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q -Blossoming for Analytic Functions

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Abstract We construct a q -analogue of the blossom for analytic functions, the analytic q -blossom. This q -analogue also extends the notion of q -blossoming from polynomials to analytic functions. We then apply this analytic q -blossom to derive identities for analytic functions represented in terms of the q -Poisson basis, including q -versions of the Marsden identity and the de Boor-Fix formula for analytic functions.

Keywords analytic function · q -blossom · q -Poisson basis · q -Marsden identity · q -derivative · q -bracket operator · q -de Boor-Fix formula

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1 Introduction

Blossoming is one of the basic tools used in the study of Bézier and B-spline curves and surfaces [5, 6]. Algorithms for evaluation, subdivision, and knot insertion can all be derived directly from blossoming. Because blossoming is so fundamental, blossoming has been extended beyond polynomial and piecewise polynomial spaces, in particular to Müntz spaces and to extended Chebyshev

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spaces [24, 12–14, 18–20, 10] (as well as [16, 17] and [1, 2]), to spaces of trigonometric polynomials [9] and trigonometric splines [15], and more recently to finite dimensional function spaces [4]. There are also theories of quantum blossoming [26, 27, 25, 7, 8] for polynomial and piecewise polynomial spaces, where the classical derivative is replaced by a quantum derivative.

An analytic blossom for analytic functions was introduced by Morin and Goldman [21, 22]. *The goal of this paper is to generalize this analytic blossom to an analytic q -blossom, where once again the classical derivative is replaced by a quantum derivative, and then to apply this analytic q -blossom to derive some fundamental identities for analytic functions.*

The analogue for analytic functions of the polynomial Bernstein bases is the Poisson basis [21]. Similarly, the analogue for analytic functions of the quantum q -Bernstein bases is the q -Poisson basis (see Section 2) which, along with the analytic q -blossom, will play a fundamental role in this paper.

This paper is organized in the following fashion. In Section 2, we introduce the q -Poisson basis functions, prove that they are limits of the q -Bernstein basis functions, and derive a relation between the q -Poisson and the q -Taylor coefficients. In Section 3 we define and establish the existence and uniqueness of the analytic q -blossom. We also show that the analytic q -blossom provides the dual functionals for the q -Poisson basis functions. In Section 4, we apply this dual functional property to derive change of basis formulas involving the q -Poisson basis functions, a q -analogue of the Marsden identity for analytic functions, a q -summation formula, and a q -analogue of Gauss' theorem. In Section 5 we apply the analytic q -blossom to construct a q -analogue of the de Boor-Fix formula for analytic functions. We close in Section 6 by exploring the connection between the homogeneous analogue of the analytic q -blossom and the q -derivative.

2 The q -Poisson Basis

Throughout this paper we will assume that $q \in (0, 1)$ is fixed and we will use the following q -calculus notation [11]:

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1, \end{cases} \quad [0]_q! = 1, \quad [n]_q! = [1]_q \cdots [n]_q, \quad n \in \mathbb{N}^+,$$

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{j=0}^{n-1} (1 - q^j a), \quad n \in \mathbb{N}^+ \cup \{\infty\},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}, \quad k = 0, \dots, n, \quad n \in \mathbb{N}.$$

Here $\mathbb{N}^+ = \{1, 2, \dots\}$ and $\mathbb{N} = \mathbb{N}^+ \cup \{0\}$. The q -binomial coefficients satisfy

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q. \quad (2.1)$$

We will also make use of the Euler identities [11, Theorem 12.2.6]

$$e_q(z) := \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} = \frac{1}{(z; q)_{\infty}}, \quad |z| < 1, \quad (2.2)$$

$$E_q(z) := \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} z^k}{(q; q)_k} = (-z; q)_{\infty}, \quad z \in \mathbb{C}. \quad (2.3)$$

The q -Bernstein basis functions on $[0, 1]$ are defined by [23]

$$B_k^n(t; q) = \begin{bmatrix} n \\ k \end{bmatrix}_q t^k (t; q)_{n-k}, \quad k = 0, \dots, n, \quad n \in \mathbb{N}, \quad (2.4)$$

and we define the q -Poisson basis functions by

$$b_k(t; q) = ((1 - q)t; q)_{\infty} t^k / [k]_q!, \quad k \in \mathbb{N}. \quad (2.5)$$

Lemma 2.1

$$B_k^n(t/[n]_q; q) \rightarrow b_k(t; q), \quad n \rightarrow \infty, \quad (2.6)$$

uniformly for $|t| \leq 1$, for every $k \in \mathbb{N}$.

Proof By (2.4) we have $B_k^n(t/[n]_q; q) = \frac{t^k}{[k]_q!} (t/[n]_q; q)_{n-k} \prod_{j=0}^{k-1} \frac{[n-j]_q}{[n]_q}$. The product tends to 1 as $n \rightarrow \infty$. Set $u = (1 - q)t$. Then $|u| \leq 1 - q$, and

$$\left| \log \frac{((1 - q)t; q)_{n-k}}{(t/[n]_q; q)_{n-k}} \right| = \left| \sum_{j=0}^{n-k-1} \log \left(1 + \frac{q^{n+j} u}{1 - q^j u - q^n} \right) \right| \rightarrow 0,$$

as $n \rightarrow \infty$. Here we used the inequality $q^{n+j}|u|/|1 - q^j u - q^n| \leq q^n/(1 - |u| - q^2) \leq q^n/(q - q^2) < 1/2$ for all sufficiently large n . We also used the inequality $|\log(1 + z)| \leq 3|z|$ for $|z| \leq 1/2$. This inequality holds since for $|z| \leq 1/2$, $\log(1 + z) = \log|1 + z| + i\text{Arg}(1 + z)$. But for $|z| \leq 1/2$ we have $\log|1 + z| \leq \log(1 + |z|) \leq |z|$ and $|\text{Arg}(1 + z)| = \tan^{-1}(|\text{Im}(z)|/(1 + \text{Re}(z))) \leq \tan^{-1}(2|z|) \leq 2|z|$.

Thus $\lim_{n \rightarrow \infty} (t/[n]_q; q)_{n-k} = ((1 - q)t; q)_{\infty}$, uniformly for $|t| \leq 1$. $\square$

From (2.5) and (2.2), it follows that $\sum_{k=0}^{\infty} b_k(t; q) = 1$, $|t| < 1/(1 - q)$. (see Section 4 for another proof of this partition of unity property).

We say that a function $F(t)$ is *analytic at $t = 0$* if $F(t)$ is analytic in an open subset of $\mathbb{C}$ containing $t = 0$. We say that a sequence of functions $\{F_n(t)\}$ analytic at $t = 0$ *converges uniformly to $F(t)$ at $t = 0$* if $\lim_{t \rightarrow 0} F_n(t) = F(t)$ uniformly in an open subset of $\mathbb{C}$ containing $t = 0$. Then by Cauchy's integral formula the limit function $F(t)$ is also analytic at $t = 0$.

Definition 2.2 Let $F(t) = \sum_{k=0}^{\infty} a_k t^k / [k]_q!$ be a function analytic at $t = 0$. The coefficients $\{a_k\}$ are called the q -Taylor coefficients of $F(t)$.

A function $F(t) = \sum_{k=0}^{\infty} P_k b_k(t; q)$ is called a q -Poisson function or a q -Poisson curve at $t = 0$ if $F(t)$ is analytic at $t = 0$. The coefficients $\{P_k\}$ are called the q -Poisson coefficients of $F(t)$.

Since the q -Poisson basis functions are non-negative on $[0, 1]$ and form a partition of unity, q -Poisson curves are affine invariant and lie in the convex hull of their control points. Moreover, since $((1-q)t; q)_\infty$ and $1/((1-q)t; q)_\infty$ are analytic at $t = 0$, every function analytic at $t = 0$ is a q -Poisson function at $t = 0$ and vice versa.

Let $F(t)$ be analytic at $t = 0$, with q -Poisson coefficients $\{P_n\}$ and q -Taylor coefficients $\{a_n\}$. It follows immediately from (2.5), (2.3), and (2.2) that

$$a_n = \sum_{l=0}^n (-1)^l q^{\binom{l}{2}} \begin{bmatrix} n \\ l \end{bmatrix}_q P_{n-l} = \sum_{l=0}^n (-1)^{n-l} q^{\binom{n-l}{2}} \begin{bmatrix} n \\ l \end{bmatrix}_q P_l, \quad n \in \mathbb{N}, \quad (2.7)$$

$$P_n = \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_q a_{n-l} = \sum_{l=0}^n \begin{bmatrix} n \\ l \end{bmatrix}_q a_l, \quad n \in \mathbb{N}. \quad (2.8)$$

Next, we derive two recursive evaluation algorithms to convert between the q -Taylor coefficients $\{a_n\}$ and the q -Poisson coefficients $\{P_n\}$.

Lemma 2.3 *Set $a_{0,s} = P_s$, $s \in \mathbb{N}$ and define recursively $a_{n,s} = a_{n-1,s+1} - q^{n-1}a_{n-1,s}$, $s \in \mathbb{N}$, $n \in \mathbb{N}^+$. Then*

$$a_{n,s} = \sum_{l=0}^n (-1)^l q^{\binom{l}{2}} \begin{bmatrix} n \\ l \end{bmatrix}_q P_{n+s-l}, \quad s, n \in \mathbb{N}. \quad (2.9)$$

Proof Let $Q_{n,s}$ denote the right-hand side of (2.9). Then $Q_{0,s} = P_s = a_{0,s}$, $s \in \mathbb{N}$. Applying (2.1) to $Q_{n,s}$ yields $Q_{n,s} = Q_{n-1,s+1} - q^{n-1}Q_{n-1,s}$, $s \in \mathbb{N}$, $n \in \mathbb{N}^+$. Therefore $a_{n,s} = Q_{n,s}$, $s \in \mathbb{N}$, $n \in \mathbb{N}^+$. In particular, (2.9) and (2.7) yield $a_{n,0} = a_n$. $\square$

Lemma 2.4 *Set $P_{0,s} = a_s$, $s \in \mathbb{N}$ and define recursively $P_{n,s} = P_{n-1,s+1} + q^s P_{n-1,s}$, $s \in \mathbb{N}$, $n \in \mathbb{N}^+$. Then*

$$P_{n,s} = \sum_{l=0}^n q^{sl} \begin{bmatrix} n \\ l \end{bmatrix}_q a_{n+s-l}, \quad s, n \in \mathbb{N}. \quad (2.10)$$

Proof Let $R_{n,s}$ denote the right-hand side of (2.10). Then $R_{0,s} = a_s = P_{0,s}$, $s \in \mathbb{N}$. Applying (2.1) to $R_{n,s}$ yields $R_{n,s} = R_{n-1,s+1} + q^s R_{n-1,s}$, $s \in \mathbb{N}$, $n \in \mathbb{N}^+$. Therefore $P_{n,s} = R_{n,s}$, $s \in \mathbb{N}$, $n \in \mathbb{N}^+$. In particular, (2.10) and (2.8) yield $P_{n,0} = P_n$. $\square$

3 The analytic q -blossom

We begin by recalling the definition of the q -blossom of a polynomial [27].

Definition 3.1 *The q -blossom of a degree n polynomial $P(t)$ is the unique symmetric multiaffine function $p(u_1, \dots, u_n; q)$ that satisfies the q -diagonal property $p(t, qt, \dots, q^{n-1}t; q) = P(t)$.*

Let $\{\phi_{n,k}(u_1, \dots, u_n)\}_{k=0}^n$ denote the elementary symmetric functions. The q -binomial formula $(z; q)_n = \sum_{k=0}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k$, [11, (12.2.27)] implies

$$\phi_{n,k}(1, q, \dots, q^{n-1}) = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q, \quad k = 0, \dots, n. \quad (3.1)$$

A q -Bézier curve of degree n on the interval $[0, 1]$ is a polynomial curve of the form $P(t) = \sum_{k=0}^n P_k B_k^n(t; q)$, [23, 27]. The control points $\{P_k\}$ can be expressed in terms of the q -blossom $p(u_1, \dots, u_n; q)$ of $P(t)$ using the following *dual functional property* [27, Theorem 4.4]:

$$P_k = p(1, q, \dots, q^{k-1}, 0, \dots, 0; q), \quad k = 0, \dots, n. \quad (3.2)$$

To simplify future expressions, we adopt the notation: $a^{(n)}$ = the n -term sequence all of whose terms are a , where $n \in \mathbb{N}^+ \cup \{\infty\}$.

Definition 3.2 Let $F(t)$ be analytic at $t = 0$. The analytic q -blossom of $F(t)$ is the unique symmetric multiaffine function $f[u_1, \dots, u_n, 0^{(\infty)}; q]$, defined for all $n \geq 1$, such that the diagonal property

$$\lim_{n \rightarrow \infty} f\left[t/[n]_q, qt/[n]_q, \dots, q^{n-1}t/[n]_q, 0^{(\infty)}; q\right] = F(t) \quad (3.3)$$

holds uniformly at $t = 0$.

Theorem 3.3 Let $F(t)$ be a function analytic at $t = 0$. Then $F(t)$ has a unique analytic q -blossom. Explicitly, if $F(t) = \sum_{k=0}^{\infty} a_k t^k / [k]_q!$, then

$$f[u_1, \dots, u_n, 0^{(\infty)}; q] = \sum_{k=0}^n a_k q^{-\binom{k}{2}} \phi_{n,k}(u_1, \dots, u_n). \quad (3.4)$$

Proof First we show that the function f in (3.4) is well-defined and satisfies the properties of the analytic q -blossom of $F(t)$. Since $\phi_{n,k}(u_1, \dots, u_{n-1}, 0) = \phi_{n-1,k}(u_1, \dots, u_{n-1})$, $k = 0, \dots, n-1$ and $\phi_{n,n}(u_1, \dots, u_{n-1}, 0) = 0$, it follows that $f[u_1, \dots, u_n, 0^{(\infty)}]_{u_n=0} = f[u_1, \dots, u_{n-1}, 0^{(\infty)}]$; hence f is well-defined. Clearly f is also symmetric and multiaffine.

Next, we establish (3.3). Fix $\epsilon > 0$. For every $n \geq m$, by (3.1) and (3.4)

$$\begin{aligned} \left| F(t) - f\left[\frac{t}{[n]_q}, \frac{qt}{[n]_q}, \dots, \frac{q^{n-1}t}{[n]_q}, 0^{(\infty)}; q\right] \right| &\leq \sum_{k=0}^m |a_k| \left| 1 - \frac{(q^{n-k+1}; q)_k}{(1-q^n)^k} \right| \frac{|t|^k}{[k]_q!} \\ &+ \sum_{k=m+1}^n |a_k| \left(1 + \frac{(q^{n-k+1}; q)_k}{(1-q^n)^k} \right) \frac{|t|^k}{[k]_q!} + \left| \sum_{k=n+1}^{\infty} a_k \frac{t^k}{[k]_q!} \right|. \end{aligned}$$

Let $I_1(m, n)$, $I_2(m, n)$, and $I_3(n)$ denote the last three sums. Consider first $I_3(n)$ and $I_2(m, n)$. Since $F(t)$ is analytic at $t = 0$, there exist constants $C > 0$ and $r > 0$ such that $|a_k|/[k]_q! \leq Cr^k$, $k \in \mathbb{N}$. Let $m \in \mathbb{N}^+$ be such that $[r/(r+1)]^m \leq \epsilon/[4C(r+1)]$. Then for $|t| \leq 1/(1+r)$ and $n \geq m$,

$$|I_3(n)| \leq \sum_{k=n+1}^{\infty} |a_k| |t|^k / [k]_q! \leq C \sum_{k=m}^{\infty} |rt|^k \leq C(r+1)[r/(r+1)]^m \leq \epsilon/4.$$

Furthermore, $(q^{n-k+1}; q)_k / (1 - q^n)^k \leq 1$ for $k \leq n$. Therefore

$$I_2(m, n) \leq 2 \sum_{k=m}^n |a_k| |t|^k / [k]_q! \leq 2C \sum_{k=m}^{\infty} |rt|^k \leq \epsilon/2.$$

Finally to bound $I_1(m, n)$, set $c_m := \max_{0 \leq k \leq m} |a_k|$ and choose $m_1 \geq m$ so that for $n \geq m_1$, $1 - (q^{n-k+1}; q)_k / (1 - q^n)^k < \epsilon / [4(m+1)c_m]$, $k = 0, \dots, m$. Then $I_1(m, n) \leq \epsilon/4$ for $|t| \leq 1$. Combining the estimates for $I_1(m, n)$, $I_2(m, n)$, and $|I_3(n)|$, we conclude that for every $n \geq m = m(\epsilon)$

$$\left| F(t) - f\left[t/[n]_q, qt/[n]_q, \dots, q^{n-1}t/[n]_q, 0^{(\infty)}; q\right] \right| \leq \epsilon,$$

uniformly in an open neighborhood of $t = 0$ independent of n and ϵ . This inequality establishes (3.3) and hence the existence of an analytic q -blossom.

To show uniqueness, let $f[u_1, \dots, u_n, 0^{(\infty)}; q]$ be any analytic q -blossom of $F(t)$. Since the q -blossom is symmetric and multiaffine,

$$f[u_1, \dots, u_n, 0^{(\infty)}; q] = \sum_{k=0}^n c_{n,k} \phi_{n,k}(u_1, \dots, u_n), \quad n \in \mathbb{N}.$$

for some constants $\{c_{n,k}\}$. Setting $u_n = 0$ yields

$$\sum_{k=0}^{n-1} c_{n,k} \phi_{n-1,k}(u_1, \dots, u_{n-1}) = \sum_{k=0}^{n-1} c_{n-1,k} \phi_{n-1,k}(u_1, \dots, u_{n-1});$$

hence $c_{n,k} = c_{n-1,k}$, $n > k$, $k \in \mathbb{N}$. Therefore $c_{n,k} = c_{k,k} := c_k$, $n \geq k \in \mathbb{N}$.

Next, let $f_\nu[u_1, \dots, u_n, 0^{(\infty)}; q] = \sum_{k=0}^n c_k^{(\nu)} \phi_{n,k}(u_1, \dots, u_n)$, $\nu = 1, 2$ be two analytic q -blossoms of $F(t)$. Set $g_n(u_1, \dots, u_n) = (f_1 - f_2)[u_1, \dots, u_n, 0^{(\infty)}; q]$. By (3.1) and (3.3) the functions

$$g_n(t) := g_n\left(\frac{t}{[n]_q}, \frac{qt}{[n]_q}, \dots, \frac{q^{n-1}t}{[n]_q}\right) = \sum_{k=0}^n (c_k^{(1)} - c_k^{(2)}) q^{\binom{k}{2}} \frac{(q^{n-k+1}; q)_k}{(1 - q^n)^k} \frac{t^k}{[k]_q!}$$

converge to 0 as $n \rightarrow \infty$ uniformly in an open neighborhood E of $t = 0$. Choose $r > 0$ so that $D_r(0) := \{t : |t| \leq r\} \subset E$. By Cauchy's integral formula,

$$c_k^{(1)} - c_k^{(2)} = q^{-\binom{k}{2}} [k]_q! \frac{(1 - q^n)^k}{(q^{n-k+1}; q)_k} \frac{1}{2\pi i} \oint_{|t|=r} \frac{g_n(t)}{t^{k+1}} dt, \quad k \leq n.$$

Letting $n \rightarrow \infty$ in the last equation and using $\lim_{n \rightarrow \infty} \frac{(1 - q^n)^k}{(q^{n-k+1}; q)_k} = 1$ and $g_n(t) \rightarrow 0$ as $n \rightarrow \infty$ uniformly for $|t| = r$ yields $c_k^{(1)} - c_k^{(2)} = 0$, $k \in \mathbb{N}$. $\square$

Two important analytic q -blossom formulas are:

$$F(t) = t^k, \quad k \in \mathbb{N}, \quad f[u_1, \dots, u_n, 0^{(\infty)}; q] = [k]_q! q^{-\binom{k}{2}} \phi_{n,k}(u_1, \dots, u_n) \quad (3.5)$$

$$F(t) = (t; q)_\infty, \quad f[u_1, \dots, u_n, 0^{(\infty)}; q] = \prod_{j=1}^n (1 - u_j / (1 - q)). \quad (3.6)$$

Formula (3.5) is a direct consequence of Theorem 3.3. Formula (3.6) follows from Definition 3.2 and the fact that $\lim_{n \rightarrow \infty} [n]_q = 1/(1-q)$ for $q \in (0, 1)$.

Lemma 3.4 *Let $F(t)$ be analytic at $t = 0$, with q -Poisson coefficients $\{P_k\}$. Let $b_k^n(u_1, \dots, u_n; q)$ be the q -blossom of $B_k^n(t; q)$. Then*

$$f[u_1, \dots, u_n, 0^{(\infty)}; q] = \sum_{k=0}^n P_k b_k^n(u_1, \dots, u_n; q). \quad (3.7)$$

Proof Let $g[u_1, \dots, u_n]$ denote the right-hand side of (3.7). By (3.1)

$$\phi_{n,k}(1, q, \dots, q^{l-1}, 0^{(n-l)}) = \begin{cases} \phi_{l,k}(1, q, \dots, q^{l-1}) = q^{\binom{k}{2}} \begin{bmatrix} l \\ k \end{bmatrix}_q, & k \leq l \\ 0, & k > l, \end{cases} \quad (3.8)$$

$l = 1, \dots, n$, and $\phi_{n,k}(0^{(n)}) = \delta_{0,k}$. By the dual functional property (3.2)

$$b_k^n(1, q, \dots, q^{l-1}, 0^{(n-l)}; q) = \delta_{k,l}, \quad k, l = 0, \dots, n. \quad (3.9)$$

Thus by (3.4), (3.8), (2.8), (3.9), and (3.7),

$$\begin{aligned} f[1, q, \dots, q^{l-1}, 0^{(\infty)}; q] &= \sum_{k=0}^n a_k q^{-\binom{k}{2}} \phi_{n,k}(1, q, \dots, q^{l-1}, 0^{(n-l)}) \\ &= \sum_{k=0}^l a_k \begin{bmatrix} l \\ k \end{bmatrix}_q = P_l = g[1, q, \dots, q^{l-1}], \quad l = 0, \dots, n. \end{aligned} \quad (3.10)$$

The recursive evaluation algorithm for symmetric multiaffine functions (for special cases of this algorithm see [25, Section 4] and Proposition 4.1 below) applied to (3.10) as the base cases yields $f[u_1, \dots, u_n, 0^{(\infty)}; q] = g[u_1, \dots, u_n]$. $\square$

Corollary 3.5 *The analytic q -blossom of the q -Poisson function $b_l(t; q)$ is*

$$b_l[u_1, \dots, u_n, 0^{(\infty)}; q] = \begin{cases} b_l^n(u_1, \dots, u_n; q), & n \geq l, \\ 0, & n < l. \end{cases} \quad (3.11)$$

In particular, $b_0[u_1, \dots, u_n, 0^{(\infty)}; q] = \prod_{j=1}^n (1 - u_j)$.

Proof Formula (3.11) follows from Lemma 3.4, since the q -Poisson coefficients of $b_l(t; q)$ are $\{P_k = \delta_{k,l}\}$. $\square$

Proposition 3.6 *(Dual Functional Property) Let $F(t)$ be analytic at $t = 0$, with q -Poisson coefficients $\{P_l\}$ and analytic q -blossom $f[u_1, \dots, u_n, 0^{(\infty)}; q]$. Then*

$$P_l = f[1, q, \dots, q^{l-1}, 0^{(\infty)}; q], \quad l \in \mathbb{N}. \quad (3.12)$$

Proof This property follows from (3.10) of Lemma 3.4. $\square$

4 Consequences of the dual functional property

Proposition 4.1 (*Recursive Evaluation of the Analytic q -Blossom*) Let $F(t)$ be a function analytic at $t = 0$, with q -Poisson coefficients $\{P_k\}$ and analytic q -blossom $f[u_1, \dots, u_n, 0^{(\infty)}; q]$. Set $Q_k^0 = P_k$, $k \in \mathbb{N}$ and define recursively

$$Q_k^{s+1} = (1 - q^{-k}u_{s+1})Q_k^s + q^{-k}u_{s+1}Q_{k+1}^s, \quad k, s \in \mathbb{N}. \quad (4.1)$$

Then

$$Q_k^s = f[u_1, \dots, u_s, 1, q, \dots, q^{k-1}, 0^{(\infty)}; q], \quad k, s \in \mathbb{N}. \quad (4.2)$$

In particular, $Q_0^n = f[u_1, \dots, u_n, 0^{(\infty)}; q]$, $n \in \mathbb{N}$.

Proof Formula (4.2) follows by induction on $s \in \mathbb{N}$ from the dual functional property (3.12), (4.1), and the multiaffine property of the analytic q -blossom. $\square$

Proposition 4.2 (*Monomial Representation*)

$$t^k = \sum_{l=k}^{\infty} \frac{[l]_q!}{[l-k]_q!} b_l(t; q), \quad k \in \mathbb{N}. \quad (4.3)$$

In particular, the q -Poisson basis functions form a partition of unity.

Proof Formula (4.3) follows immediately from (3.5), (3.12), and (3.8). $\square$

Proposition 4.3 Let $F(t)$ be a function analytic at $t = 0$, with q -Taylor coefficients $\{a_k\}$. For a fixed r , let $F_r(t) = F(rt)$. Then

$$f_r[u_1, \dots, u_n, 0^{(\infty)}; q] = f[ru_1, \dots, ru_n, 0^{(\infty)}; q]. \quad (4.4)$$

Proof Equation (4.4) follows immediately from (3.4). $\square$

Proposition 4.4 (*q -Marsden Identity*)

$$(xt; q)_{\infty} = \sum_{k=0}^{\infty} (x/(1-q); q)_k b_k(t; q). \quad (4.5)$$

Proof Let $F(t) = (t; q)_{\infty}$. Fix $x \in \mathbb{C}$ and set $F_x(t) = (xt; q)_{\infty}$. By (3.6) and Proposition 4.3, $f_x[u_1, \dots, u_n, 0^{(\infty)}; q] = \prod_{j=0}^n (1 - xu_j/(1-q))$. Therefore by the dual functional property (3.12), the q -Poisson coefficients of $F_x(t)$ are $P_k = f_x[1, q, \dots, q^{k-1}, 0^{(\infty)}; q] = (x/(1-q); q)_k$, $k \in \mathbb{N}$. $\square$

Corollary 4.5 (*Change of Basis Formula*)

$$b_l(rt; q) = \sum_{k=l}^{\infty} B_l^k(r; q) b_k(t; q). \quad (4.6)$$

Proof Formula (4.6) follows from (2.5), (4.5) with $x = (1 - q)r$, and (2.4). $\square$

Proposition 4.6 (*q -Poisson subdivision*) *Let $F(t)$ be analytic at $t = 0$, with q -Poisson coefficients $\{P_k\}$. Then for any $r \in \mathbb{C}$ the q -Poisson coefficients of $F_r(t) = F(rt)$ are given by*

$$P_k(r) = \sum_{l=0}^k P_l B_l^k(r; q), \quad k \in \mathbb{N}. \quad (4.7)$$

Moreover, if we set $P_k^0(r) = P_k$, $k \in \mathbb{N}$, and define recursively

$$P_k^m(r) = (1 - rq^{m-1-k})P_k^{m-1}(r) + rq^{m-1-k}P_{k+1}^{m-1}(r), \quad (4.8)$$

$k \in \mathbb{N}$, $m \in \mathbb{N}^+$, then

$$P_k^m(r) = f[1, q, \dots, q^{k-1}, r, qr, \dots, q^{m-1}r, 0^{(\infty)}; q], \quad k, m \in \mathbb{N}, \quad (4.9)$$

$$P_0^m(r) = f[r, qr, \dots, q^{m-1}r, 0^{(\infty)}; q] = P_m(r), \quad m \in \mathbb{N}. \quad (4.10)$$

Proof Equation (4.7) follows from (4.6), since

$$F(rt) = \sum_{l=0}^{\infty} P_l b_l(rt; q) = \sum_{k=0}^{\infty} \left\{ \sum_{l=0}^k P_l B_l^k(r; q) \right\} b_k(t; q).$$

Next, by (3.12), $P_k = f[1, q, \dots, q^{k-1}, 0^{(\infty)}; q]$, $k \in \mathbb{N}$. Now (4.9) follows by induction on m from (4.8) and the multiaffine property of the analytic q -blossom. Equation (4.10) follows from (4.9), Proposition 4.3, and (3.12). $\square$

Proposition 4.7 *The q -Marsden identity (4.5) is equivalent to the q -identity ([11, (12.2.22)])*

$$\frac{(az; q)_{\infty}}{(z; q)_{\infty}} = \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k, \quad |z| < 1. \quad (4.11)$$

Formula (4.11) is the case $b = 0$ of the q -Gauss theorem ([11, (12.2.18)])

$$\sum_{k=0}^{\infty} \frac{(a; q)_k (b; q)_k}{(abz; q)_k (q; q)_k} z^k = \frac{(az; q)_{\infty} (bz; q)_{\infty}}{(abz; q)_{\infty} (z; q)_{\infty}}, \quad |z| < 1. \quad (4.12)$$

Proof Setting $x = a(1 - q)$ and $t = z/(1 - q)$ with $|z| < 1$ in (4.5) shows that (4.5) is equivalent to (4.11).

Formula (4.12) follows from (4.11) by multiplying $(a; q)_k z^k / (q; q)_k$ on the right-hand side of (4.11) by $1 = \sum_{l=0}^k B_l^k(b; q)$ and simplifying the double sum. $\square$

5 The q -Bracket operator and the q -de Boor-Fix formula for analytic functions

The q -bracket operator and the q -de Boor-Fix formula for analytic functions will be expressed in terms of the q -difference operator defined by

$$D_q F(t) = D_{q,t} F = (F(qt) - F(t)) / ((q - 1)t). \quad (5.1)$$

There is a limited q -chain rule. Let $F_r(t) = F(rt)$. Then

$$D_{q,t} F_r = r(D_q F)(rt). \quad (5.2)$$

Now let $\lambda_q F(t) = F(qt)$. Then the product rule for D_q is

$$D_q(F_1 F_2) = F_1 D_q F_2 + (D_q F_1) \lambda_q F_2. \quad (5.3)$$

Let $\lambda_q^s F(t) = F(q^s t)$. Then by (5.2), $D_q \lambda_q^s F = q^s \lambda_q^s D_q F$. Using this property, (5.3), induction on m , and (2.1), one can easily derive the q -Leibniz rule:

$$D_q^m (F_1 F_2) = \sum_{s=0}^m \begin{bmatrix} m \\ s \end{bmatrix}_q (D_q^s F_1) \lambda_q^s D_q^{m-s} F_2. \quad (5.4)$$

Given two functions $F_1(t)$ and $F_2(t)$ that are analytic at $t = 0$, we define the q -bracket operator $[F_1(u), F_2(u)]^q$ by

$$[F_1(u), F_2(u)]^q = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q!} q^{\binom{k}{2}} (D_{1/q,u}^k F_1)|_{u=0} (D_{1/q,u}^k F_2)|_{u=0}. \quad (5.5)$$

A similar q -bracket operator was defined in [3]. Since in (5.5) we use $D_{1/q}$, we will need the identities:

$$D_{1/q}(t^n) = [n]_{1/q} t^{n-1} = [n]_q (t/q)^{n-1}, \quad D_{1/q}(t; q)_n = -[n]_q (t; q)_{n-1}, \quad (5.6)$$

$$D_{1/q}(t; q)_{\infty} = (t; q)_{\infty} / (q - 1).$$

Lemma 5.1 *Let $F(u) = \sum_{k=0}^{\infty} a_k u^k / [k]_q!$ be analytic at $u = 0$. Then*

$$(D_{1/q,u}^k F)|_{u=0} = a_k q^{-\binom{k}{2}}, \quad k \in \mathbb{N}. \quad (5.7)$$

Proof Formula (5.7) follows immediately from (5.6) since by (5.6)

$$D_{1/q}^k u^l = \frac{[l]_{1/q}!}{[l-k]_{1/q}!} u^{l-k} = q^{-kl + \binom{k+1}{2}} \frac{[l]_q!}{[l-k]_q!} u^{l-k}, \quad k \leq l. \quad (5.8)$$

□

Proposition 5.2 *Let $F(u) = \sum_{n=0}^{\infty} P_n b_n(u; q)$ be analytic at $u = 0$. Then*

$$[F(u), (u; q)_n]^q = P_n, \quad n \in \mathbb{N}. \quad (5.9)$$

Proof It follows from (5.6) by induction on k that

$$D_{1/q}^k(u; q)_n = (-1)^k \frac{[n]_q!}{[n-k]_q!} (u; q)_{n-k}, \quad k \leq n. \quad (5.10)$$

Now let $F(u) = \sum_{k=0}^{\infty} a_k u^k / [k]_q!$. By (5.7), (5.10), and (2.8), we get

$$[F(u), (u; q)_n]^q = \sum_{k=0}^n \frac{(-1)^k}{[k]_q!} q^{\binom{k}{2}} a_k q^{-\binom{k}{2}} (-1)^k \frac{[n]_q!}{[n-k]_q!} = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q a_k = P_n.$$

□

To avoid misunderstanding, we shall also denote the analytic q -blossom of a function $F(t)$ involving parameters by $F[u_1, \dots, u_n, 0^{(\infty)}; q]$.

Proposition 5.3 *Let $F(u)$ be a function analytic at $u = 0$. Then*

$$f[x_1, \dots, x_n, 0^{(\infty)}; q] = [F(u), \prod_{j=1}^n (1 - x_j u)]^q. \quad (5.11)$$

Proof By Lemma 3.4 the left-hand side of (5.11) is $\sum_{k=0}^n P_k b_k^n(x_1, \dots, x_n; q)$, where $\{P_k\}$ are the q -Poisson coefficients of $F(u)$. Moreover, the q -blossom of $(ux; q)_n$ in x is $(ux; q)_n(x_1, \dots, x_n; q) = \prod_{j=1}^n (1 - ux_j)$. Hence by the dual functional property (3.2), $(ux; q)_n = \sum_{k=0}^n (u; q)_k B_k^n(x; q)$. Therefore

$$\prod_{j=1}^n (1 - ux_j) = (ux; q)_n(x_1, \dots, x_n; q) = \sum_{k=0}^n (u; q)_k b_k^n(x_1, \dots, x_n; q).$$

By this equation and (5.9) the right-hand side of (5.11) is also

$$\sum_{k=0}^n [F(u), (u; q)_k]^q b_k^n(x_1, \dots, x_n; q) = \sum_{k=0}^n P_k b_k^n(x_1, \dots, x_n; q).$$

□

Proposition 5.4 (*q -de Boor-Fix Formula*) *Let $F(u)$ be analytic at $u = 0$. Then $[F(u), ((1-q)xu; q)_{\infty}]^q = F(x)$. Moreover*

$$[F(u), D_{1/q,x}^m((1-q)xu; q)_{\infty}]^q = D_{1/q,x}^m F(x), \quad m \in \mathbb{N}. \quad (5.12)$$

Proof A straightforward calculation shows that

$$D_{1/q,x}((1-q)xu; q)_{\infty} = -u((1-q)xu; q)_{\infty}. \quad (5.13)$$

Therefore by (5.13) and (5.5)

$$\begin{aligned} [F(u), D_{1/q,x}^m((1-q)xu; q)_{\infty}]^q &= (-1)^m [F(u), u^m((1-q)xu; q)_{\infty}]^q \\ &= (-1)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_q!} q^{\binom{k}{2}} (D_{1/q,u}^k F(u))|_{u=0} (D_{1/q,u}^k \{u^m((1-q)xu; q)_{\infty}\})|_{u=0}. \end{aligned}$$

By the q -Leibniz rule (5.4) and (5.8), $(D_{1/q,u}^k \{u^m((1-q)xu; q)_{\infty}\})|_{u=0} = 0$ for $k < m$. For $k \geq m$, by (5.4), (5.8), and (5.13),

$$(D_{1/q,u}^k \{u^m((1-q)xu; q)_{\infty}\})|_{u=0} = ([k]_{1/q}! / [k-m]_{1/q}!) (-x)^{k-m}.$$

Now (5.12) follows from the last three equations, (5.7), and (5.8).

□

6 The homogeneous analytic q -blossom

Let $F(t)$ be a function analytic at $t = 0$ with q -Taylor coefficients $\{a_k\}$. Set $\tilde{u}_j = (u_j, w_j)$. We define the *homogeneous analytic q -blossom* of $F(t)$ to be the homogenization of the analytic q -blossom of $F(t)$ given by (3.4), that is

$$f[\tilde{u}_1, \dots, \tilde{u}_n, (0, 1)^{\langle \infty \rangle}; q] = \sum_{k=0}^n a_k q^{-\binom{k}{2}} w_1 \cdots w_n \phi_{n,k} \left(\frac{u_1}{w_1}, \dots, \frac{u_n}{w_n} \right). \quad (6.1)$$

Setting $\tilde{u}_j = \delta := (1, 0)$, $j = n - m + 1, \dots, n$ in (6.1) yields

$$\begin{aligned} & f[\tilde{u}_1, \dots, \tilde{u}_{n-m}, \delta^{\langle m \rangle}, (0, 1)^{\langle \infty \rangle}; q] \\ &= \sum_{k=0}^{n-m} a_{m+k} q^{-\binom{m+k}{2}} w_1 \cdots w_{n-m} \phi_{n-m,k} (u_1/w_1, \dots, u_{n-m}/w_{n-m}). \end{aligned} \quad (6.2)$$

Proposition 6.1 *Let $F(t) = \sum_{k=0}^{\infty} a_k t^k / [k]_q!$ be analytic at $t = 0$. Then*

$$a_k = q^{\binom{k}{2}} f[\delta^{\langle k \rangle}, (0, 1)^{\langle \infty \rangle}; q], \quad k \in \mathbb{N}. \quad (6.3)$$

Proof Formula (6.3) is the case $m = n$ of (6.2). □

Proposition 6.2 *Let $F(u)$ be a function analytic at $u = 0$. Then for $m \in \mathbb{N}$,*

$$(D_{1/q}^m F)[u_1, \dots, u_n, 0^{\langle \infty \rangle}; q] = f[(u_1, 1), \dots, (u_n, 1), \delta^{\langle m \rangle}, (0, 1)^{\langle \infty \rangle}; q]. \quad (6.4)$$

Proof Let $F(u) = \sum_{k=0}^{\infty} a_k u^k / [k]_q!$. From (5.8) it follows that

$$D_{1/q}^m F(u) = \sum_{k=0}^{\infty} a_{m+k} q^{-mk - \binom{m}{2}} u^k / [k]_q!.$$

Now by (3.4), (6.1), and (6.2), we obtain

$$(D_{1/q}^m F)[\tilde{u}_1, \dots, \tilde{u}_{n-m}, (0, 1)^{\langle \infty \rangle}; q] = f[\tilde{u}_1, \dots, \tilde{u}_{n-m}, \delta^{\langle m \rangle}, (0, 1)^{\langle \infty \rangle}; q].$$

Setting $w_j = 1$, $j = 1, \dots, n - m$, and $n := n + m$ yields (6.4).

Corollary 6.3 *Let $F(u)$ be a function analytic at $u = 0$. Then the q -Poisson coefficients of $D_{1/q}^m F(u)$ are given by*

$$P_k^m = f[(1, 1), (q, 1), \dots, (q^{k-1}, 1), \delta^{\langle m \rangle}, (0, 1)^{\langle \infty \rangle}; q], \quad k \in \mathbb{N}. \quad (6.5)$$

In particular, the q -Poisson coefficients $\{P_k\}$ of $F(u)$ are given by

$$P_k = P_k^0 = f[(1, 1), (q, 1), \dots, (q^{k-1}, 1), (0, 1)^{\langle \infty \rangle}; q], \quad k \in \mathbb{N}. \quad (6.6)$$

Moreover, $\{P_k^m\}$ satisfy the recurrence relation

$$P_k^{m+1} = q^{-k} (P_{k+1}^m - P_k^m), \quad k, m \in \mathbb{N}. \quad (6.7)$$

Proof Equation (6.5) follows from Proposition 6.2 and (3.12). Equation (6.7) follows from (6.5) and the multilinearity of the homogeneous q -blossom. $\square$

Proposition 6.4 *Let $F(u)$ be a function analytic at $u = 0$ and let $\{P_k^m\}$ be the q -Poisson coefficients of $D_{1/q}^m F(u)$. Then*

$$P_k^m = q^{-mk} \sum_{l=0}^m (-1)^{m-l} q^{-ml + \binom{l+1}{2}} \begin{bmatrix} m \\ l \end{bmatrix}_q P_{k+l}^0, \quad k, m \in \mathbb{N}. \quad (6.8)$$

Proof The case $m = 1$ of (6.8) follows from (6.7). Then induction on m using (6.7) and (2.1) verifies (6.8) for arbitrary $m \in \mathbb{N}$. $\square$

We end this section with two algorithms for converting between the q -Taylor and the q -Poisson coefficients of a function analytic at $t = 0$. These algorithms are q -blossom analogues of the algorithms in Lemmas 2.3 and 2.4.

Proposition 6.5 *Let $F(t)$ be a function analytic at $t = 0$, with q -Taylor coefficients $\{a_n\}$ and q -Poisson coefficients $\{P_n\}$. Set $a_{0,s} = P_s$, $s \in \mathbb{N}$, and define recursively $a_{n,s} = a_{n-1,s+1} - q^{n-1}a_{n-1,s}$, $s \in \mathbb{N}$, $n \in \mathbb{N}^+$. Then*

$$a_{n,s} = q^{\binom{n}{2} + ns} f[\delta^{(n)}, (1, 1), (q, 1), \dots, (q^{s-1}, 1), (0, 1)^{(\infty)}; q], \quad (6.9)$$

$s, n \in \mathbb{N}$. In particular, $a_{n,0} = a_n$, $n \in \mathbb{N}$.

Proof The case $n = 0$ of (6.9) is (6.6). For $n \geq 1$, (6.9) follows from the recurrence relation for $\{a_{n,s}\}$ by induction on n . Finally, $a_{n,0} = a_n$ follows from (6.9) and (6.3). $\square$

Proposition 6.6 *Let $F(t)$ be a function analytic at $t = 0$, with q -Taylor coefficients $\{a_n\}$ and q -Poisson coefficients $\{P_n\}$. Set $P_{0,s} = a_s$, $s \in \mathbb{N}$, and define recursively $P_{n,s} = P_{n-1,s+1} + q^s P_{n-1,s}$, $s \in \mathbb{N}$, $n \in \mathbb{N}^+$. Then*

$$P_{n,s} = q^{\binom{s}{2} + ns} f[\delta^{(s)}, (1, 1), (q, 1), \dots, (q^{n-1}, 1), (0, 1)^{(\infty)}; q], \quad (6.10)$$

$s, n \in \mathbb{N}$. In particular, $P_{n,0} = P_n$, $n \in \mathbb{N}$.

Proof The case $n = 0$ of (6.10) is (6.3). For $n \geq 1$, (6.10) follows from the recurrence relation for $\{P_{n,s}\}$ by induction on n . Finally, $P_{n,0} = P_n$ follows from (6.10) and (6.6). $\square$

Remark 6.7 *Notice that (6.9) and (6.10) imply $P_{n,s} = a_{s,n}$ for all $s, n \in \mathbb{N}$. This property is not evident from Lemmas 2.3 and 2.4.*

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May 1, 2018

RESPONSE TO THE REFEREE REPORTS FOR NUMA-D-16-00699

This letter is a response to the referee reports for our manuscript: *q*-Blossoming for Analytic Functions, NUMA-D-16-00699, submitted to *Numerical Algorithms*. We revised the manuscript following the recommendations and suggestions of the two reviewers and the associate editor. Below we outline the changes made in the manuscript.

Reviewer 1.

1. Reviewer suggested adding the reference: J. Comp. Appl. Math. 151 (1), 1-12 (2003), as the original paper on *q*-Bézier curves.

Response: We agree with the reviewer and we cited this reference and added it to the bibliography.

2. Reviewer asked: What if $q > 1$?

Response: In this case the *q*-Poisson basis is undefined. It is at least required that $|q| < 1$ for the infinite products to converge. At the start of the paper, we fix $q \in (0, 1)$, which is a standard assumption for *q*-Bézier curves and surfaces.

3. Reviewer suggested that another proof of Lemma 2.1 using a theorem from *Theory of Partitions* by G. Andrews is possible.

Response: Perhaps this is true and there are likely other related results. Since our work is not related to partitions and number theory, we kept but shortened our proof. Our proof is within the context of *q*-Bézier and *q*-Poisson bases.

4. Reviewer suggested that we say "using convolution of sequences" after (2.7).

Response: We removed this equation and (2.9) since the reader can easily check the details. We only left formulas relating a_n and P_n with a brief statement saying that they follow immediately from the definition of the *q*-Poisson basis and the two Euler formulas.

5. Reviewer noticed that the proofs of Lemmas 2.4 and 2.5 are very similar.

Response: Yes, we agree. We shortened these already very short proofs, which omit any details. But we think that it is better to keep these algorithms for converting between the *q*-Taylor and *q*-Poisson control points separated for clarity. The two

recursions and the two conversion formulas cannot be combined.

6. Reviewer suggested adding a reference to: J. Comp. Appl. Math. 172 (2004), 49-64, for formula (3.4).

Response: In our opinion the q -binomial formula is the most direct and appropriate source for this simple identity. It is also a consequence of many other more general results, but we prefer to give the most direct one. We noticed that the results from the suggested article are more general than this identity and citing them will be a distraction. So we decided not to refer to this article.

7. Reviewer suggested citing the reference: J. Comp. Appl. Math. 151 (1), 1-12 (2003), as the original paper on q -Bézier curves.

Response: We agree with the reviewer and we cited this paper as suggested.

8. Reviewer suggested using Farin's notation $a^{\langle n \rangle}$ for sequences with repeated terms.

Response: We agree with the reviewer and we implemented this notation throughout the manuscript.

9. Reviewer suggested that we define the value $\phi_{n,k}(0^{\langle n \rangle})$ explicitly.

Response: We defined this value as suggested.

10. Reviewer suggested that the monomial representation identity also follows from results in *Theory of Partitions* by G. Andrews.

Response: While we agree with the reviewer, we are aware that this identity likely follows from a large number of other identities and methods. Our point is to prove all our identities using the dual functional property of q -analytic blossom and other related techniques introduced in our work.

11. Reviewer observed that formulas (4.11) and (4.5) are in *Theory of Partitions* by G. Andrews.

Response: Both (4.11) and (4.5) are equivalent to a standard q -summation formula for a ${}_2\phi_1$ basic hypergeometric series, which can be found in any book on special functions. We do not have room to cite all possible sources for this simple formula. We wanted to make the point that (4.11) is equivalent to our q -Marsden identity, which as well as the more general q -Gauss identity follow from the dual functional property of the analytic q -blossom. We had already cited (12.2.22) from M. Ismail's book on orthogonal polynomials, which is a very comprehensive source for such identities.

12. Reviewer suggested adding: A similar q -bracket operator is defined in J. Comp. Appl. Math. 282, 17-33 (2015).

Response: We agree with the reviewer. We added this sentence and the suggested reference to the bibliography.

13. Reviewer suggested to remove the second line from equation (5.9).

Response: This line was removed as suggested.

14. Reviewer noticed a text error in line 2, page 18.

Response: We fixed this error by moving the word "multilinearity" to a single line.

15. Reviewer observed that the proofs of Propositions 6.5 and 6.6 are very similar.

Response: We agree with the reviewer. But these propositions have very short statements and proofs (3 lines each) and all technical details are omitted. Although similar, these propositions describe recursive evaluation for two very different sets of control points (q -Taylor and q -Poisson). Mixing or combining these two propositions into one will confuse the reader. We prefer to keep them separate.

16. Reviewer asked: What about subdivision for q -Poisson?

Response: The manuscript contains several formulas for the analytic q -blossom and algorithms for computing the control points of $F(rt)$ recursively. We did not investigate the convergence of the corresponding subdivided control polygons in this work. This type of convergence was investigated in the classical case by G. Morin in her Ph.D thesis. We leave this investigation for future work as it could be lengthy and technical, and we do not have room for this analysis in this paper.

17. Reviewer suggested that q -Poisson distribution has been in the literature and it may be suitable to mention it in the introduction.

Response: We were unable to find any direct reference to the q -Poisson distribution, which we defined as a natural q -analog of the classical Poisson distribution, despite our searches on the Internet and in the literature available to us. Since the reviewer did not suggest any references to this q -Poisson distribution, we were unable to add any.

Reviewer 2.

In light of the comment by this reviewer on the excessive length of our manuscript, we have shortened our manuscript from 19 pages to 14 pages. The 20-th page was generated during the submission by the electronic editor of the journal and is not part of our manuscript.

We kept the all results as they are essential for the completeness of our work. But most of the proofs, except the proof of the main theorem in Section 3, were shortened to just a few lines and sometimes 2-3 lines only, leaving all details to be filled by the reader. For example in any proof that uses an induction argument the details were omitted. We think that now the manuscript is very concise. It is still somewhat technical but it contains some important algorithms, properties, identities, and definitions.

The reviewer suggested that only the q -Poisson form needs to be discussed. However, both forms are needed since depending on the result, algorithm, or identity, one of the forms is much more suitable and straightforward than the other. If only the q -Poisson form is used, this will make some results and their proofs longer and less elegant.

Other Changes.

We added 11 new references to the bibliography and removed 2 old references, increasing the total number of references from 18 to 27.

We proofread the manuscript and made some minor corrections, improvements, and reordering of some statements and results.

We would like to thank the reviewers and the associate editor for helpful suggestions and comments.

The Authors.