motions: translating and rotating the two generating curves translates and rotates the translational surface by the same amount. We construct three special syzygies for a translational surface from a μ -basis of one of the generating space curves, and we show how to compute the implicit equation of a translational surface from these three special syzygies. Examples are provided to illustrate our theorems and flesh out our algorithms.

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1. Introduction

A translational surface is a rational tensor product surface generated from two rational space curves by translating either one of these curves parallel to itself in such a way that each of its points describes a curve that is a translation along the other curve. Translational surfaces (Benkő and Várady, 2000; Liu, 1999; Munteanu and Nistor, 2011; Pérez-Díaz and Shen, 2017; Verstraeten et al., 1994; Vršek and Lávička, 2016; Yoon, 2002), ruled surfaces (Chen et al., 2011; Shen and Pérez-Díaz, 2014), swept surfaces (Schroeder et al., 1994; Weinert et al., 2014), surface of revolution (Gray, 1998; Vršek and Lávička, 2015), along with low degree surfaces such as quadratic surfaces (Goldman, 1983; Wang, 2002; Wang and Joe, 1997), Steiner surfaces (Coffman et al., 1996; Wang and Chen, 2012), cubic surfaces (Bajaj 1990, 1992), and cyclides (Boehm, 1990; Jia, 2014) are basic modeling surfaces that are widely used in computer aided geometric design and geometric modeling.

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Since translational surfaces are generated from two space curves, translational surfaces have simple representations. The simplest and perhaps the most common representation of a translational surface is given by the rational parametric representation $\mathbf{h}^*(s; t) = \mathbf{f}^*(s) + \mathbf{g}^*(t)$, where $\mathbf{f}^*(s)$ and $\mathbf{g}^*(t)$ are two rational space curves. Translational surfaces represented by $\mathbf{h}^*(s; t) = \mathbf{f}^*(s) + \mathbf{g}^*(t)$ have been investigated by differential geometers (Gray, 1998; Liu, 1999; Munteanu and Nistor, 2011; Verstraelen et al., 1994; Yoon, 2002). For example, Liu (1999) classifies the translational surfaces with constant mean curvature or constant Gaussian curvature in both 3-dimensional Euclidean space and 3-dimensional Minkowski space. Munteanu and Nistor (2011) study the second fundamental form of translational surfaces. Researchers (Farin, 2014; Pérez-Díaz, 2006; Pérez-Díaz and Shen, 2017; Schroeder et al., 1994) investigate translational surfaces from a geometric modeling point of view. For instance, translational surfaces are viewed as solutions to the following interpolation problem: given two intersecting curves, find a surface that contain them both as boundary curves (Farin, 2014). Pérez-Díaz and Shen (2017) provide a necessary and sufficient condition for algebraic surfaces to be translational surfaces.

Translational surfaces defined by $\mathbf{h}^*(s; t) = \mathbf{f}^*(s) + \mathbf{g}^*(t)$ are not translation invariant: translating both curves $\mathbf{f}^*$ and $\mathbf{g}^*$ by the vector $\mathbf{v}$ translates the surface $\mathbf{h}^*$ by the vector $2\mathbf{v}$. One would like to define translational surfaces in such a way that translating the two generating curves by the same vector $\mathbf{v}$, also translates every point on the surface by the vector $\mathbf{v}$. Recently, Vršek and Lávička (2016) offer an alternative definition of translational surfaces given by the rational parametric representation $\mathbf{h}^*(s; t) = \frac{\mathbf{f}^*(s) + \mathbf{g}^*(t)}{2}$, where $\mathbf{f}^*(s)$ and $\mathbf{g}^*(t)$ are two rational space curves. Under this definition, these translational surfaces consist of all the midpoints of all the lines joining a point on $\mathbf{f}^*$ to a point on $\mathbf{g}^*$, so these translational surfaces are invariant under rigid motions: translating and rotating the two generating curves translates and rotates these translational surfaces by the same amount. Hence, applying a rigid motion to a translational surface can be achieved by applying the same rigid motion to the two rational space curves that generate the surface. In fact, these translational surfaces are invariant under all affine transformations. Therefore, one can control these translational surfaces simply by manipulating their generating curves. Vršek and Lávička focus on the geometry of these translation surfaces and study their geometric properties. In particular, they show that all minimal surfaces are translational surfaces where the generating curves are isotropic curves.

In this paper, we investigate the translational surfaces given by the rational parametric representation $\mathbf{h}^*(s; t) = \frac{\mathbf{f}^*(s) + \mathbf{g}^*(t)}{2}$. We begin by examining the number of base points (counting multiplicity) and the implicit degree of these translational surfaces. Our main goal, however, is to utilize syzygies to study translational surfaces. We construct three special syzygies for a translational surface from a μ -basis of one of the generating space curves, and we show how to compute the implicit equation of a translational surface from these three special syzygies.

This paper is structured in the following fashion. In Section 2 we provide an introduction to translational surfaces. We fix our notation, define translational surfaces, and establish a few properties of translational surfaces in Section 2.1. We start Section 2.2 by recalling the notions of syzygies and μ -bases for rational curves and surfaces. Then we study syzygies of translational surfaces, and relate the syzygies of the generating curves to the syzygies of the corresponding translational surface. We also construct three special syzygies for a translational surface from a μ -basis of one of the generating space curves. Section 3 contains our main results. Here we show how to use these three special syzygies to compute the implicit equation of a translational surface. In Section 4, we observe that the techniques used in this paper can be applied with only minor modifications to the translational surfaces defined by $\mathbf{h}^*(s; t) = a\mathbf{f}^*(s) + b\mathbf{g}^*(t)$, where $a, b \in \mathbb{R}$ and $ab \neq 0$. We close in Section 5 by proposing two open problems for future research. Throughout this paper, we provide simple, straightforward examples to illustrate and clarify our theorems and algorithms.

2. An introduction to translational surfaces

Throughout this paper, the following expressions denote the homogeneous, non-homogeneous and rational forms of the same vector valued function $\mathbf{a}$:

$$\mathbf{a}(s, u) = (a_0(s, u), a_1(s, u), a_2(s, u), a_3(s, u)) \in \mathbb{R}^4[s, u], \quad \text{homogeneous form,}$$

$\mathbf{a}(s) = \mathbf{a}(s, 1) = (a_0(s), a_1(s), a_2(s), a_3(s)) \in \mathbb{R}^4[s]$, non-homogeneous form,

$\mathbf{a}^*(s) = \left(\frac{a_1(s)}{a_0(s)}, \frac{a_2(s)}{a_0(s)}, \frac{a_3(s)}{a_0(s)} \right) \in \mathbb{R}^3(s)$, rational form.

2.1. Properties of translational surfaces

In affine 3-space, the rational surface

$$\mathbf{h}^*(s; t) = \frac{\mathbf{f}^*(s) + \mathbf{g}^*(t)}{2}, \quad (1)$$

is called the *translational surface* generated by the rational space curves $\mathbf{f}^*(s)$ and $\mathbf{g}^*(t)$

$$\mathbf{f}^*(s) = \left(\frac{f_1(s)}{f_0(s)}, \frac{f_2(s)}{f_0(s)}, \frac{f_3(s)}{f_0(s)} \right), \quad \mathbf{g}^*(t) = \left(\frac{g_1(t)}{g_0(t)}, \frac{g_2(t)}{g_0(t)}, \frac{g_3(t)}{g_0(t)} \right) \quad (2)$$

where $f_i \in \mathbb{R}[s]$ and $g_i \in \mathbb{R}[t]$. If $\max(\deg(f_i(s))) = m$, $\max(\deg(g_i(t))) = n$, and $\gcd(f_i) = \gcd(g_i) = 1$ for $i = 0, 1, 2, 3$, then $\mathbf{h}^*(s, t)$ is a rational tensor product surface of bidegree (m, n) .

Translational surfaces are typical modeling surfaces in architecture and computer aided design. Among the non-degenerate quadratic surfaces, ellipsoids, elliptical (and circular) cones, and hyperboloids of one or two sheets cannot be constructed as translational surfaces, that is these surfaces do not admit a rational parametrization given by Equation (1). However, elliptical (and circular) cylinders, parabolic cylinders, hyperbolic cylinders, elliptical (and circular) paraboloids and hyperbolic paraboloids can be constructed as translational surfaces. Indeed it is easy to picture that elliptical (and circular), parabolic, or hyperbolic cylinders are translational surfaces generated by translating an ellipse (and circle), parabola, or hyperbola along a straight line.

There are many other interesting surfaces that can be constructed as translational surfaces generated by two rational space curves. The collection of translational surfaces increases if we allow the generating curves to be complex curves. For example, Vršek and Lávička (2016) generate the Enneper surface from two complex curves.

In affine 3-space, a rigid motion is a transformation that when acting on any point p , generates a transformed point $T(p) = Rp + \mathbf{t}$, where R is a 3×3 orthogonal matrix representing the rotation with $\det(R) = 1$, and $\mathbf{t} \in \mathbb{R}^3$ a 3-dimensional translation vector. Since R represents a linear transformation

$$\begin{aligned} T(\mathbf{h}^*(s; t)) &= R \left(\frac{\mathbf{f}^*(s) + \mathbf{g}^*(t)}{2} \right) + \mathbf{t} = \frac{(R(\mathbf{f}^*(s)) + \mathbf{t}) + (R(\mathbf{g}^*(t)) + \mathbf{t})}{2} \\ &= \frac{T(\mathbf{f}^*(s)) + T(\mathbf{g}^*(t))}{2}. \end{aligned}$$

Thus in affine 3-space a translational surface $\mathbf{h}^*(s; t)$ given by Equation (1) is invariant under rigid motions: translating and rotating the two generating curves translates and rotates the translational surface $\mathbf{h}^*(s; t)$ by the same amount. In fact, the same argument shows that translational surfaces are invariant under all affine transformations.

To understand the geometry of rational surfaces, it is sometimes necessary to consider homogeneous forms with a homogeneous parametrization of these rational surfaces. Homogenizations of Equations (1) and (2) yield a tensor product parametrization $\mathbf{h} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$:

$$\begin{aligned} \mathbf{h}(s, u; t, v) &= [h_0(s, u; t, v), h_1(s, u; t, v), h_2(s, u; t, v), h_3(s, u; t, v)] \\ &= [2f_0(s, u)g_0(t, v), f_0(s, u)g_1(t, v) + f_1(s, u)g_0(t, v), \\ &\quad f_0(s, u)g_2(t, v) + f_2(s, u)g_0(t, v), f_0(s, u)g_3(t, v) + f_3(s, u)g_0(t, v)] \\ &= f_0(s, u)\mathbf{g}(t, v) + g_0(t, v)\mathbf{f}(s, u) \end{aligned}$$

generated by two generic one-to-one homogeneous parametrizations $\mathbf{f}$ and $\mathbf{g}$:

$$\mathbf{f}: (s, u) \in \mathbb{P}^1 \rightarrow (x, y, z, w) \in \mathbb{P}^3, \quad \mathbf{f}(s, u) = [f_0(s, u), f_1(s, u), f_2(s, u), f_3(s, u)],$$

$$\mathbf{g}: (t, v) \in \mathbb{P}^1 \rightarrow (x, y, z, w) \in \mathbb{P}^3, \quad \mathbf{g}(t, v) = [g_0(t, v), g_1(t, v), g_2(t, v), g_3(t, v)],$$

where f_i and g_i are homogeneous polynomials in the standard $\mathbb{Z}$ -graded rings $\mathbb{R}[s, u]$ and $\mathbb{R}[t, v]$ of $\deg(f_i) = m$, $\deg(g_i) = n$, and $\gcd(f_i) = \gcd(g_i) = 1$ for $i = 1, 2, 3, 4$. Note that the functions h_i are bihomogeneous polynomials in the bigraded ring $\mathbb{R}[s, u; t, v]$ of $\text{bidegree}(h_i) = (m, n)$, where $\text{bidegree}(s) = \text{bidegree}(u) = (1, 0)$ and $\text{bidegree}(t) = \text{bidegree}(v) = (0, 1)$. From now on, we shall write $\mathbf{h}(s, u; t, v)$ as the bihomogeneous parametrization of the translational surfaces given by Equation (1).

Notice that the parametrization of a translational surface generated by any two generic one-to-one rational curves need not be a generic one-to-one parametrization. For example, consider

$$\mathbf{f}(s, u) = (s^4, s^3u, s^2u^2, u^4), \quad \mathbf{g}(t, v) = (t^4, t^3v, t^2v^2, v^4),$$

$$\mathbf{h}(s, u; t, v) = (2s^4t^4, s^4t^3v + s^3ut^4, s^4t^2v^2 + s^2u^2t^4, s^4v^4 + u^4t^4).$$

Since $\mathbf{h}(a, 1; b, 1) = \mathbf{h}(b, 1; a, 1)$ for all $a, b \in \mathbb{R}$, $\mathbf{h}(s, u; t, v)$ is a 2-to-1 tensor product parametrization.

One would like to find the necessary and sufficient conditions which guarantee that a translational surface has a proper parametrization using only the parametrizations of the generating curves $\mathbf{f}(s, u)$ and $\mathbf{g}(t, v)$. Currently, we do not have a complete solution for this problem. Hence, in this paper, we focus on generic one-to-one parametrizations of $\mathbf{h}(s, u; t, v)$.

Since translational surfaces are a special type of rational tensor product surface, translational surfaces are equipped with all the properties of rational tensor product surfaces. In addition, translational surfaces have their own special characteristics. Next, we discuss some of these special characteristics in more detail.

First, observe that $\gcd(h_0, h_1, h_2, h_3) = 1$. Otherwise, suppose that

$$\gcd(h_0(s, u; t, v), h_1(s, u; t, v), h_2(s, u; t, v), h_3(s, u; t, v)) = \gamma(s, u; t, v) \neq 1.$$

Then since $h_0(s, u; t, v) = 2f_0(s, u)g_0(t, v)$, it follows that $\gamma(s, u; t, v) \mid f_0(s, u)g_0(t, v)$. Hence, $\gamma(s, u; t, v) = \alpha(s, u)\beta(t, v)$, where $\alpha(s, u) \mid f_0(s, u)$ and $\beta(t, v) \mid g_0(t, v)$. Assume without loss of generality that $\alpha(s, u) \neq 1$. Then since $\gamma(s, u; t, v) \mid h_2(s, u; t, v)$, we have $\alpha(s, u) \mid (f_0(s, u)g_1(t, v) + g_0(t, v)f_1(s, u))$. But $\alpha(s, u) \mid f_0(s, u)$ and $\alpha(s, u) \nmid g_0(t, v)$; therefore $\alpha(s, u) \mid f_1(s, u)$. A similar argument shows that $\alpha(s, u) \mid f_2(s, u)$ and $\alpha(s, u) \mid f_3(s, u)$, so $\gcd(f_0(s, u), f_1(s, u), f_2(s, u), f_3(s, u)) \neq 1$. Contradiction. Hence

$$\gcd(h_0(s, u; t, v), h_1(s, u; t, v), h_2(s, u; t, v), h_3(s, u; t, v)) = 1.$$

Second, even with the assumption $\gcd(h_0, h_1, h_2, h_3) = 1$, it can happen that there are parameters $(s, u; t, v)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ where $h_i(s, u; t, v) = 0$, $i = 0, 1, 2, 3$. These parameters, where the map $\mathbf{h}: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ is not defined, are called *base points*. Some base points can be quite complicated. A base point is called a *local complete intersection* if the ideal $\langle h_0, h_1, h_2, h_3 \rangle$ is generated by two polynomials in a neighborhood of $(s_0, u_0; t_0, v_0)$. Base points that are local complete intersections are the simplest, tamest types of base points. Although we shall see shortly that translational surfaces have lots of base points, we will see in Section 3 that none of these base points are local complete intersections.

Lemma 2.1. *The base points of a translational surface $\mathbf{h}(s, u; t, v)$ are exactly the parameters $(s, u; t, v)$ for which $f_0(s, u) = 0$ and $g_0(t, v) = 0$.*

Proof. Clearly if $f_0(s_0, u_0) = 0$ and $g_0(t_0, v_0) = 0$, then $(s_0, u_0; t_0, v_0)$ is a base point of $\mathbf{h}(s, u; t, v)$, since $\mathbf{h}(s, u; t, v) = f_0(s, u)\mathbf{g}(t, v) + g_0(t, v)\mathbf{f}(s, u)$. Conversely, if $(s_0, u_0; t_0, v_0)$ is a base point of $\mathbf{h}(s, u; t, v)$, then since $h_0(s, u; t, v) = 2f_0(s, u)g_0(t, v) = 0$ either $f_0(s, u) = 0$ or $g_0(t, v) = 0$. If $f_0(s, u) = 0$, then $\gcd(\mathbf{f}(s, u)) = 1$ means one of $f_i(s, u) \neq 0$ for some $i = 1, 2, 3$. But then $h_i(s, u; t, v) = g_0(t, v)f_i(s, u) + f_0(s, u)g_i(t, v) = g_0(t, v)f_i(s, u) = 0$ forces $g_0(t, v) = 0$. A similar argument shows that if $g_0(t, v) = 0$, then $f_0(s, u) = 0$. Therefore, the base points of a translational surface $\mathbf{h}(s, u; t, v)$ are exactly the parameters $(s, u; t, v)$ for which $f_0(s, u) = 0$ and $g_0(t, v) = 0$. $\square$

Thus by Lemma 2.1 unlike arbitrary rational tensor product surfaces, the base points of translational surfaces are very easy to find simply by solving for the roots of two univariate polynomials. Moreover, the following corollary is a direct consequence of Lemma 2.1.

Corollary 2.2. *Counting multiplicity, translational surfaces have at least mn base points.*

The implicit degree of a generic one-to-one rational tensor product surface of bidegree (m, n) is given by Adkins et al. (2005)

implicit degree of the surface = $2mn$ – total multiplicity of the base points.

Since translational surfaces of bidegree (m, n) have at least mn base points, their implicit degree is at most mn . In Section 3 we shall give sufficient conditions for the implicit degree of a translational surface to be exactly mn . So the good news about translational surfaces is that these surfaces tend to have low implicit degree.

The search for techniques for implicitizing rational surfaces with base points is a very active area of research because base points show up quite frequently in practical industrial design. It is often difficult to implicitize a surface that has a complicated collection of base points. In Section 3, we will use the algorithm in Shen and Goldman (2017a) to compute the implicit equation of a translational surface $\mathbf{h}$ from the resultant of a μ -basis for $\mathbf{h}$.

2.2. Syzygies and μ -bases for translational surfaces

In this section, we study syzygies of translational surfaces, and relate the syzygies of the two generating curves to the syzygies of the translational surface. We begin with a brief review of syzygies and μ -bases for rational space curves and rational surfaces. For additional details and results concerning syzygies and μ -bases for rational curves and rational surfaces, see Chen et al. (2005) and Cox et al. (1998).

2.2.1. Syzygies and μ -bases for rational curves and rational surfaces

Consider a rational space curve $\mathbf{F} \in \mathbb{R}^4[s, u]$ of the following form

$$\mathbf{F}(s, u) = [F_0(s, u), F_1(s, u), F_2(s, u), F_3(s, u)], \text{ where } \deg(F_i(s, u)) = m \text{ and } \gcd(\mathbf{F}) = 1.$$

A syzygy of $\mathbf{F}$ is a polynomial vector

$$\mathbf{l}(s, u) = (a_0(s, u), a_1(s, u), a_2(s, u), a_3(s, u)),$$

where a_i are homogeneous polynomials of the same degree in (s, u) such that

$$\mathbf{l} \cdot \mathbf{F} = \sum_{i=0}^3 a_i(s, u) F_i(s, u) = a_0 F_0 + a_1 F_1 + a_2 F_2 + a_3 F_3 \equiv 0.$$

The set of all syzygies of $\mathbf{F}$ is a module over the ring $\mathbb{R}[s, u]$ called the *syzygy module* of $\mathbf{F}$, and is denoted by $\text{Syz}(\mathbf{F})$. It is known that $\text{Syz}(\mathbf{F})$ is a free module over the ring $\mathbb{R}[s, u]$ generated by three elements (Cox et al., 1998).

Three syzygies $\mathbf{p}_0(s, u), \mathbf{p}_1(s, u), \mathbf{p}_2(s, u)$ are called a μ -basis for a rational space curve $\mathbf{F}(s, u)$ if $\mathbf{p}_0(s, u), \mathbf{p}_1(s, u), \mathbf{p}_2(s, u)$ form a basis of $\text{Syz}(\mathbf{F})$ – that is,

$$\mathbf{l} \in \text{Syz}(\mathbf{F}) \Rightarrow \mathbf{l} = \sum_{i=0}^2 \alpha_i(s, u) \mathbf{p}_i(s, u), \text{ where } \alpha_i \in \mathbb{R}[s, u], i = 0, 1, 2.$$

If $\mathbf{p}_i = (p_{i0}, p_{i1}, p_{i2}, p_{i3}), i = 0, 1, 2$, is a μ -basis for a rational space curve $\mathbf{F}$, then

$$\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\} = \lambda \mathbf{F}(s; t), \quad \lambda \text{ is a non-zero constant (Cox et al., 1998),}$$

where $\{\cdot\}$ denotes the outer product of $\mathbf{p}_0, \mathbf{p}_1$ and $\mathbf{p}_2$, defined by

$$\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\} = \left(\begin{vmatrix} p_{01} & p_{02} & p_{03} \\ p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{vmatrix}, - \begin{vmatrix} p_{00} & p_{02} & p_{03} \\ p_{10} & p_{12} & p_{13} \\ p_{20} & p_{22} & p_{23} \end{vmatrix}, \right. \\ \left. \begin{vmatrix} p_{00} & p_{01} & p_{03} \\ p_{10} & p_{11} & p_{13} \\ p_{20} & p_{21} & p_{23} \end{vmatrix}, - \begin{vmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{vmatrix} \right).$$

Similarly, consider a rational tensor product surface $\mathbf{H} \in \mathbb{R}^4[s, u; t, v]$ of the following form

$$\mathbf{H}(s, u; t, v) = [H_0(s, u; t, v), H_1(s, u; t, v), H_2(s, u; t, v), H_3(s, u; t, v)],$$

where $H_i(s, u; t, v)$ are bihomogeneous polynomials of bidegree (m, n) in s, u and t, v , and $\gcd(\mathbf{H}) = 1$. A syzygy of $\mathbf{H}(s, u; t, v)$ is a polynomial vector

$$\mathbf{L}(s, u; t, v) = (A_0(s, u; t, v), A_1(s, u; t, v), A_2(s, u; t, v), A_3(s, u; t, v)),$$

where A_i are bihomogeneous polynomials of the same bidegree in $(s, u; t, v)$ such that

$$\mathbf{L} \cdot \mathbf{H} = \sum_{i=0}^3 A_i(s, u; t, v) H_i(s, u; t, v) = A_0 H_0 + A_1 H_1 + A_2 H_2 + A_3 H_3 \equiv 0.$$

The set of all syzygies of $\mathbf{H}$ is a module over the ring $\mathbb{R}[s, u; t, v]$ called the *syzygy module* of $\mathbf{H}$, and is denoted by $\text{Syz}(\mathbf{H})$. If $\deg(A_i) = (a, b)$, then we say $\mathbf{L}$ is a syzygy of bidegree (a, b) .

Geometrically, a syzygy $\mathbf{L}$ of $\mathbf{H}$ can be viewed as a *moving plane*, that is a family of planes with each set of parameters $(s, u; t, v)$ corresponding to the implicit expression for a plane:

$$L(\mathbf{x}; s, u; t, v) = A_0(s, u; t, v)w + A_1(s, u; t, v)x + A_2(s, u; t, v)y + A_3(s, u; t, v)z.$$

Moreover, this moving plane has the property that

$$L(\mathbf{H}; s, u; t, v) = \sum_{i=0}^3 A_i(s, u; t, v) H_i(s, u; t, v) = A_0 H_0 + A_1 H_1 + A_2 H_2 + A_3 H_3 \equiv 0,$$

that is, the point on the surface $\mathbf{H}$ at the parameter $(s, u; t, v)$ lies on the plane $\mathbf{L}$ at the parameter $(s, u; t, v)$. Hence, a syzygy of $\mathbf{H}$ is in fact a moving plane $\mathbf{L}$ that *follows the surface* $\mathbf{H}$.

It is known that $\text{Syz}(\mathbf{H})$ is not a free module over the ring $\mathbb{R}[s, u; t, v]$ (Cox et al., 1998). But in the affine setting, that is, if $u = v = 1$, then $\text{Syz}(\mathbf{H}(s; t))$ is a free module over the ring $\mathbb{R}[s; t]$ with a basis consisting of three elements (Chen et al., 2005). Three syzygies $\mathbf{p}_0(s; t), \mathbf{p}_1(s; t), \mathbf{p}_2(s; t)$ are called a μ -basis for a rational tensor product surface $\mathbf{H}(s; t)$ if

$$\{\mathbf{p}_0(s; t), \mathbf{p}_1(s; t), \mathbf{p}_2(s; t)\} = \lambda \mathbf{H}(s; t), \quad \lambda \text{ is a non-zero constant,}$$

where $\{\cdot\}$ denotes the outer product of $\mathbf{p}_0, \mathbf{p}_1$ and $\mathbf{p}_2$. Three syzygies that satisfy this formula always form a basis for the syzygy module $\text{Syz}(\mathbf{H}(s; t))$ (Chen et al., 2005).

Notice that for rational space curves, μ -bases are defined relative to homogeneous parameters, but for rational surfaces, μ -bases are defined relative to affine parameters. For a μ -basis $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ and homogeneous parameters $(s, u; t, v)$

$$\{\mathbf{p}_0(s, u; t, v), \mathbf{p}_1(s, u; t, v), \mathbf{p}_2(s, u; t, v)\} = \lambda(s, u; t, v) \mathbf{H}(s, u; t, v),$$

where $\lambda(s, u; t, v) \in \mathbb{R}[s, u; t, v]$.

2.2.2. Syzygies for translational surfaces

Next, we investigate syzygies for translational surfaces, and we relate the syzygies for the generating curves to the syzygies for the translational surface. We shall see that the matrix representations of a translational surface $\mathbf{h}$ provide some easy to see relations between $\text{Syz}(\mathbf{f})$ and $\text{Syz}(\mathbf{h})$, or $\text{Syz}(\mathbf{g})$ and $\text{Syz}(\mathbf{h})$.

To begin, observe that the homogeneous parametrization $\mathbf{h}(s, u; t, v)$ has matrix representations:

$$\begin{aligned}\mathbf{h} &= [2f_0g_0, f_0g_1 + f_1g_0, f_0g_2 + f_2g_0, f_0g_3 + f_3g_0] \\ &= [f_0 \ f_1 \ f_2 \ f_3] \begin{bmatrix} 2g_0 & g_1 & g_2 & g_3 \\ 0 & g_0 & 0 & 0 \\ 0 & 0 & g_0 & 0 \\ 0 & 0 & 0 & g_0 \end{bmatrix} = \mathbf{f}M_{\mathbf{g}}, \\ &= [g_0 \ g_1 \ g_2 \ g_3] \begin{bmatrix} 2f_0 & f_1 & f_2 & f_3 \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & f_0 & 0 \\ 0 & 0 & 0 & f_0 \end{bmatrix} = \mathbf{g}M_{\mathbf{f}}.\end{aligned}$$

Thus a translational surface has the following matrix representations

$$\mathbf{h}(s, u; t, v) = \mathbf{f}(s, u)M_{\mathbf{g}}(t, v) = \mathbf{g}(t, v)M_{\mathbf{f}}(s, u).$$

Moreover, if $\mathbf{a} \in \text{Syz}(\mathbf{h})$, then

$$\mathbf{h} = \mathbf{f}M_{\mathbf{g}} = \mathbf{g}M_{\mathbf{f}} \Rightarrow M_{\mathbf{g}}\mathbf{a} \in \text{Syz}(\mathbf{f}) \text{ and } M_{\mathbf{f}}\mathbf{a} \in \text{Syz}(\mathbf{g}).$$

Now let

$$N_{\mathbf{g}} = \begin{bmatrix} \frac{g_0}{2} & -\frac{g_1}{2} & -\frac{g_2}{2} & -\frac{g_3}{2} \\ 0 & g_0 & 0 & 0 \\ 0 & 0 & g_0 & 0 \\ 0 & 0 & 0 & g_0 \end{bmatrix} \text{ and } N_{\mathbf{f}} = \begin{bmatrix} \frac{f_0}{2} & -\frac{f_1}{2} & -\frac{f_2}{2} & -\frac{f_3}{2} \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & f_0 & 0 \\ 0 & 0 & 0 & f_0 \end{bmatrix}.$$

Then

$$M_{\mathbf{g}}N_{\mathbf{g}} = N_{\mathbf{g}}M_{\mathbf{g}} = g_0^2\mathbf{I} \text{ and } M_{\mathbf{f}}N_{\mathbf{f}} = N_{\mathbf{f}}M_{\mathbf{f}} = f_0^2\mathbf{I}, \text{ where } \mathbf{I} \text{ is the } 4 \times 4 \text{ identity matrix.}$$

Furthermore,

$$\mathbf{a} \in \text{Syz}(\mathbf{f}) \Rightarrow N_{\mathbf{g}}\mathbf{a} \in \text{Syz}(\mathbf{h}), \text{ and } \mathbf{b} \in \text{Syz}(\mathbf{g}) \Rightarrow N_{\mathbf{f}}\mathbf{b} \in \text{Syz}(\mathbf{h}).$$

To better understand the syzygies of $\mathbf{h}$, observe that if $(a_0, a_1, a_2, a_3) \in \text{Syz}(\mathbf{h})$, then

$$\begin{aligned}\sum_{i=0}^3 a_i h_i &= 2a_0 f_0 g_0 + a_1(f_0 g_1 + f_1 g_0) + a_2(f_0 g_2 + f_2 g_0) + a_3(f_0 g_3 + f_3 g_0) = 0 \\ &\Leftrightarrow f_0(a_0 g_0 + a_1 g_1 + a_2 g_2 + a_3 g_3) + g_0(a_0 f_0 + a_1 f_1 + a_2 f_2 + a_3 f_3) = 0 \\ &\Leftrightarrow a_0 f_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 = f_0 c \text{ and } a_0 g_0 + a_1 g_1 + a_2 g_2 + a_3 g_3 = -g_0 c \\ &\Leftrightarrow (a_0 - c)f_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 = 0 \text{ and } (a_0 + c)g_0 + a_1 g_1 + a_2 g_2 + a_3 g_3 = 0 \\ &\Leftrightarrow (a_0 - c, a_1, a_2, a_3) \in \text{Syz}(\mathbf{f}), \text{ and } (a_0 + c, a_1, a_2, a_3) \in \text{Syz}(\mathbf{g}).\end{aligned}$$

Thus,

$$\text{Syz}(\mathbf{h}) = \{(a_0 = \frac{a+b}{2}, a_1, a_2, a_3) \mid (a, a_1, a_2, a_3) \in \text{Syz}(\mathbf{f}) \text{ and } (b, a_1, a_2, a_3) \in \text{Syz}(\mathbf{g})\}. \quad (3)$$

From now on, $\mathbf{p}, \mathbf{q}, \mathbf{r}$ will denote a μ -basis for a parametrization $\mathbf{f}(s, u)$ of degree (μ_1, μ_2, μ_3) where $\mu_3 = m - \mu_1 - \mu_2$; and $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ will denote a μ -basis for a parametrization $\mathbf{g}(t, v)$ of degree (ν_1, ν_2, ν_3) where $\nu_3 = n - \nu_1 - \nu_2$. In addition, we will set $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{P}, \mathbf{Q}, \mathbf{R}$ to column vectors with the following entries:

$$\begin{aligned}\mathbf{p} &= (p_0, p_1, p_2, p_3)^T, \quad \mathbf{q} = (q_0, q_1, q_2, q_3)^T, \quad \mathbf{r} = (r_0, r_1, r_2, r_3)^T; \\ \mathbf{P} &= (P_0, P_1, P_2, P_3)^T, \quad \mathbf{Q} = (Q_0, Q_1, Q_2, Q_3)^T, \quad \mathbf{R} = (R_0, R_1, R_2, R_3)^T.\end{aligned}$$

Lemma 2.3.

$$\begin{aligned}\text{Syz}(\mathbf{h}) &= \left\{ \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^4[s, u; t, v] : \right. \\ &\quad \left. \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{cases} \frac{N_{\mathbf{f}}(\Gamma \mathbf{P} + \Delta \mathbf{Q} + \Theta \mathbf{R})}{f_0}, & \Gamma, \Delta, \Theta \in \mathbb{R}[s, u; t, v], \\ \text{or} \\ \frac{N_{\mathbf{g}}(\gamma \mathbf{p} + \delta \mathbf{q} + \theta \mathbf{r})}{g_0}, & \gamma, \delta, \theta \in \mathbb{R}[s, u; t, v] \end{cases} \right\}.\end{aligned}$$

Proof. Recall that the outer products $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\} = \mathbf{f}$ and $\{\mathbf{P}, \mathbf{Q}, \mathbf{R}\} = \mathbf{g}$. Therefore

$$\det \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix} = f_0, \quad \text{and} \quad \det \begin{bmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{bmatrix} = g_0.$$

Let $(a_0, a_1, a_2, a_3) \in \text{Syz}(\mathbf{h})$. By Equation (3), $a_0 = \frac{a+b}{2}$, where $(a, a_1, a_2, a_3) \in \text{Syz}(\mathbf{f})$ and $(b, a_1, a_2, a_3) \in \text{Syz}(\mathbf{g})$. Moreover, since $\mathbf{p}, \mathbf{q}, \mathbf{r}$ is a μ -basis for $\text{Syz}(\mathbf{f})$ and $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ is a μ -basis for $\text{Syz}(\mathbf{g})$, there exist polynomials γ, δ, θ and Γ, Δ, Θ such that

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \\ \theta \end{bmatrix} = \begin{bmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{bmatrix} \begin{bmatrix} \Gamma \\ \Delta \\ \Theta \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} \gamma \\ \delta \\ \theta \end{bmatrix} = \frac{1}{f_0} \begin{bmatrix} \begin{vmatrix} q_2 & r_2 \\ q_3 & r_3 \end{vmatrix} & -\begin{vmatrix} p_2 & r_2 \\ p_3 & r_3 \end{vmatrix} & \begin{vmatrix} p_2 & q_2 \\ p_3 & q_3 \end{vmatrix} \\ -\begin{vmatrix} q_1 & r_1 \\ q_3 & r_3 \end{vmatrix} & \begin{vmatrix} p_1 & r_1 \\ p_3 & r_3 \end{vmatrix} & -\begin{vmatrix} p_1 & q_1 \\ p_3 & q_3 \end{vmatrix} \\ \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} & -\begin{vmatrix} p_1 & r_1 \\ p_2 & r_2 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{bmatrix}^T \begin{bmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{bmatrix} \begin{bmatrix} \Gamma \\ \Delta \\ \Theta \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} a \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} p_0 & q_0 & r_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \\ \theta \end{bmatrix}$$

$$\begin{aligned}
&= \frac{1}{f_0} \begin{bmatrix} p_0 & q_0 & r_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix} \begin{bmatrix} \begin{vmatrix} q_2 & r_2 \\ q_3 & r_3 \end{vmatrix} & -\begin{vmatrix} q_1 & r_1 \\ q_3 & r_3 \end{vmatrix} & \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} \\ -\begin{vmatrix} p_2 & r_2 \\ p_3 & r_3 \end{vmatrix} & \begin{vmatrix} p_1 & r_1 \\ p_3 & r_3 \end{vmatrix} & -\begin{vmatrix} p_1 & r_1 \\ p_2 & r_2 \end{vmatrix} \\ \begin{vmatrix} p_2 & q_2 \\ p_3 & q_3 \end{vmatrix} & -\begin{vmatrix} p_1 & q_1 \\ p_3 & q_3 \end{vmatrix} & \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} \end{bmatrix} \begin{bmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{bmatrix} \begin{bmatrix} \Gamma \\ \Delta \\ \Theta \end{bmatrix} \\
&= \frac{1}{f_0} \begin{bmatrix} -f_1 & -f_2 & -f_3 \\ f_0 & 0 & 0 \\ 0 & f_0 & 0 \\ 0 & 0 & f_0 \end{bmatrix} \begin{bmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{bmatrix} \begin{bmatrix} \Gamma \\ \Delta \\ \Theta \end{bmatrix}, \quad \Gamma, \Delta, \Theta \in \mathbb{R}[s, u; t, v].
\end{aligned}$$

But

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \frac{a+b}{2} \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \text{where} \quad \begin{bmatrix} b \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} P_0 & Q_0 & R_0 \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{bmatrix} \begin{bmatrix} \Gamma \\ \Delta \\ \Theta \end{bmatrix}.$$

Thus, since $(a, a_1, a_2, a_3) \in \text{Syz}(\mathbf{f})$,

$$\begin{aligned}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} &= \begin{bmatrix} \frac{a+b}{2} \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{f_0} \begin{bmatrix} \frac{f_0}{2} & -\frac{f_1}{2} & -\frac{f_2}{2} & -\frac{f_3}{2} \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & f_0 & 0 \\ 0 & 0 & 0 & f_0 \end{bmatrix} \begin{bmatrix} P_0 & Q_0 & R_0 \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{bmatrix} \begin{bmatrix} \Gamma \\ \Delta \\ \Theta \end{bmatrix} \\
&= \frac{N_{\mathbf{f}}(\Gamma \mathbf{P} + \Delta \mathbf{Q} + \Theta \mathbf{R})}{f_0}.
\end{aligned}$$

Similarly, for some $\gamma, \delta, \theta \in \mathbb{R}[s, u; t, v]$,

$$\begin{aligned}
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} &= \begin{bmatrix} \frac{a+b}{2} \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \frac{1}{g_0} \begin{bmatrix} \frac{g_0}{2} & -\frac{g_1}{2} & -\frac{g_2}{2} & -\frac{g_3}{2} \\ 0 & g_0 & 0 & 0 \\ 0 & 0 & g_0 & 0 \\ 0 & 0 & 0 & g_0 \end{bmatrix} \begin{bmatrix} p_0 & q_0 & r_0 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{bmatrix} \begin{bmatrix} \gamma \\ \delta \\ \theta \end{bmatrix} \\
&= \frac{N_{\mathbf{g}}(\gamma \mathbf{p} + \delta \mathbf{q} + \theta \mathbf{r})}{g_0}. \quad \square
\end{aligned}$$

Remark 2.4. By Lemma 2.3 finding elements in $\text{Syz}(\mathbf{h})$ is equivalent to searching for Γ, Δ, Θ , or $\gamma, \delta, \theta \in \mathbb{R}[s, u; t, v]$ where

$$[f_0, -f_1, -f_2, -f_3] \cdot (\Gamma \mathbf{P} + \Delta \mathbf{Q} + \Theta \mathbf{R}) \equiv 0 \pmod{f_0}, \quad \text{or}$$

$$[g_0, -g_1, -g_2, -g_3] \cdot (\gamma \mathbf{p} + \delta \mathbf{q} + \theta \mathbf{r}) \equiv 0 \pmod{g_0}.$$

If we choose

$$\begin{bmatrix} \Gamma \\ \Delta \\ \Theta \end{bmatrix} = \begin{bmatrix} f_0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 \\ f_0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 \\ 0 \\ f_0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \gamma \\ \delta \\ \theta \end{bmatrix} = \begin{bmatrix} g_0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 \\ g_0 \\ 0 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 \\ 0 \\ g_0 \end{bmatrix},$$

then we have the following two sets of three syzygies represented by the columns of the following matrices

$$N_{\mathbf{f}}[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}] = \begin{bmatrix} \frac{f_0 P_0 - f_1 P_1 - f_2 P_2 - f_3 P_3}{2} & \frac{f_0 Q_0 - f_1 Q_1 - f_2 Q_2 - f_3 Q_3}{2} & \frac{f_0 R_0 - f_1 R_1 - f_2 R_2 - f_3 R_3}{2} \\ f_0 P_1 & f_0 Q_1 & f_0 R_1 \\ f_0 P_2 & f_0 Q_2 & f_0 R_2 \\ f_0 P_3 & f_0 Q_3 & f_0 R_3 \end{bmatrix},$$

$$N_g[\mathbf{p} \ \mathbf{q} \ \mathbf{r}] = \begin{bmatrix} \frac{g_0 p_0 - g_1 p_1 - g_2 p_2 - g_3 p_3}{2} & \frac{g_0 q_0 - g_1 q_1 - g_2 q_2 - g_3 q_3}{2} & \frac{g_0 r_0 - g_1 r_1 - g_2 r_2 - g_3 r_3}{2} \\ g_0 p_1 & g_0 q_1 & g_0 r_1 \\ g_0 p_2 & g_0 q_2 & g_0 r_2 \\ g_0 p_3 & g_0 q_3 & g_0 r_3 \end{bmatrix}.$$

Lemma 2.5. The columns of $N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$ are three linearly independent syzygies of $\mathbf{h}$ of bidegree (m, v_i) for $i = 1, 2, 3$ over the ring $\mathbb{R}[s, u; t, v]$, and $\{N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]\} = f_0^2 \mathbf{h}$.

Similarly, the columns of $N_g[\mathbf{p} \ \mathbf{q} \ \mathbf{r}]$ are three linearly independent syzygies of $\mathbf{h}$ of bidegree (μ_i, n) for $i = 1, 2, 3$ over the ring $\mathbb{R}[s, u; t, v]$, and $\{N_g[\mathbf{p} \ \mathbf{q} \ \mathbf{r}]\} = g_0^2 \mathbf{h}$.

Proof. To see that the columns of $N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$ are three linearly independent syzygies of $\mathbf{h}$ over the ring $\mathbb{R}[s, u; t, v]$, observe that since $\det(N_f) = \frac{f_0^4}{2} \neq 0$,

$$N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow [\mathbf{P} \ \mathbf{Q} \ \mathbf{R}] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ are a μ -basis for the curve $\mathbf{g}$, we must have $a = b = c = 0$. Therefore, the columns of $N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$ are three linearly independent syzygies of $\mathbf{h}$ over the ring $\mathbb{R}[s, u; t, v]$. Moreover, since $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ is a μ -basis for $\mathbf{g}(t, v)$,

$$\begin{aligned} \{N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]\} &= \left\{ \begin{array}{ccc} \frac{f_0 p_0 - f_1 p_1 - f_2 p_2 - f_3 p_3}{2} & \frac{f_0 q_0 - f_1 q_1 - f_2 q_2 - f_3 q_3}{2} & \frac{f_0 r_0 - f_1 r_1 - f_2 r_2 - f_3 r_3}{2} \\ f_0 p_1 & f_0 q_1 & f_0 r_1 \\ f_0 p_2 & f_0 q_2 & f_0 r_2 \\ f_0 p_3 & f_0 q_3 & f_0 r_3 \end{array} \right\} \\ &= f_0^2 \left[f_0 g_0 \quad \frac{f_0 g_1 + g_0 f_1}{2} \quad \frac{f_0 g_2 + g_0 f_2}{2} \quad \frac{f_0 g_3 + g_0 f_3}{2} \right]^T \\ &\equiv f_0^2 \left[2 f_0 g_0 \quad f_0 g_1 + g_0 f_1 \quad f_0 g_2 + g_0 f_2 \quad f_0 g_3 + g_0 f_3 \right]^T = f_0^2 [h_0 \ h_1 \ h_2 \ h_3] \\ &= f_0^2 \mathbf{h}. \end{aligned}$$

A similar argument proves the claim for $N_g[\mathbf{p} \ \mathbf{q} \ \mathbf{r}]$. $\square$

Lemma 2.5 shows that the outer product of the columns of $N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$ is a polynomial multiple of the translational surface $\mathbf{h}$. This observation suggests that the columns of $N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$ are very special syzygies. Therefore we would like to extract information from $N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$ to construct a μ -basis for the translational surface $\mathbf{h}$. We can do so using the following theorem.

Theorem 2.6. (Primitive Factorization Theorem (Bose, 1995)) Suppose that A is an $\alpha \times \beta$ matrix ($\alpha \leq \beta$) with entries in a polynomial ring $E[X]$ in the variable X , where E is a Euclidean domain. Let $d(X), d \in E$ denote the greatest common divisor of the α -th order minors of A . Then A can be factored as the product $A = LB$ where L is an $\alpha \times \alpha$ matrix, B is an $\alpha \times \beta$ matrix, the entries of L and B are in $E[X]$, and $\det(L) = d(X)$. Furthermore, the Primitive Factorization Algorithm in Deng et al. (2005) provides a means for finding these factorizations.

Setting $u = v = 1$, the Primitive Factorization Theorem with $\alpha = 3$ implies that

$$(N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}])^T = M \begin{bmatrix} \tilde{\mathbf{P}}^T \\ \tilde{\mathbf{Q}}^T \\ \tilde{\mathbf{R}}^T \end{bmatrix}, \text{ where } \det M = f_0^2(s) \text{ and } \tilde{\mathbf{P}}, \tilde{\mathbf{Q}}, \tilde{\mathbf{R}} \text{ form a } \mu\text{-basis for } \mathbf{h}(s; t).$$

To see that the column vectors $\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}, \tilde{\mathbf{R}}$ form a μ -basis for $\mathbf{h}(s; t)$, recall a generalization of the Cauchy–Binet formula (Broida and Williamson, 1989), a statement about the minors of a product of

two matrices: Suppose that A is an $\alpha \times \beta$ matrix, B is an $\beta \times \rho$ matrix, I is a subset of $\{1, \dots, \alpha\}$ with k elements, and J is a subset of $\{1, \dots, \rho\}$ with k elements. Let $[A]_{I,J}$ be the minor of A associated to the ordered sequences of indexes I and J . Then the Cauchy–Binet formula is

$$[AB]_{I,J} = \sum_K [A]_{I,K} [B]_{K,J},$$

where the sum extends over all subsets K of $\{1, \dots, \beta\}$ with k elements.

To apply the Cauchy–Binet formula to our situation, let $A = [\tilde{\mathbf{P}}(s; t) \quad \tilde{\mathbf{Q}}(s; t) \quad \tilde{\mathbf{R}}(s; t)]$, $B = M^T(s)$, I a subset of $\{1, 2, 3, 4\}$ with three elements, and $K = J = \{1, 2, 3\}$. Then by the Primitive Factorization Theorem

$$\begin{aligned} f_0^2(s) \mathbf{h}(s; t) &= \{N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]\} = \{[\tilde{\mathbf{P}}(s; t) \quad \tilde{\mathbf{Q}}(s; t) \quad \tilde{\mathbf{R}}(s; t)] M^T(s)\} \\ &= \det(M) \{ \tilde{\mathbf{P}}(s; t) \quad \tilde{\mathbf{Q}}(s; t) \quad \tilde{\mathbf{R}}(s; t) \}. \end{aligned}$$

Since $\det(M) = f_0^2(s) \neq 0$, it follows that $\mathbf{h}(s; t) = \{ \tilde{\mathbf{P}}(s; t) \quad \tilde{\mathbf{Q}}(s; t) \quad \tilde{\mathbf{R}}(s; t) \}$. Thus, $\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}, \tilde{\mathbf{R}}$ form a μ -basis for $\mathbf{h}(s; t)$. Similarly,

$$(N_g[\mathbf{p} \ \mathbf{q} \ \mathbf{r}])^T = N \begin{bmatrix} \tilde{\mathbf{p}}^T \\ \tilde{\mathbf{q}}^T \\ \tilde{\mathbf{r}}^T \end{bmatrix}, \text{ where } \det N = g_0^2(t) \text{ and } \tilde{\mathbf{p}}, \tilde{\mathbf{q}}, \tilde{\mathbf{r}} \text{ form a } \mu\text{-basis for } \mathbf{h}(s; t).$$

3. Implicit equations of translational surfaces

In this section, we shall retrieve the implicit equation of a translational surface $\mathbf{h}(s, u; t, v) = f_0(s, u)\mathbf{g}(t, v) + g_0(t, v)\mathbf{f}(s, u)$ from the resultant of the three moving planes formed from the columns of $N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$. These techniques and results can also be applied to the columns of $N_g[\mathbf{p} \ \mathbf{q} \ \mathbf{r}]$.

To begin, recall that

$$N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}] = \begin{bmatrix} \frac{f_0 P_0 - f_1 P_1 - f_2 P_2 - f_3 P_3}{f_0^2 P_1} & \frac{f_0 Q_0 - f_1 Q_1 - f_2 Q_2 - f_3 Q_3}{f_0^2 Q_1} & \frac{f_0 R_0 - f_1 R_1 - f_2 R_2 - f_3 R_3}{f_0^2 R_1} \\ f_0 P_2 & f_0 Q_2 & f_0 R_2 \\ f_0 P_3 & f_0 Q_3 & f_0 R_3 \end{bmatrix}.$$

Thus, if $f_0(s_0, u_0) = 0$, then

$$N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)} = \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ where } a, b, c \in \mathbb{R}. \quad (4)$$

We shall retrieve the implicit equation $F(\mathbf{x}) \equiv F(w, x, y, z) = 0$, of the parametrized surface $\mathbf{h}(s, u; t, v)$ from the three moving planes formed from the columns of $N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$. We will consider two cases:

$$\text{Res}(N_f \mathbf{P} \cdot \mathbf{x}, N_f \mathbf{Q} \cdot \mathbf{x}, N_f \mathbf{R} \cdot \mathbf{x}) \neq 0, \text{ and } \text{Res}(N_f \mathbf{P} \cdot \mathbf{x}, N_f \mathbf{Q} \cdot \mathbf{x}, N_f \mathbf{R} \cdot \mathbf{x}) \equiv 0.$$

First, we shall characterize these two conditions in terms of the base points of the translational surface using the curves $\mathbf{f}(s, u)$ and $\mathbf{g}(t, v)$.

Lemma 3.1. *Let $(s_0, u_0; t_0, v_0)$ be a base point of $\mathbf{h}(s, u; t, v)$. Then*

1. $\text{rank}(N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)}) \leq 1$.
2. $\text{rank}(N_g[\mathbf{p} \ \mathbf{q} \ \mathbf{r}]_{(s_0, u_0; t_0, v_0)}) \leq 1$.

Thus none of the base points of $\mathbf{h}(s, u; t, v)$ are local complete intersections.

Proof. Statement 1 follows immediately from Equation (4) and Statement 2 follows from a similar analysis. Furthermore, by Lemma 2.5, the columns of $N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$ and $N_g[\mathbf{p} \ \mathbf{q} \ \mathbf{r}]$ are three linearly independent syzygies. Hence the conclusion that none of the base points of $\mathbf{h}(s, u; t, v)$ are local complete intersections follows immediately by Shen and Goldman (2017a), Lemma 2.1 or by Chen et al. (2005), Lemma 3.2. $\square$

Proposition 3.2. *The following conditions are equivalent:*

1. $\text{Res}(N_f \mathbf{P} \cdot \mathbf{x}, N_f \mathbf{Q} \cdot \mathbf{x}, N_f \mathbf{R} \cdot \mathbf{x}) \equiv 0$.
2. $\text{rank}(N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)}) = 0$ for at least one base point $(s_0, u_0; t_0, v_0)$ of $\mathbf{h}$.
3. $\mathbf{f}(s_0, u_0) = \lambda \mathbf{g}(t_0, v_0)$ for some $\lambda \in \mathbb{R}$ for at least one base point $(s_0, u_0; t_0, v_0)$ of $\mathbf{h}$.
4. $\text{Res}(N_g \mathbf{p} \cdot \mathbf{x}, N_g \mathbf{q} \cdot \mathbf{x}, N_g \mathbf{r} \cdot \mathbf{x}) \equiv 0$.
5. $\text{rank}(N_g[\mathbf{p} \ \mathbf{q} \ \mathbf{r}]_{(s_0, u_0; t_0, v_0)}) = 0$ for at least one base point $(s_0, u_0; t_0, v_0)$ of $\mathbf{h}$.

Proof. (1. $\Leftrightarrow$ 2.) It is clear that if there is a base point $(s_0, u_0; t_0, v_0)$ such that $\text{rank}(N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)}) = 0$, then $\text{Res}(N_f \mathbf{P} \cdot \mathbf{x}, N_f \mathbf{Q} \cdot \mathbf{x}, N_f \mathbf{R} \cdot \mathbf{x}) \equiv 0$. Conversely, suppose that $\text{rank}(N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)}) \neq 0$ for all the base points $(s_0, u_0; t_0, v_0)$ of $\mathbf{h}$. Then by Lemma 3.1, $\text{rank}(N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)}) = 1$. Therefore, the solution set of the system of equations

$$N_f \mathbf{P} \cdot \mathbf{x} = 0, \quad N_f \mathbf{Q} \cdot \mathbf{x} = 0, \quad N_f \mathbf{R} \cdot \mathbf{x} = 0,$$

is of codimension one. Hence, $\text{Res}(N_f \mathbf{P} \cdot \mathbf{x}, N_f \mathbf{Q} \cdot \mathbf{x}, N_f \mathbf{R} \cdot \mathbf{x}) \neq 0$.

(2. $\Leftrightarrow$ 3.) Recall from Lemma 2.1 that the base points of $\mathbf{h}(s, u; t, v)$ are exactly the points $(s_0, u_0; t_0, v_0)$ satisfying $f_0(s_0, u_0) = 0$ and $g_0(t_0, v_0) = 0$. Since $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ is a μ -basis of the curve $\mathbf{g}(t, v)$, Equation (4) implies that $\text{rank}(N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)}) = 0$ for at least one base point $(s_0, u_0; t_0, v_0)$ of $\mathbf{h}$ if and only if

$$\begin{aligned} & -f_0(s_0, u_0)P_0(t_0, v_0) - f_1(s_0, u_0)P_1(t_0, v_0) - f_2(s_0, u_0)P_2(t_0, v_0) \\ & - f_3(s_0, u_0)P_3(t_0, v_0) = 0 \\ & -f_0(s_0, u_0)Q_0(t_0, v_0) - f_1(s_0, u_0)Q_1(t_0, v_0) - f_2(s_0, u_0)Q_2(t_0, v_0) \\ & - f_3(s_0, u_0)Q_3(t_0, v_0) = 0 \\ & -f_0(s_0, u_0)R_0(t_0, v_0) - f_1(s_0, u_0)R_1(t_0, v_0) - f_2(s_0, u_0)R_2(t_0, v_0) \\ & - f_3(s_0, u_0)R_3(t_0, v_0) = 0, \end{aligned}$$

which is equivalent to $\mathbf{f}(s_0, u_0) = \lambda \mathbf{g}(t_0, v_0)$ for some $\lambda \in \mathbb{R}$ for at least one base point $(s_0, u_0; t_0, v_0)$ of $\mathbf{h}$.

A similar argument shows 4. $\Leftrightarrow$ 5. $\Leftrightarrow$ 3.. Therefore all five conditions are equivalent. $\square$

The following corollary is a direct consequence of Proposition 3.2.

Corollary 3.3. *The following conditions are equivalent:*

1. $\text{Res}(N_f \mathbf{P} \cdot \mathbf{x}, N_f \mathbf{Q} \cdot \mathbf{x}, N_f \mathbf{R} \cdot \mathbf{x}) \neq 0$.
2. $\text{rank}(N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)}) = 1$ for all base points $(s_0, u_0; t_0, v_0)$ of $\mathbf{h}$.
3. $\mathbf{f}(s_0, u_0) \neq \lambda \mathbf{g}(t_0, v_0)$ for any $\lambda \in \mathbb{R}$ for all base point $(s_0, u_0; t_0, v_0)$ of $\mathbf{h}$.
4. $\text{Res}(N_g \mathbf{p} \cdot \mathbf{x}, N_g \mathbf{q} \cdot \mathbf{x}, N_g \mathbf{r} \cdot \mathbf{x}) \neq 0$.
5. $\text{rank}(N_g[\mathbf{p} \ \mathbf{q} \ \mathbf{r}]_{(s_0, u_0; t_0, v_0)}) = 1$ for all base points $(s_0, u_0; t_0, v_0)$ of $\mathbf{h}$.

Proof. (1. $\Leftrightarrow$ 2.) By Proposition 3.2, $\text{Res}(N_f \mathbf{P} \cdot \mathbf{x}, N_f \mathbf{Q} \cdot \mathbf{x}, N_f \mathbf{R} \cdot \mathbf{x}) \neq 0$ if and only if $\text{rank}(N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)}) \neq 0$ for all base points $(s_0, u_0; t_0, v_0)$ of $\mathbf{h}(s, u; t, v)$. But

rank($N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)}$) $\neq 0$ if and only if rank($N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)}$) = 1, since by Lemma 3.1, rank($N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)}$) ≤ 1 at all base point $(s_0, u_0; t_0, v_0)$.

(2. $\Leftrightarrow$ 3.) This result follows directly from the equivalence 2. $\Leftrightarrow$ 3. in Proposition 3.2.

A similar argument shows 4. $\Leftrightarrow$ 5. $\Leftrightarrow$ 3. Therefore all five conditions are equivalent. $\square$

By Proposition 3.2 and Corollary 3.3, we can now discuss how to find the implicit equation of a translational surface based simply on the types of the base points $(s_0, u_0; t_0, v_0)$: either $\mathbf{f}(s_0, u_0) \neq \lambda \mathbf{g}(t_0, v_0)$ or $\mathbf{f}(s_0, u_0) = \lambda \mathbf{g}(t_0, v_0)$ – that is, either the curves $\mathbf{f}$ and $\mathbf{g}$ intersect at a base point, introducing a bad base point, or the curves $\mathbf{f}$ and $\mathbf{g}$ do not intersect at any base point. Moreover, an implicitization algorithm will work using the syzygies $N_f\mathbf{P}$, $N_f\mathbf{Q}$, $N_f\mathbf{R}$ if and only if the same implicitization algorithm will work using the syzygies $N_g\mathbf{p}$, $N_g\mathbf{q}$, $N_g\mathbf{r}$. Thus, from now on, we shall focus only on the syzygies $N_f\mathbf{P}$, $N_f\mathbf{Q}$, $N_f\mathbf{R}$.

Next we discuss the case when all the base points $(s_0, u_0; t_0, v_0)$ satisfy $\mathbf{f}(s_0, u_0) \neq \lambda \mathbf{g}(t_0, v_0)$.

Theorem 3.4. Suppose that the parametrization of the translational surface $\mathbf{h}(s, u; t, v)$ is a generic 1-1 parametrization, and that one of the conditions in Corollary 3.3 is satisfied. Then

$$\text{Res}(N_f\mathbf{P} \cdot \mathbf{x}, N_f\mathbf{Q} \cdot \mathbf{x}, N_f\mathbf{R} \cdot \mathbf{x}) = w^\tau F(\mathbf{x}), \quad \tau \in \mathbb{Z}, \quad \tau > 0.$$

Proof. Since at least one of the conditions in Corollary 3.3 is satisfied, it follows by Corollary 3.3 that $\text{Res}(N_f\mathbf{P} \cdot \mathbf{x}, N_f\mathbf{Q} \cdot \mathbf{x}, N_f\mathbf{R} \cdot \mathbf{x}) \neq 0$.

Now consider the system of equations

$$N_f\mathbf{P} \cdot \mathbf{x} = 0, \quad N_f\mathbf{Q} \cdot \mathbf{x} = 0, \quad N_f\mathbf{R} \cdot \mathbf{x} = 0. \quad (5)$$

Since $N_f\mathbf{P}$, $N_f\mathbf{Q}$, $N_f\mathbf{R}$ are syzygies of $\mathbf{h}$, the system (5) has a common root at all points $\mathbf{x}$ on the surface $\mathbf{h}(s, u; t, v)$. Hence $\text{Res}(N_f\mathbf{P} \cdot \mathbf{x}, N_f\mathbf{Q} \cdot \mathbf{x}, N_f\mathbf{R} \cdot \mathbf{x}) = 0$ for every point $\mathbf{x}$ on the surface $\mathbf{h}(s, u; t, v)$. Therefore $F(\mathbf{x})$ is a factor of $\text{Res}(N_f\mathbf{P} \cdot \mathbf{x}, N_f\mathbf{Q} \cdot \mathbf{x}, N_f\mathbf{R} \cdot \mathbf{x})$.

Moreover, since $f_0(s_0, u_0) = 0$ at all the bases points, it follows by Equation (4) that

$$N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]_{(s_0, u_0; t_0, v_0)} = \begin{bmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{where } a, b, c \in \mathbb{R}, \text{ not all zero.}$$

Thus for each base point the system of equations (5) has a common root at the points $\mathbf{x}_0 = \{(w, x, y, z) \mid w = 0\}$. Therefore, w^τ is a factor of $\text{Res}(N_f\mathbf{P} \cdot \mathbf{x}, N_f\mathbf{Q} \cdot \mathbf{x}, N_f\mathbf{R} \cdot \mathbf{x})$ for some power τ .

It remains to show that there are no other factors of $\text{Res}(N_f\mathbf{P} \cdot \mathbf{x}, N_f\mathbf{Q} \cdot \mathbf{x}, N_f\mathbf{R} \cdot \mathbf{x})$. Recall from Lemma 2.5 that $\{N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]\} = f_0^2(s, u) \mathbf{h}(s, u; t, v)$. Now if $f_0(s, u) \neq 0$, then by Lemma 2.1

$$\mathbf{h}(s, u; t, v) = f_0(s, u) \mathbf{g}(t, v) + g_0(t, v) \mathbf{f}(s, u) \neq \mathbf{0}.$$

Therefore when $f_0(s, u) \neq 0$, the syzygies $N_f\mathbf{P}$, $N_f\mathbf{Q}$, $N_f\mathbf{R}$ are linearly independent at (s, u) . Hence when $f_0(s, u) \neq 0$, the only common roots of (5) are multiples of $\mathbf{x} = \mathbf{h}(s, u; t, v)$, since these vectors are the only vectors simultaneously perpendicular to $N_f\mathbf{P}$, $N_f\mathbf{Q}$, $N_f\mathbf{R}$. But the vectors $\mathbf{x} = \mathbf{h}(s, u; t, v)$ correspond to points on the surface $F(\mathbf{x}) = 0$. Thus there are no other roots of (5) and hence no other factors of $\text{Res}(N_f\mathbf{P} \cdot \mathbf{x}, N_f\mathbf{Q} \cdot \mathbf{x}, N_f\mathbf{R} \cdot \mathbf{x})$. $\square$

Notice that in Theorem 3.4, we have assumed that the parametrization $\mathbf{h}(s, u; t, v)$ is a generic 1-1 parametrization. If the parametrization is n to 1, then standard arguments show that $F^n(\mathbf{x})$ is a factor of the resultant.

By Lemma 2.5 the columns of $N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$ represent three syzygies of bidegrees (m, v_1) , (m, v_2) and (m, v_3) , so we can construct a specialized resultant for the three bivariate polynomials

$$\rho(s; t) = N_f\mathbf{P}(s, 1; t, 1) \cdot \mathbf{x}, \quad \eta(s; t) = N_f\mathbf{Q}(s, 1; t, 1) \cdot \mathbf{x}, \quad \omega(s; t) = N_f\mathbf{R}(s, 1; t, 1) \cdot \mathbf{x}$$

of bidegrees (m, v_1) , (m, v_2) and (m, v_3) . Consider the coefficient matrix of the polynomials:

$$\rho, s\rho, t\rho, \dots, s^{m-1}t^{v_2+v_3-1}\rho, \eta, s\eta, t\eta, \dots, s^{m-1}t^{v_1+v_3-1}\eta, \omega, s\omega, t\omega, \dots, s^{m-1}t^{v_1+v_2-1}\omega.$$

Let $n = v_1 + v_2 + v_3$. Then this coefficient matrix has $2m(v_1 + v_2 + v_3) = 2mn$ rows indexed by these polynomials, and $2mn$ columns indexed by the monomials $1, s, \dots, s^{2m-1}, t, \dots, s^{2m-1}t, \dots, s^{2m-1}t^{n-1}$. Thus this matrix is a square matrix. The resultant of the three bivariate polynomials ρ, η, ω is precisely the determinant of this coefficient matrix (Shi et al., 2013). When $v_1 = v_2 = v_3 = v$, this resultant reduces to the Dixon resultant for three bivariate polynomials of bidegree (m, v) (Dixon, 1908).

Corollary 3.5. Suppose that the parametrization of the translational surface $\mathbf{h}(s, u; t, v)$ is a generic one-to-one parametrization, and that one of the conditions in Corollary 3.3 is satisfied. Then

1. $\text{Res}(N_{\mathbf{f}}\mathbf{P} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{Q} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{R} \cdot \mathbf{x}) = w^{mn}F(\mathbf{x})$.
2. $\deg(F(\mathbf{x})) = mn$.

Proof. By construction $\text{Res}(N_{\mathbf{f}}\mathbf{P} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{Q} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{R} \cdot \mathbf{x})$ is a matrix of size $2mn \times 2mn$ whose entries are linear in w, x, y, z ; hence $\deg(\text{Res}(N_{\mathbf{f}}\mathbf{P} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{Q} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{R} \cdot \mathbf{x})) = 2mn$. Moreover, by the proof of Theorem 3.4, there is exactly one factor of w in $\text{Res}(N_{\mathbf{f}}\mathbf{P} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{Q} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{R} \cdot \mathbf{x})$ for each base point counting multiplicity. Since there are exactly mn base points counting multiplicity, there are exactly mn factors of w . Therefore by Theorem 3.4, $\text{Res}(N_{\mathbf{f}}\mathbf{P} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{Q} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{R} \cdot \mathbf{x}) = w^{mn}F(\mathbf{x})$. Since $\deg(\text{Res}(N_{\mathbf{f}}\mathbf{P} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{Q} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{R} \cdot \mathbf{x})) = 2mn$, it follows that $\deg(F(\mathbf{x})) = mn$. $\square$

Notice that by Corollary 3.5, if $\text{Res}(N_{\mathbf{f}}\mathbf{P} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{Q} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{R} \cdot \mathbf{x}) \neq 0$, then we can find the implicit equation of a translational surface in affine space simply by dehomogenizing $\text{Res}(N_{\mathbf{f}}\mathbf{P} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{Q} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{R} \cdot \mathbf{x})$. For arbitrary rational tensor product surfaces, base points that are not local complete intersections generate linear extraneous factors in the resultant (Shen and Goldman, 2017b). Translational surfaces are very special; the only extraneous factors in the resultant generated by the base points are powers of w . The following example illustrates Corollary 3.5, and the construction of the specialized resultant for three bivariate polynomials of bidegrees (m, v_1) , (m, v_2) and (m, v_3) .

Example 3.6. Consider the translational surface $\mathbf{h} = f_0\mathbf{g} + g_0\mathbf{f}$, where

$$\mathbf{f}(s, u) = (s^2, u^2, su, su), \quad \mathbf{g}(t, v) = (t^3, t^2v, tv^2, v^3),$$

$$\mathbf{h}(s, u; t, v) = (2s^2t^3, s^2t^2v + t^3u^2, s^2tv^2 + t^3su, s^2v^3 + t^3su).$$

Thus $m = \deg(\mathbf{f}) = 2$ and $n = \deg(\mathbf{g}) = 3$. It is easy to see that the point $p = (s, u; t, v) = (0, 1; 0, 1)$ is the only base point of the surface $\mathbf{h}(s, u; t, v)$, and $\mathbf{f}(0, 1) = (0, 1, 0, 0) \neq (0, 0, 0, 1) = \mathbf{g}(0, 1)$. By Corollary 3.5, the degree of the implicit equation corresponding to the translational surface $\mathbf{h}$ should be $mn = 6$. A computation via Macaulay 2 (<http://www.math.uiuc.edu/Macaulay2/>) shows that the total multiplicity of this base point is 6. We can verify that a μ -basis for $\mathbf{g}(t, v)$ is

$$[\mathbf{PQR}] = \begin{bmatrix} -v & 0 & 0 \\ t & -v & 0 \\ 0 & t & -v \\ 0 & 0 & t \end{bmatrix}, \quad \text{so} \quad N_{\mathbf{f}}[\mathbf{PQR}] = \begin{bmatrix} \frac{-s^2v-tu^2}{2} & \frac{vu^2-sut}{2} & \frac{-tsu+svv}{2} \\ s^2t & -s^2v & 0 \\ 0 & s^2t & -s^2v \\ 0 & 0 & s^2t \end{bmatrix}.$$

Let $u = v = 1$. Since the syzygies $N_{\mathbf{f}}\mathbf{P}$, $N_{\mathbf{f}}\mathbf{Q}$, $N_{\mathbf{f}}\mathbf{R}$ are each of bidegree $(2, 1)$, it follows that $\text{Res}(N_{\mathbf{f}}\mathbf{P} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{Q} \cdot \mathbf{x}, N_{\mathbf{f}}\mathbf{R} \cdot \mathbf{x})$ is the determinant of the 12×12 coefficient matrix constructed by multiplying the moving planes $N_{\mathbf{f}}\mathbf{P} \cdot \mathbf{x}$, $N_{\mathbf{f}}\mathbf{Q} \cdot \mathbf{x}$, $N_{\mathbf{f}}\mathbf{R} \cdot \mathbf{x}$ by the monomials $1, s, t, st$. Thus

$$\text{Res}(N_f \mathbf{P} \cdot \mathbf{x}, N_f \mathbf{Q} \cdot \mathbf{x}, N_f \mathbf{R} \cdot \mathbf{x})$$

$$= \det \begin{bmatrix} 0 & 0 & -\frac{w}{2} & 0 & -\frac{w}{2} & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{w}{2} & 0 & -\frac{w}{2} & 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{w}{2} & 0 & -\frac{w}{2} & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{w}{2} & 0 & -\frac{w}{2} & 0 & x \\ \frac{w}{2} & 0 & -x & 0 & 0 & -\frac{w}{2} & y & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{w}{2} & 0 & -x & 0 & 0 & -\frac{w}{2} & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{w}{2} & 0 & -x & 0 & 0 & -\frac{w}{2} & y & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{w}{2} & 0 & -x & 0 & 0 & -\frac{w}{2} & y \\ 0 & \frac{w}{2} & -y & 0 & 0 & -\frac{w}{2} & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{w}{2} & -y & 0 & 0 & -\frac{w}{2} & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{w}{2} & -y & 0 & 0 & -\frac{w}{2} & z & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{w}{2} & -y & 0 & 0 & -\frac{w}{2} & z \end{bmatrix}$$

$$= \frac{w^6}{1026} F(\mathbf{x}),$$

where

$$\begin{aligned} F(\mathbf{x}) = & 16y^6 - 24xy^4w - 16y^3z^2w + 12x^2y^2w^2 - 8xy^3w^2 - 20y^4w^2 \\ & + 48xy^2zw^2 + 24y^3zw^2 - 24xyz^2w^2 + 4z^4w^2 - 2x^3w^3 - 12x^2yw^3 + 8xy^2w^3 \\ & + 8x^2zw^3 - 12xyzw^3 + 8y^2zw^3 - 4xz^2w^3 - 14yz^2w^3 + 6z^3w^3 + 2x^2w^4 - xyw^4 \\ & + 5y^2w^4 + xzw^4 - 8yzw^4 + 3z^2w^4 - yw^5 + zw^5. \end{aligned}$$

The extraneous factor w^6 is induced by the base point $(s, u; t, v) = (0, 1; 0, 1)$, and $F(\mathbf{x}) = 0$ is the implicit equation of the translational surface $\mathbf{h}$. Notice that, as expected, $\deg(F(\mathbf{x})) = 6$.

Recall from Corollary 2.2 that translational surfaces of bidegree (m, n) have at least mn base points; therefore, their implicit degree is at most mn . However, the resultant matrix constructed above is of size $2mn \times 2mn$, so it is natural to seek an alternative method to generate a smaller size matrix whose determinant is the implicit equation. To construct a smaller matrix, we would have to compute syzygies of the surface not necessarily based on syzygies of the generating curves, which would require solving a system of linear equations, while computing syzygies of the curves can be done by Gaussian elimination without solving a system of linear equations. The goal of this paper is not necessarily to find the smallest matrix that gives the implicit equation, but rather to show how to use syzygies of the generating curves to find the implicit equation in a simple way.

Moreover, to find the resultant using syzygies of the curves, we use the Sylvester dialytic approach to build the resultant. Often where there is a Sylvester construction, there is a corresponding Bézoutian construction. When the bidegrees of the syzygies are all the same, such a Bézoutian always exists. But in our case the syzygies we construct may have different bidegree, so it is not clear how to construct a smaller Bézoutian corresponding to the Sylvester resultant.

Next, we discuss the case where one of the base points $(s_0, u_0; t_0, v_0)$ satisfies $\mathbf{f}(s_0, u_0) = \lambda \mathbf{g}(t_0, v_0)$ – that is, the generating curves $\mathbf{f}, \mathbf{g}$ intersect at a base point of the translational surface $\mathbf{h}$. (This case corresponds to a base point that blows up to a space curve lying on the surface; for further details and examples, see Shen and Goldman, 2017b.) In this case, we cannot use Corollary 3.5 to find the implicit equation of the translational surface $\mathbf{h}$, since by Proposition 3.2, $\text{Res}(N_f \mathbf{P} \cdot \mathbf{x}, N_f \mathbf{Q} \cdot \mathbf{x}, N_f \mathbf{R} \cdot \mathbf{x}) \equiv 0$. Rather in this case there are often two other simple, direct methods for finding the implicit equation (Shen and Goldman, 2017b): perturbing the syzygies or computing the GCD of the maximal minors of the resultant matrix generated by the columns of $N_f[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$. Next, we provide an example to illustrate these two methods.

Example 3.7. Consider the translational surface $\mathbf{h} = f_0\mathbf{g} + g_0\mathbf{f}$, where

$$\mathbf{f}(s, u) = (s^3, 0, s^2u, u^3), \quad \mathbf{g}(t, v) = (t^2, tv, 0, -v^2),$$

$$\mathbf{h}(s, u; t, v) = (2s^3t^2, s^3tv, t^2s^2u, -s^3v^2 + u^3t^2).$$

It is easy to see that the point $p = (s, u; t, v) = (0, 1; 0, 1)$ is the only base point of the surface $\mathbf{h}(s, u; t, v)$. Note that $\mathbf{f}(0, 1) = (0, 0, 0, 1)$, $\mathbf{g}(0, 1) = (0, 0, 0, -1)$, so $\mathbf{f}(0, 1) = -\mathbf{g}(0, 1)$. A computation via Macaulay 2 (<http://www.math.uiuc.edu/Macaulay2/>) shows that the total multiplicity of this base point is 9. We can verify that a μ -basis for $\mathbf{g}(t, v)$ is

$$[\mathbf{PQR}] = \begin{bmatrix} -v & 0 & 0 \\ t & v & 0 \\ 0 & 0 & 1 \\ 0 & t & 0 \end{bmatrix}, \quad \text{so } N_{\mathbf{f}}[\mathbf{PQR}] = \begin{bmatrix} \frac{-s^3v}{2} & \frac{-tu^3}{2} & \frac{-s^2u}{2} \\ s^3t & s^3v & 0 \\ 0 & 0 & s^3 \\ 0 & s^3t & 0 \end{bmatrix}.$$

To apply a perturbation, we first construct a 12×12 coefficient matrix M with respect to the polynomials $N_{\mathbf{f}}\mathbf{P} \cdot \mathbf{x}$, $N_{\mathbf{f}}\mathbf{Q} \cdot \mathbf{x}$, $N_{\mathbf{f}}\mathbf{R} \cdot \mathbf{x}$, where

$$M = \begin{bmatrix} 0 & 0 & 0 & -\frac{w}{2} & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{w}{2} & 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{w}{2} & 0 & 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & x & 0 & 0 & -\frac{w}{2} & 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & 0 & x & 0 & 0 & -\frac{w}{2} & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & -\frac{w}{2} & 0 & 0 & z \\ 0 & 0 & -\frac{w}{2} & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{w}{2} & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{w}{2} & y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{w}{2} & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{w}{2} & y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{w}{2} & y \end{bmatrix}.$$

Next, we compute a perturbation $\det(M - \lambda I)$, a non-zero polynomial in λ whose coefficients are polynomials in w, x, y, z . Taking the coefficient of the term with lowest degree with respect to λ yields $w^7(2wx^2 - 4y^3 + w^2z)$, where w^7 is an extraneous factor, and the implicit equation of this translational surface $\mathbf{h}(s, u; t, v)$ is

$$F(\mathbf{x}) = F(w, x, y, z) = 2wx^2 - 4y^3 + w^2z.$$

Alternatively, we check that $\text{rank} M = 10$, and the GCD of the maximal minors is $w^5(2wx^2 - 4y^3 + w^2z)$. Here, w^5 is the extraneous factor, and the implicit equation of this translational surface $\mathbf{h}(s, u; t, v)$ is

$$F(\mathbf{x}) = F(w, x, y, z) = 2wx^2 - 4y^3 + w^2z.$$

Since one of the goals of this paper is to show how to use the syzygies of the generating curves to extract information about the translational surface, we will provide an alternative method to find the implicit equation of translational surfaces by extracting a μ -basis from the syzygies $N_{\mathbf{f}}[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}]$.

To do so, we shall apply the Primitive Factorization Algorithm in Deng et al. (2005) to factor

$$(N_{\mathbf{f}}[\mathbf{P} \ \mathbf{Q} \ \mathbf{R}])^T = M \begin{bmatrix} \tilde{\mathbf{P}}^T \\ \tilde{\mathbf{Q}}^T \\ \tilde{\mathbf{R}}^T \end{bmatrix}.$$

Let $u = v = 1$; then as we showed in Section 2.2 $\tilde{\mathbf{P}}(s; t)$, $\tilde{\mathbf{Q}}(s; t)$, $\tilde{\mathbf{R}}(s; t)$ is a μ -basis for $\mathbf{h}(s; t)$. Let $\tilde{\mathbf{P}}'$, $\tilde{\mathbf{Q}}'$, $\tilde{\mathbf{R}}'$ be the homogenizations of $\tilde{\mathbf{P}}(s; t)$, $\tilde{\mathbf{Q}}(s; t)$, $\tilde{\mathbf{R}}(s; t)$. Then

$$\{\tilde{\mathbf{P}}', \tilde{\mathbf{Q}}', \tilde{\mathbf{R}}'\} = au^i v^j \mathbf{h}(s, u; t, v), \quad 0 \neq a \in \mathbb{R}, i + j > 0.$$

Next we will retrieve the implicit equation of the parametrized surface $\mathbf{h}(s, u; t, v)$ from the resultant of the three moving planes formed from the μ -basis $\tilde{\mathbf{P}}', \tilde{\mathbf{Q}}', \tilde{\mathbf{R}}'$. Again we consider two cases:

$$\text{Res}(\tilde{\mathbf{P}}' \cdot \mathbf{x}, \tilde{\mathbf{Q}}' \cdot \mathbf{x}, \tilde{\mathbf{R}}' \cdot \mathbf{x}) \neq 0, \text{ and } \text{Res}(\tilde{\mathbf{P}}' \cdot \mathbf{x}, \tilde{\mathbf{Q}}' \cdot \mathbf{x}, \tilde{\mathbf{R}}' \cdot \mathbf{x}) \equiv 0.$$

Theorem 3.8. Suppose that the parametrization of the translational surface $\mathbf{h}(s, u; t, v)$ is a generic 1-1 parametrization. If $\text{Res}(\tilde{\mathbf{P}}' \cdot \mathbf{x}, \tilde{\mathbf{Q}}' \cdot \mathbf{x}, \tilde{\mathbf{R}}' \cdot \mathbf{x}) \neq 0$, then

$$\text{Res}(\tilde{\mathbf{P}}' \cdot \mathbf{x}, \tilde{\mathbf{Q}}' \cdot \mathbf{x}, \tilde{\mathbf{R}}' \cdot \mathbf{x}) = F(\mathbf{x})E(\mathbf{x}),$$

where $F(\mathbf{x}) = 0$ is the implicit equation of the surface $\mathbf{h}$, and $E(\mathbf{x})$ is an extraneous factor. For each base point $(s_0, u_0; t_0, v_0)$ assume without loss of generality that $\tilde{\mathbf{P}}'(s_0, u_0; t_0, v_0) \neq \mathbf{0}$. Then the extraneous factor $E(\mathbf{x})$ contains $\tilde{\mathbf{P}}'(s_0, u_0; t_0, v_0) \cdot \mathbf{x}$ as an extraneous linear factor induced by the base point $(s_0, u_0; t_0, v_0)$, as well as extraneous factors associated to infinity, that is, to points where $uv = 0$. Refer to Theorem 17 (Shen and Goldman, 2017a) to compute the extraneous factors associated to infinity.

Proof. If $\text{Res}(\tilde{\mathbf{P}}' \cdot \mathbf{x}, \tilde{\mathbf{Q}}' \cdot \mathbf{x}, \tilde{\mathbf{R}}' \cdot \mathbf{x}) \neq 0$, then by Corollary 3.3, $\text{rank}[\tilde{\mathbf{P}}', \tilde{\mathbf{Q}}', \tilde{\mathbf{R}}']_{(s_0, u_0; t_0, v_0)} = 1$ for every base point $(s_0, u_0; t_0, v_0)$. Now the claim follows directly from Shen–Goldman (2017a). $\square$

If $\text{Res}(\tilde{\mathbf{P}}' \cdot \mathbf{x}, \tilde{\mathbf{Q}}' \cdot \mathbf{x}, \tilde{\mathbf{R}}' \cdot \mathbf{x}) \equiv 0$, we can no longer apply resultants with our special syzygies or μ -bases for translational surface to compute the implicit equation. Instead we can use the general implicitization algorithm for tensor product surface in Shen and Goldman (2017b), or apply the previously discussed methods such as perturbing the syzygies or computing the GCD of the maximal minors of the resultant matrix. We may also apply MU-BASIS-IMP Algorithm (Chen et al., 2005) to the μ -basis $\tilde{\mathbf{P}}', \tilde{\mathbf{Q}}', \tilde{\mathbf{R}}'$ to compute the implicit equation. Alternatively we can compute the Gröbner basis of the ideal generated by $\tilde{\mathbf{P}}' \cdot \mathbf{x}, \tilde{\mathbf{Q}}' \cdot \mathbf{x}, \tilde{\mathbf{R}}' \cdot \mathbf{x}$.

In general, the bidegrees of the μ -basis $\tilde{\mathbf{P}}', \tilde{\mathbf{Q}}', \tilde{\mathbf{R}}'$ vary from case to case, so the special form of the resultant in Shi et al. (2013) presented before Example 3.6 cannot be used to compute the resultant of $\tilde{\mathbf{P}}' \cdot \mathbf{x}, \tilde{\mathbf{Q}}' \cdot \mathbf{x}, \tilde{\mathbf{R}}' \cdot \mathbf{x}$. Therefore, the construction of a general resultant matrix as in Cox et al. (1998), Dickenstein and Emiris (2003), Gelfand et al. (1994) must be used here. Software such as Maple (2016), Singular (Decker et al., 2015), and Macaulay 2 (<http://www.math.uiuc.edu/Macaulay2/>) have implemented packages to compute multivariate resultants.

Geometrically, the case when the resultant is identically zero is related to the complexity of the base points. We refer readers to the papers by Shen and Goldman (2017a) and (2017b) for a detailed study of implicitization of tensor product surfaces with bad base points using the resultant of three moving planes.

Notice that in Theorem 3.8, the parametrization $\mathbf{h}(s, u; t, v)$ is a generic 1-1 parametrization. If the parametrization is n to 1, then standard arguments show that $F^n(\mathbf{x})$ is a factor of the resultant. We illustrate Theorem 3.8 and the correspondence between the power of the implicit equation and the improneness of the parametrization in the following example.

Example 3.9. Consider the translational surface $\mathbf{h} = f_0 \mathbf{g} + g_0 \mathbf{f}$, where

$$\mathbf{f}(s, u) = (s^4, s^3 u, s^2 u^2, u^4), \quad \mathbf{g}(t, v) = (t^4, t^3 v, t^2 v^2, v^4),$$

$$\mathbf{h}(s, u; t, v) = (2s^4 t^4, s^4 t^3 v + s^3 u t^4, s^4 t^2 v^2 + s^2 u^2 t^4, s^4 v^4 + u^4 t^4).$$

It is easy to see that the point $p = (s, u; t, v) = (0, 1; 0, 1)$ is the only base point of the surface $\mathbf{h}(s, u; t, v)$, and $\mathbf{f}(0, 1) = (0, 0, 0, 1) = \mathbf{g}(0, 1)$. A computation via Macaulay 2 (<http://www.math.uiuc.edu/Macaulay2/>) shows that the total multiplicity of this base point is 24.

We can verify that a μ -basis for $\mathbf{g}(t, v)$ is

$$[\mathbf{PQR}] = \begin{bmatrix} -v & 0 & 0 \\ t & -v & 0 \\ 0 & t & -v^2 \\ 0 & 0 & t^2 \end{bmatrix}, \text{ so } N_f[\mathbf{PQR}] = \begin{bmatrix} \frac{-s^3tu-s^4v}{2} & \frac{-s^2tu^2+s^3uv}{2} & \frac{-t^2u^4+s^2u^2v^2}{2} \\ s^4t & -s^4v & 0 \\ 0 & s^4t & -s^4v^2 \\ 0 & 0 & s^4t^2 \end{bmatrix}.$$

Let $u = v = 1$. Then the Primitive Factorization Algorithm in Deng et al. (2005) gives

$$(N_f[\mathbf{PQR}])^T = \begin{bmatrix} \frac{-s^3t}{2} & s^3 & 0 \\ \frac{(2s-t)s^2}{2} & s^2 & 0 \\ \frac{t(s-t)}{2} & s+t & s^2 \end{bmatrix} \begin{bmatrix} 1 & -s-t & st & 0 \\ -\frac{s}{2} & \frac{t(s-t)}{2} & \frac{st^2}{2} & 0 \\ 1 & 0 & -s^2-t^2 & s^2t^2 \end{bmatrix} = M \begin{bmatrix} \tilde{\mathbf{P}}^T \\ \tilde{\mathbf{Q}}^T \\ \tilde{\mathbf{R}}^T \end{bmatrix}$$

where $\det M = -s^8$, and $\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}, \tilde{\mathbf{R}}$ form a μ -basis for $\mathbf{h}(s; t)$.

If we homogenize this u -basis, then the outer product

$$\begin{pmatrix} (\tilde{\mathbf{P}}')^T \\ (\tilde{\mathbf{Q}}')^T \\ (\tilde{\mathbf{R}}')^T \end{pmatrix} = \begin{pmatrix} \frac{-sv^2+2uv^2}{2} & \frac{stv-t^2u}{2} - sv^2 - tvu & \frac{st^2}{2} + stv & 0 \\ -\frac{sv^2}{2} & \frac{stv-t^2u}{2} & \frac{st^2}{2} & 0 \\ u^2v^2 & 0 & -s^2v^2 - t^2u^2 & s^2t^2 \end{pmatrix} = v^2 \mathbf{h}.$$

Let $\mathbf{x} = (w, x, y, z)$. Then

$$\text{Res}(\tilde{\mathbf{P}}' \cdot \mathbf{x}, \tilde{\mathbf{Q}}' \cdot \mathbf{x}, \tilde{\mathbf{R}}' \cdot \mathbf{x}) = 4w^2(zx^2 - y^3)^2(16zw^3 - 4w^2y^2 - 4wx^2y + x^4)^2.$$

The base point $(s, u; t, v) = (0, 1; 0, 1)$ induces the extraneous factor w , and the infinity point $(t, v) = (1, 0)$ induces the extraneous factor $zx^2 - y^3$. The implicit equation of this surface is $16zw^3 - 4w^2y^2 - 4wx^2y + x^4 = 0$, and the powers of two appear because the parametrization is 2-to-1.

In geometric modeling, the most important examples of translational surfaces are those where the generating curves are in orthogonal planes. Without loss of generality, we may assume that the curve $\mathbf{f}(s, u)$ is in the yz -plane, and the curve $\mathbf{g}(t, v)$ is in the xz -plane, that is

$$\mathbf{f}(s, u) = (f_0(s, u), 0, f_2(s, u), f_3(s, u)), \quad \mathbf{g}(t, v) = (g_0(t, v), g_1(t, v), 0, g_3(t, v)).$$

Now a μ -basis for the curve $\mathbf{g}(t, v)$ will have the following form $[\mathbf{PQR}] = \begin{bmatrix} P_0 & Q_0 & 0 \\ P_1 & Q_1 & 0 \\ 0 & 0 & 1 \\ P_3 & Q_3 & 0 \end{bmatrix}$. Hence,

$$N_f[\mathbf{PQR}] = \begin{bmatrix} \frac{f_0P_0-f_3P_3}{2} & \frac{f_0Q_0-f_3Q_3}{2} & \frac{-f_2}{2} \\ f_0P_1 & f_0Q_1 & 0 \\ 0 & 0 & f_0 \\ f_0P_3 & f_0Q_3 & 0 \end{bmatrix}.$$

Notice that if $\gcd(f_0(s, u), f_2(s, u)) = 1$, then $f_2(s_0, u_0) \neq 0$ for all base points $(s_0, u_0; t_0, v_0)$. Otherwise, if $f_2(s_0, u_0) = 0$, then since $f_0(s_0, u_0) = 0$ for any base point, $(su_0 - us_0)$ would be a common factor for both $f_0(s, u)$ and $f_2(s, u)$ contradicting the condition that $\gcd(f_0(s, u), f_2(s, u)) = 1$. Therefore, $\text{rank} N_f[\mathbf{PQR}]_{(s_0, u_0; t_0, v_0)} \neq 0$, and the implicit equation of the translational surface $\mathbf{h}(s, u; t, v)$ generated by the curves $\mathbf{f}(s, u)$ and $\mathbf{g}(t, v)$ can be computed by Theorem 3.4.

On the other hand, if $\gcd(f_0(s, u), f_2(s, u)) = d(s, u)$ with $\deg(d(s, u)) = k \neq 1$, let

$$f'_0(s, u) = \frac{f_0(s, u)}{d(s, u)}, \quad f'_2(s, u) = \frac{f_2(s, u)}{d(s, u)},$$

$$[\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3] = \begin{bmatrix} \frac{f_0 P_0 - f_3 P_3}{2} & \frac{f_0 Q_0 - f_3 Q_3}{2} & \frac{-f'_2}{2} \\ f_0 P_1 & f_0 Q_1 & 0 \\ 0 & 0 & f'_0 \\ f_0 P_3 & f_0 Q_3 & 0 \end{bmatrix}.$$

Then $f'_2(s_0, u_0) \neq 0$ for all base points $(s_0, u_0; t_0, v_0)$. Otherwise, if $f'_2(s_0, u_0) = 0$, then since $f_0(s_0, u_0) = 0$ for any base point, $(su_0 - us_0)$ would be a common factor for both $f_0(s, u)$ and $f'_2(s, u)$ contradicting the condition that $\gcd(f_0(s, u), f'_2(s, u)) = 1$. Hence, $\text{rank}[\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3]_{(s_0, u_0; t_0, v_0)} \neq 0$. We may construct a square matrix of size $(2m - k)n \times (2m - k)n$ for the three bivariate polynomials

$$\xi(s; t) = \mathbf{m}_1 \cdot \mathbf{x}, \quad \zeta(s; t) = \mathbf{m}_2 \cdot \mathbf{x}, \quad \kappa(s; t) = \mathbf{m}_3 \cdot \mathbf{x}$$

of bidegrees (m, v_1) , $(m, n - v_1)$ and $(m - k, 0)$. Consider the coefficient matrix M of the polynomials:

$$\xi, s\xi, t\xi, \dots, s^{m-k-1}t^{n-v_1-1}\xi, \zeta, s\zeta, t\zeta, \dots, s^{m-k-1}t^{v_1-1}\zeta, \kappa, s\kappa, t\kappa, \dots, s^{m-1}t^{n-1}\kappa.$$

This coefficient matrix has $(2m - k)n$ rows indexed by these polynomials, and $(2m - k)n$ columns indexed by the monomials $1, s, \dots, s^{2m-k-1}, t, \dots, s^{2m-k-1}t, \dots, s^{2m-k-1}t^{n-1}$. Thus this matrix is a square matrix. By an argument similar to the proof of Theorem 3.4, together with the result $\text{Res}(\mathbf{m}_1 \cdot \mathbf{x}, \mathbf{m}_2 \cdot \mathbf{x}, \mathbf{m}_3 \cdot \mathbf{x}) = \det(M)$ proved in [Theorem 7, (Shi et al., 2012)], we can show that

$$\text{Res}(\mathbf{m}_1 \cdot \mathbf{x}, \mathbf{m}_2 \cdot \mathbf{x}, \mathbf{m}_3 \cdot \mathbf{x}) = \det(M) = w^\tau F(\mathbf{x}), \quad \tau \in \mathbb{Z}, \tau > 0,$$

where $F(\mathbf{x}) \equiv F(w, x, y, z) = 0$ is the implicit equation of the surface $\mathbf{h}(s, u; t, v)$.

To illustrate this approach, we recall Example 3.7. From $N_f[\mathbf{PQR}]$, we find that $[\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3] =$

$$\begin{bmatrix} \frac{-s^3 v}{2} & \frac{-tu^3}{2} & \frac{-u}{2} \\ s^3 t & s^3 v & 0 \\ 0 & 0 & s \\ 0 & s^3 t & 0 \end{bmatrix}.$$

Now we construct the 8×8 coefficient matrix M with respect to the polynomials $\mathbf{m}_1 \cdot \mathbf{x}, \mathbf{m}_2 \cdot \mathbf{x}, \mathbf{m}_3 \cdot \mathbf{x}$,

$$M = \begin{bmatrix} 0 & 0 & 0 & -\frac{w}{2} & 0 & 0 & 0 & x \\ 0 & 0 & 0 & x & -\frac{w}{2} & 0 & 0 & z \\ -\frac{w}{2} & y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{w}{2} & y & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{w}{2} & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{w}{2} & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{w}{2} & y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{w}{2} & y \end{bmatrix}, \quad \text{and}$$

$$\det(M) = \frac{w^5}{128} (2wx^2 - 4y^3 + w^2z).$$

Thus, $\text{Res}(\mathbf{m}_1 \cdot \mathbf{x}, \mathbf{m}_2 \cdot \mathbf{x}, \mathbf{m}_3 \cdot \mathbf{x}) = \frac{w^5}{128} (2wx^2 - 4y^3 + w^2z)$, where w^5 is the extraneous factor corresponding to the base point $(0, 1; 0, 1)$, and the implicit equation of this translational surface $\mathbf{h}(s, u; t, v)$ is $F(\mathbf{x}) = F(w, x, y, z) = 2wx^2 - 4y^3 + w^2z$, which is exactly the expression we found in Example 3.7.

4. Alternative definitions of translational surfaces

An alternative definition of translational surfaces is given by the rational parametric representation

$$\mathbf{h}^*(s; t) = a\mathbf{f}^*(s) + b\mathbf{g}^*(t), \quad a, b \in \mathbb{R}, ab \neq 0. \quad (6)$$

Typically $a = b = 1$, Farin (2014), Pérez-Díaz (2006), Pérez-Díaz and Shen (2017) and Schroeder et al. (1994). Notice, however, the translational surfaces defined by Equation (6) are translation invariant if and only if $a + b = 1$.

The homogeneous parametrization of the translational surfaces given by Equation (6) is

$$\mathbf{h}(s, u; t, v) = [f_0 g_0, \quad b f_0 g_1 + a f_1 g_0, \quad b f_0 g_2 + a f_2 g_0, \quad b f_0 g_3 + a f_3 g_0],$$

a tensor product surface of bidegree (m, n) with base points of total multiplicity at least mn .

Furthermore,

$$\begin{aligned} \mathbf{h} &\equiv \left[\frac{f_0 g_0}{a}, \quad \frac{b}{a} f_0 g_1 + f_1 g_0, \quad \frac{b}{a} f_0 g_2 + f_2 g_0, \quad \frac{b}{a} f_0 g_3 + f_3 g_0 \right] = \mathbf{f}(s, u) \mathbf{M}_g(t, v), \\ &\equiv \left[\frac{f_0 g_0}{b}, \quad f_0 g_1 + \frac{a}{b} f_1 g_0, \quad f_0 g_2 + \frac{a}{b} f_2 g_0, \quad f_0 g_3 + \frac{a}{b} f_3 g_0 \right] = \mathbf{g}(t, v) \mathbf{M}_f(s, u), \end{aligned}$$

$$\text{where } \mathbf{M}_g = \begin{bmatrix} \frac{g_0}{a} & \frac{b}{a} g_1 & \frac{b}{a} g_2 & \frac{b}{a} g_3 \\ 0 & g_0 & 0 & 0 \\ 0 & 0 & g_0 & 0 \\ 0 & 0 & 0 & g_0 \end{bmatrix} \text{ and } \mathbf{M}_f = \begin{bmatrix} \frac{f_0}{b} & \frac{a}{b} f_1 & \frac{a}{b} f_2 & \frac{a}{b} f_3 \\ 0 & f_0 & 0 & 0 \\ 0 & 0 & f_0 & 0 \\ 0 & 0 & 0 & f_0 \end{bmatrix}.$$

$$\text{Let } \mathbf{N}_g = \begin{bmatrix} \frac{a^2}{b} g_0 & -a g_1 & -a g_2 & -a g_3 \\ 0 & \frac{a}{b} g_0 & 0 & 0 \\ 0 & 0 & \frac{a}{b} g_0 & 0 \\ 0 & 0 & 0 & \frac{a}{b} g_0 \end{bmatrix} \text{ and } \mathbf{N}_f = \begin{bmatrix} \frac{b^2}{a} f_0 & -b f_1 & -b f_2 & -b f_3 \\ 0 & \frac{b}{a} f_0 & 0 & 0 \\ 0 & 0 & \frac{b}{a} f_0 & 0 \\ 0 & 0 & 0 & \frac{b}{a} f_0 \end{bmatrix}.$$

Then

$$\mathbf{M}_g \mathbf{N}_g = \mathbf{N}_g \mathbf{M}_g = \frac{a}{b} g_0^2 \mathbf{I} \text{ and } \mathbf{M}_f \mathbf{N}_f = \mathbf{N}_f \mathbf{M}_f = \frac{b}{a} f_0^2 \mathbf{I}, \text{ where } \mathbf{I} \text{ is the } 4 \times 4 \text{ identity matrix.}$$

Hence, all the techniques and results in this paper can be applied to the translational surfaces given by (6) simply by replacing $\mathbf{M}_f$, $\mathbf{M}_g$, $\mathbf{N}_f$ and $\mathbf{N}_g$ with the corresponding matrices defined above.

5. Conclusion and future work

In this paper, we utilize syzygies to study translational surfaces. In particular, we construct three special syzygies for a translational surface from a μ -basis of one of the generating space curves, and we show how to compute the implicit equation of a translational surface from these three special syzygies. We close this paper by proposing the following two open problems for future research.

First, can we derive an algorithm for computing the singular loci of a translational surface from the three special syzygies for a translational surface constructed from a μ -basis of one of the generating space curves?

Second, the matrix representation of a translational surface connects the syzygies of the generating curves to the syzygies of the corresponding translational surface. Through this connection, it is possible to study translational surfaces via a μ -basis of one of the generating space curves. Can we also apply this technique to any other special types of surfaces?

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