A Two-grid Method for Coupled Free Flow with Porous Media Flow

Prince Chidyagwai^a and Béatrice Rivière^{a,1}

^a*Department of Computational and Applied Mathematics, Rice University, 6100
Main Street, Houston, TX 77005, U.S.A.*

Abstract

This paper presents a two-grid method for solving systems of partial differential equations modelling free flow coupled with porous media flow. This work considers both the coupled Stokes and Darcy as well as the coupled Navier-Stokes and Darcy problems. The numerical schemes proposed are based on combinations of the continuous finite element method and the discontinuous Galerkin method. Numerical errors and convergence rates for solutions obtained from the two-grid method are presented. CPU times for the two-grid algorithm are shown to be significantly less than those obtained by solving the fully coupled problem.

Key words: Navier-Stokes equations, Darcy's law, discontinuous Galerkin, finite element method, Beavers-Joseph-Saffman, decoupled and linearized problems, efficiency.

1 Introduction

The coupling of free flow with porous media flow has been studied extensively in the literature. Interest in this coupling is mostly due to the wide range of applications for example in biosciences, geosciences and industrial applications. The Navier-Stokes and Darcy coupling has been analyzed for example in [1, 2, 3, 4]. The Stokes and Darcy model has been studied for example in

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[5, 6, 7, 8, 9, 10]. The coupled Navier-Stokes and Darcy schemes studied in [1, 4] result in large non-linear coupled systems which may be time consuming to solve. In this paper, we apply the two-grid technique for both the coupled Stokes/Darcy and the Navier-Stokes/Darcy problems. We show that the two-grid method is attractive for these systems because it decouples the coupled model into two relatively smaller problems. The first is an Oseen problem (or Stokes problem) to solve for the fluid velocity and pressure in the free flow domain. The second is a modified Darcy problem to solve for the Darcy pressure in the porous medium. The two grid technique has several advantages. First it allows for easy parallelization. Second, legacy codes exist that have been optimized to solve the Stokes, Navier-Stokes or Darcy problems separately. These codes can be used independently with the two-grid approach. In this work, we use a multi-numeric scheme that combines the continuous finite element method with the discontinuous Galerkin method. The use of the latter in the subsurface offers many advantages: local mass conservation, local and high order accuracy, robustness of the method with respect to the heterogeneities of the medium. Recently, the two-grid method applied to a first order continuous finite element method is analyzed for solving the Stokes/Darcy [11] and Navier-Stokes/Darcy problems [12].

An outline of the paper is as follows. The mathematical model and its fully coupled numerical discretization are described in Section 2 and Section 3 respectively. The two-grid method is introduced in Section 4. It yields decoupled, linearized and smaller problems to solve. Numerical examples are shown in Section 5. Conclusions follow.

2 Fluid Flow Models

Let $\Omega \subset \mathbb{R}^d$ be a bounded polygonal domain subdivided into a finite number of non-overlapping subdomains Ω_1^i and Ω_2^i . The free flow domain is $\Omega_1 = \cup_i \Omega_1^i$ in which the flow is modelled by the Stokes or Navier-Stokes equations. The domain $\Omega_2 = \cup_i \Omega_2^i$ is the porous medium region where the flow is described by Darcy's Law. The free flow and porous media domains are separated by a polygonal interface Γ_{12} . First we state the equations that govern the flow in each domain. The flow in Ω_1 can be described by either the Stokes equations:

$$-\nabla \cdot (2\nu \mathbf{D}(\mathbf{u}_1)) + \nabla p_1 = \mathbf{f}_1, \quad \text{in } \Omega_1, \quad (1)$$

$$\nabla \cdot \mathbf{u}_1 = 0, \quad \text{in } \Omega_1, \quad (2)$$

$$\mathbf{u}_1 = 0, \quad \text{on } \Gamma_1 = \partial\Omega \cap \partial\Omega_1, \quad (3)$$

or the Navier-Stokes equations:

$$-\nabla \cdot (2\nu \mathbf{D}(\mathbf{u}_1)) + \nabla p_1 + \mathbf{u}_1 \cdot \nabla \mathbf{u}_1 = \mathbf{f}_1, \quad \text{in } \Omega_1, \quad (4)$$

$$\nabla \cdot \mathbf{u}_1 = 0, \quad \text{in } \Omega_1, \quad (5)$$

$$\mathbf{u}_1 = 0, \quad \text{on } \Gamma_1. \quad (6)$$

The fluid velocity and pressure in Ω_1 are denoted by $\mathbf{u}_1$ and p_1 respectively. The coefficient $\nu > 0$ is the fluid kinematic viscosity, the function $\mathbf{f}_1$ is an external force acting on the fluid, and $\mathbf{D}(\mathbf{u}_1)$ is the rate of strain matrix:

$$\mathbf{D}(\mathbf{u}_1) = \frac{1}{2}(\nabla \mathbf{u}_1 + \nabla \mathbf{u}_1^T). \quad (7)$$

The flow in Ω_2 is modeled by Darcy's Law:

$$-\nabla \cdot \mathbf{K} \nabla p_2 = f_2, \quad \text{in } \Omega_2, \quad (8)$$

$$-\mathbf{K} \nabla p_2 = \mathbf{u}_2, \quad \text{in } \Omega_2, \quad (9)$$

$$p_2 = g_D, \quad \text{on } \Gamma_{2D}, \quad (10)$$

$$\mathbf{K} \nabla p_2 \cdot \mathbf{n}_2 = g_N, \quad \text{on } \Gamma_{2N}. \quad (11)$$

We note that the boundary of the porous medium $\Gamma_2 = \partial\Omega \cap \partial\Omega_2$ is partitioned into disjoint sets Γ_{2N} and Γ_{2D} , the Neumann and Dirichlet boundary respectively. We assume that $|\Gamma_{2D}| > 0$. The fluid velocity and pressure in Ω_2 are denoted by $\mathbf{u}_2$ and p_2 respectively. The function f_2 is an external force acting on the fluid, the functions g_D and g_N are the prescribed value and flux respectively. The vector $\mathbf{n}_2$ denotes the unit vector normal to Γ_2 and the coefficient $\mathbf{K}$ is a symmetric positive definite tensor representing the hydraulic conductivity of the fluid.

The coupled models are completed by specifying the following coupling conditions on the interface Γ_{12} :

- (1) The continuity of the normal component of velocity arises from the incompressibility condition. Let $\mathbf{n}_{12}$ be the unit normal vector to Γ_{12} directed from Ω_1 to Ω_2 .

$$\mathbf{u}_1 \cdot \mathbf{n}_{12} = -\mathbf{K} \nabla p_2 \cdot \mathbf{n}_{12}. \quad (12)$$

- (2) The second condition is on the tangential component of the velocity in the free flow region. Let $\boldsymbol{\tau}_{12}$ be the unit tangent vector on the interface Γ_{12} then the Beavers-Joseph-Saffman law [13, 14, 15] will be written as:

$$\mathbf{u}_1 \cdot \boldsymbol{\tau}_{12} = -2\nu G(\mathbf{D}(\mathbf{u}_1) \mathbf{n}_{12}) \cdot \boldsymbol{\tau}_{12}. \quad (13)$$

- (3) The last condition relates to the balance of forces on the interface:

$$-2\nu \mathbf{D}(\mathbf{u}_1) \mathbf{n}_{12} \cdot \mathbf{n}_{12} + p_1 = p_2. \quad (14)$$

3 Numerical Schemes

Let $\mathcal{E}_h = \mathcal{E}_1^h \cup \mathcal{E}_2^h$ be a regular triangulation of the computational domain Ω with elements E [16]. The velocity and pressure solutions on Ω_1 belong to the conforming MINI finite element spaces $\mathbf{X}_1^h$ and M_1^h respectively. We recall that the space $\mathbf{X}_1^h$ consists of continuous piecewise linears plus cubic bubbles and the space M_1^h consists of continuous piecewise linears [17]. In Ω_1 , this yields a first order method. The Darcy pressure belongs to the space of discontinuous piecewise polynomials of degree $k_2 \geq 1$:

$$M_2^h = \{q_2 \in L^2(\Omega_2) : \forall E \in \mathcal{E}_2^h, \quad q_2|_E \in \mathbb{P}_{k_2}(E)\}.$$

We denote by Γ_2^h the set of interior edges of $\mathcal{E}_2^h$. In addition, a unit normal vector $\mathbf{n}_e$ is associated with each edge e . In the case that e lies on the boundary $\partial\Omega_2$, then $\mathbf{n}_e$ is taken to be the outward unit normal vector to $\partial\Omega_2$. For any two elements E_i and E_j (with $i < j$) of $\mathcal{E}_2^h$ that share a common edge e , we define the jump $[v]$ and average $\{v\}$ of a piecewise discontinuous polynomial v by

$$[v] = (v|_{E_i}) - (v|_{E_j}), \text{ and } \{v\} = \frac{1}{2}(v|_{E_i}) + \frac{1}{2}(v|_{E_j})$$

respectively. The choice of the coupling of the continuous finite element method (FEM) with the discontinuous Galerkin method is due to the fact that the continuous finite element method is adequate for the free flow region for problems considered in this work. On the other hand, the discontinuous Galerkin method is better suited to handle heterogeneities such as kinks and faults that arise in the porous medium. We refer the reader to [18] for a detailed discussion on this matter. In order to define the numerical scheme for the Stokes/Darcy and Navier-Stokes/Darcy couplings, we first introduce the following bilinear forms.

- For the discretization of the viscous term in the Stokes or Navier-Stokes equations:

$$\forall \mathbf{v}, \mathbf{w} \in \mathbf{X}_1^h, \quad a_S(\mathbf{v}, \mathbf{w}) = 2\nu(\mathbf{D}(\mathbf{v}), \mathbf{D}(\mathbf{w}))_{\Omega_1}.$$

- For the discretization of the pressure term in the Stokes or Navier-Stokes equations:

$$\forall \mathbf{v} \in \mathbf{X}_1^h, \forall q_1 \in M_1^h, \quad b_S(\mathbf{v}, q_1) = -(q_1, \nabla \cdot \mathbf{v})_{\Omega_1}.$$

- For the discretization of the non-linear term in the Navier-Stokes equations:

$$\begin{aligned} & \forall \mathbf{z}, \mathbf{v}, \mathbf{w} \in \mathbf{X}_1^h, \\ c_{NS}(\mathbf{z}, \mathbf{v}, \mathbf{w}) &= \frac{1}{2}(\mathbf{z} \cdot \nabla \mathbf{v}, \mathbf{w})_{\Omega_1} - \frac{1}{2}(\mathbf{z} \cdot \nabla \mathbf{w}, \mathbf{v})_{\Omega_1} + \frac{1}{2}(\mathbf{z} \cdot \mathbf{n}_{12}, \mathbf{v} \cdot \mathbf{w})_{\Gamma_{12}}. \end{aligned}$$

- For the discretization of the diffusion term in Darcy's Law:

$$\begin{aligned} \forall q_2, t_2 \in M_2^h, \quad a_D(q_2, t_2) &= \sum_{E \in \mathcal{E}_h^2} (\mathbf{K} \nabla q_2, \nabla t_2)_E + \sum_{e \in \Gamma_2^h \cup \Gamma_{2D}} \frac{\sigma}{|e|} ([q_2], [t_2])_e \\ &- \sum_{e \in \Gamma_2^h \cup \Gamma_{2D}} (\{\mathbf{K} \nabla q_2 \cdot \mathbf{n}_e\} [t_2])_e + \epsilon_2 \sum_{e \in \Gamma_2^h \cup \Gamma_{2D}} (\{\mathbf{K} \nabla t_2 \cdot \mathbf{n}_e\}, [q_2])_e. \end{aligned}$$

The parameters σ and ϵ_2 are the penalty and symmetrization parameters respectively. The case $\epsilon_2 = -1$ yields the Symmetric Interior Penalty Galerkin (SIPG) method whereas the case $\epsilon_2 = 1$ yields the Non-Symmetric Interior Penalty Galerkin (NIPG) method [19]. We also note that the form a_D depends strongly on the underlying mesh. We may use the notation $a_D = a_{D,h}$ in the definition of the two-grid method to avoid any confusion.

- For the discretization of the interface terms:

$$\begin{aligned} \forall \mathbf{v}, \mathbf{w} \in \mathbf{X}_1^h, \quad \forall q_2, t_2 \in M_2^h, \quad \gamma(\mathbf{v}, q_2; \mathbf{w}, t_2) &= (q_2, \mathbf{w} \cdot \mathbf{n}_{12})_{\Gamma_{12}} - (\mathbf{v} \cdot \mathbf{n}_{12}, t_2)_{\Gamma_{12}} \\ &+ \frac{1}{G} (\mathbf{v} \cdot \boldsymbol{\tau}_{12}, \mathbf{w} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}}. \end{aligned}$$

- For the discretization of the data terms:

$$\begin{aligned} \forall t_2 \in M_2^h, \forall \mathbf{v} \in \mathbf{X}_1^h, \quad L(\mathbf{v}, t_2) &= (\mathbf{f}_1, \mathbf{v})_{\Omega_1} + (f_2, t_2)_{\Omega_2} \\ &+ \epsilon_2 \sum_{e \in \Gamma_{2D}} (\mathbf{K} \nabla t_2 \cdot \mathbf{n}_e, g_D)_e + \sum_{e \in \Gamma_{2D}} \frac{\sigma}{|e|} (g_D, t_2)_e + \sum_{e \in \Gamma_{2N}} (g_N, t_2)_e. \end{aligned}$$

We now define the numerical schemes for the coupled Stokes/Darcy and coupled Navier-Stokes/Darcy problems (W_S^h) and (W_{NS}^h) respectively:

$$\begin{aligned} (W_S^h) \quad &\left\{ \begin{array}{l} \text{Find } \mathbf{U}_1^h \in \mathbf{X}_1^h, P_1^h \in M_1^h, P_2^h \in M_2^h, \text{ s.t} \\ \forall \mathbf{v}_1 \in \mathbf{X}_1^h, q_2 \in M_2^h, a_S(\mathbf{U}_1^h, \mathbf{v}_1) + b_S(\mathbf{v}_1, P_1^h) + a_D(P_2^h, q_2) \\ + \gamma(\mathbf{U}_1^h, P_2^h; \mathbf{v}_1, q_2) = L(\mathbf{v}_1, q_2), \\ \forall q_1 \in M_1^h \quad b_S(\mathbf{U}_1^h, q_1) = 0, \end{array} \right. \\ (W_{NS}^h) \quad &\left\{ \begin{array}{l} \text{Find } \mathbf{U}_1^h \in \mathbf{X}_1^h, P_1^h \in M_1^h, P_2^h \in M_2^h, \text{ s.t} \\ \forall \mathbf{v}_1 \in \mathbf{X}_1^h, q_2 \in M_2^h, a_S(\mathbf{U}_1^h, \mathbf{v}_1) + b_S(\mathbf{v}_1, P_1^h) + c_{NS}(\mathbf{U}_1^h; \mathbf{U}_1^h, \mathbf{v}_1) \\ + a_D(P_2^h, q_2) + \gamma(\mathbf{U}_1^h, P_2^h; \mathbf{v}_1, q_2) = L(\mathbf{v}_1, q_2), \\ \forall q_1 \in M_1^h \quad b_S(\mathbf{U}_1^h, q_1) = 0. \end{array} \right. \end{aligned}$$

In order to solve the coupled problem (W_S^h) , we have to assemble a large linear system of dimension $\dim \mathbf{X}_1^h + \dim M_1^h + \dim M_2^h$. For realistic domains

this system can be very large and time consuming to solve. Similarly, solving (W_{NS}^h) is time consuming as it involves a Picard iteration and large linear systems. In the next section we present a two-grid algorithm that reduces the computational cost by decoupling the coupled problems into two relatively smaller problems in the free flow and porous media domains.

4 Two Grid Algorithm

The objective of the two-grid method is to decouple the fully coupled problem into separate problems in the free flow and porous media domains. The technique involves two steps. First, a fully coupled problem, with relatively small cost, is solved on a coarse mesh $\mathcal{E}_H$ of size H . Second, the surface problem and subsurface problems are solved in parallel and independently on a fine mesh $\mathcal{E}_h$ of size h . The two-grid algorithm is summarized below for both linear and nonlinear cases.

4.1 Two-grid algorithm for coupled Stokes/Darcy problem

- (1) First we solve a fully coupled Stokes and Darcy problem on a coarse grid of size H :

$$(W_S^H) \left\{ \begin{array}{l} \text{Find } \mathbf{U}_1^H \in \mathbf{X}_1^H, P_1^H \in M_1^H, P_2^H \in M_2^H, \text{ s.t.} \\ \forall \mathbf{v}_1 \in \mathbf{X}_1^H, \forall q_2 \in M_2^H, \quad a_S(\mathbf{U}_1^H, \mathbf{v}_1) + b_S(\mathbf{v}_1, P_1^H) + a_{D,H}(P_2^H, q_2) \\ + \gamma(\mathbf{U}_1^H, P_2^H; \mathbf{v}_1, q_2) = L(\mathbf{v}_1, q_2), \\ \forall q_1 \in M_1^H, \quad b_S(\mathbf{U}_1^H, q_1) = 0. \end{array} \right.$$

- (2) We then solve a decoupled problem on a fine grid of size h :

$$(W_{h,\text{SD}}) \left\{ \begin{array}{l} \text{Find } \mathbf{U}_{h,1} \in \mathbf{X}_1^h, P_{h,1} \in M_1^h, P_{h,2} \in M_2^h, \text{ s.t.} \\ \forall \mathbf{v}_1 \in \mathbf{X}_1^h, \forall q_2 \in M_2^h, \quad a_S(\mathbf{U}_{h,1}, \mathbf{v}_1) + \frac{1}{G}(\mathbf{U}_{h,1} \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} + b_S(\mathbf{v}_1, P_{h,1}) \\ + a_{D,h}(P_{h,2}, q_2) = L(\mathbf{v}_1, q_2) - (P_2^H, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}} + (\mathbf{U}_1^H \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}}, \\ \forall q_1 \in M_1^h, \quad b_S(\mathbf{U}_{h,1}, q_1) = 0. \end{array} \right.$$

The decoupled fine grid problem is equivalent to solving modified Stokes and Darcy problems in Ω_1 and Ω_2 respectively. The modified problems can be

stated as follows: in Ω_1 ,

$$(W_{h,S}) \begin{cases} \text{Find } \mathbf{U}_{h,1} \in \mathbf{X}_1^h, P_{h,1} \in M_1^h, \text{ s.t.} \\ \forall \mathbf{v}_1 \in \mathbf{X}_1^h, \forall q_2 \in M_2^h, \quad a_S(\mathbf{U}_{h,1}, \mathbf{v}_1) + b_S(\mathbf{v}_1, P_{h,1}) + \frac{1}{G}(\mathbf{U}_{h,1} \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} \\ = (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} - (P_2^H, \mathbf{v}_1 \cdot \mathbf{n}_{12})_{\Gamma_{12}}, \\ \forall q_1 \in M_1^h, \quad b_S(\mathbf{U}_{h,1}, q_1) = 0, \end{cases}$$

and in Ω_2 ,

$$(W_{h,D}) \begin{cases} \text{Find } P_{h,2} \in M_2^h, \text{ s.t.} \\ \forall q_2 \in M_2^h, \quad a_{D,h}(P_{h,2}, q_2) = (f_2, q_2)_{\Omega_2} + (\mathbf{U}_1^H \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} + (g_N, q_2)_{\Gamma_{2N}} \\ + \epsilon_2 \sum_{e \in \Gamma_{2D}} (\mathbf{K} \nabla q_2 \cdot \mathbf{n}_e, g_D)_e + \sum_{e \in \Gamma_{2D}} \frac{\sigma}{|e|} (g_D, q_2)_e. \end{cases}$$

4.2 Two-grid algorithm for coupled Navier-Stokes and Darcy problem

(1) First we solve a nonlinear problem on a coarse grid of size H :

$$(W_{NS}^H) \begin{cases} \text{Find } \mathbf{U}_1^H \in \mathbf{X}_1^H, P_1^H \in M_1^H, P_2^H \in M_2^H, \text{ s.t.} \\ \forall \mathbf{v}_1 \in \mathbf{X}_1^H, \forall q_2 \in M_2^H, \quad a_S(\mathbf{U}_1^H, \mathbf{v}_1) + b_S(\mathbf{v}_1, P_1^H) + c_{NS}(\mathbf{U}_1^H; \mathbf{U}_1^H; \mathbf{v}_1) \\ + a_{D,H}(P_2^H, q_2) + \gamma(\mathbf{U}_1^H, P_2^H; \mathbf{v}_1, q_2) = L(\mathbf{v}_1, q_2), \\ \forall q_1 \in M_1^H, \quad b_S(\mathbf{U}_1^H, q_1) = 0. \end{cases}$$

The solution $(\mathbf{U}_1^H, P_1^H, P_2^H)$ for the problem (W_{NS}^H) is obtained using a Picard iteration. At each iteration, given $\mathbf{U}_1^{H,n-1}$, we solve for $(\mathbf{U}_1^{H,n}, P_1^H, P_2^H)$ satisfying:

$$\begin{aligned} \forall \mathbf{v}_1 \in \mathbf{X}_1^H, \forall q_2 \in M_2^H, \quad & a_S(\mathbf{U}_1^{H,n}, \mathbf{v}_1) + b_S(\mathbf{v}_1, P_1^H) + c_{NS}(\mathbf{U}_1^{H,n-1}; \mathbf{U}_1^{H,n}; \mathbf{v}_1) \\ & + a_{D,H}(P_2^H, q_2) + \gamma(\mathbf{U}_1^{H,n}, P_2^H; \mathbf{v}_1, q_2) = L(\mathbf{v}_1, q_2), \\ & \forall q_1 \in M_1^H, \quad b_S(\mathbf{U}_1^{H,n}, q_1) = 0. \end{aligned}$$

We use the stopping criterion: $\|\mathbf{U}_1^{H,n} - \mathbf{U}_1^{H,n-1}\|_{L^2(\Omega_1)} \leq \delta$, where δ is a given tolerance.

(2) We then solve a decoupled problem on a fine grid of size h :

$$(W_{h,\text{NSD}}) \left\{ \begin{array}{l} \text{Find } \mathbf{U}_{h,1} \in \mathbf{X}_1^h, P_{h,1} \in M_1^h, P_{h,2} \in M_2^h, \text{ s.t.} \\ \forall \mathbf{v}_1 \in \mathbf{X}_1^h, \forall q_2 \in M_2^h, \quad a_S(\mathbf{U}_{h,1}, \mathbf{v}_1) + b_S(\mathbf{v}_1, P_{h,1}) + c_{\text{NS}}(\mathbf{U}_1^H; \mathbf{U}_{h,1}, \mathbf{v}_1) \\ + \frac{1}{G}(\mathbf{U}_{h,1} \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} + a_{\text{D},h}(P_{h,2}, q_2) \\ = L(\mathbf{v}_1, q_2) - (P_2^H, \mathbf{v}_1 \cdot \mathbf{n})_{\Gamma_{12}} + (\mathbf{U}_1^H \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}}, \\ \forall q_1 \in M_1^h, \quad b_S(\mathbf{U}_{h,1}, q_1) = 0. \end{array} \right.$$

We note that the resulting two-grid problem $(W_{h,\text{NSD}})$ does not require a Picard iteration because the non-linear term c_{NS} has been linearized. The decoupled fine grid problem is equivalent to solving modified Oseen and Darcy problems in Ω_1 and Ω_2 respectively. The modified problems can be stated as follows: in Ω_1 ,

$$(W_{h,\text{NS}}) \left\{ \begin{array}{l} \text{Find } \mathbf{U}_{h,1} \in \mathbf{X}_1^h, P_{h,1} \in M_1^h, \text{ s.t.} \\ \forall \mathbf{v}_1 \in \mathbf{X}_1^h, \forall q_2 \in M_2^h, \quad a_S(\mathbf{U}_{h,1}, \mathbf{v}_1) + b_S(\mathbf{v}_1, P_{h,1}) + c_{\text{NS}}(\mathbf{U}_1^H; \mathbf{U}_{h,1}, \mathbf{v}_1) \\ + \frac{1}{G}(\mathbf{U}_{h,1} \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} = (\mathbf{f}_1, \mathbf{v}_1)_{\Omega_1} - (P_2^H, \mathbf{v}_1 \cdot \mathbf{n})_{\Gamma_{12}}, \\ \forall q_1 \in M_1^h, \quad b_S(\mathbf{U}_{h,1}, q_1) = 0, \end{array} \right.$$

and in Ω_2 ,

$$(W_{h,\text{D}}) \left\{ \begin{array}{l} \text{Find } P_{h,2} \in M_2^h, \text{ s.t.} \\ \forall q_2 \in M_2^h, \quad a_{\text{D},h}(P_{h,2}, q_2) = (f_2, q_2)_{\Omega_2} + (\mathbf{U}_1^H \cdot \mathbf{n}_{12}, q_2)_{\Gamma_{12}} + (g_N, q_2)_{\Gamma_{2N}} \\ + \epsilon_2 \sum_{e \in \Gamma_{2D}} (\mathbf{K} \nabla q_2 \cdot \mathbf{n}_e, g_D)_e + \sum_{e \in \Gamma_{2D}} \frac{\sigma}{|e|} (g_D, q_2)_e. \end{array} \right.$$

Remark 1 For all schemes described above, the Darcy velocity is obtained by the following formula:

$$\mathbf{U}_{h,2} = -\mathbf{K} \nabla P_{h,2}.$$

4.3 Convergence Result

Under the assumption that the solution to the dual coupled problem is regular enough, one can show that there exists a constant C independent of H and h such that

$$\left(\sum_{E \in \mathcal{E}_2^h} \|\nabla(P_2^h - P_{h,2})\|_{L^2(E)}^2 + \sum_{e \in \Gamma_h^2} \frac{1}{|e|} \|[P_2^h - P_{h,2}]\|_{L^2(e)}^2 \right)^{1/2} \leq CH^2,$$

$$\|\mathbf{D}(\mathbf{U}_1^h - \mathbf{U}_{h,1})\|_{L^2(\Omega_1)} + \|P_1^h - P_{h,1}\|_{L^2(\Omega_1)} \leq CH^{3/2}.$$

Therefore, if one chooses $h = H^2$, we obtain an overall theoretical rate $\mathcal{O}(h^{3/4})$. In the following section, we study the numerical convergence and we observe improved numerical rates $\mathcal{O}(h)$.

5 Numerical Examples

Fig. 1. Computational meshes $H = 0.5, h = 0.25$.

We compute numerical convergence rates for the two-grid algorithm applied to both the coupled Stokes/Darcy and coupled Navier-Stokes/Darcy problems. We also compare the accuracy of the two-grid solution with the solution obtained by solving the fully coupled problem. The computational domain $\Omega \subset \mathbb{R}^2$ is subdivided into $\Omega_1 = (0, 1) \times (0, 1)$ and $\Omega_2 = (0, 1) \times (1, 2)$. For each given mesh of size h , we solve the fully coupled problem on a coarse mesh of size $H = \sqrt{h}$ and the decoupled problems on a fine mesh of size h . An example of such meshes is given in Fig. 1. The boundary conditions are chosen in such a way that the exact solution to the coupled problem for both models is:

$$\mathbf{u}_1(x, y) = (y^2 - 2y + 2x, x^2 - x - 2y), \quad p_1(x, y) = x^2y + xy + y^2 - 4.0,$$

$$p_2(x, y) = -x^2y + xy + y^2.$$

For simplicity we set $\mathbf{K} = \mathbf{I}$ and $\nu = 1$.

5.1 Convergence Study for Decoupled Stokes and Darcy problem

In this section we compare the accuracy of the two-grid method for the coupled Stokes and Darcy model. First we solve the fully coupled problem on a sequence of fine meshes of size h . For the subsurface part, the NIPG method is used with penalty $\sigma = 1$ and with order of approximation $k_2 = 1$. Table 1 shows the errors and convergence rates for the fully coupled problem. As expected from the theory [4] we observe order one convergence rate in the Stokes and Darcy velocities in the energy norm. The Stokes pressure has a slightly higher convergence rate than the expected order one. We also show the rates for the Stokes velocity and Darcy pressure in the L^2 norm. We recover the optimal rate of two.

h	$\ \mathbf{U}_1^h - \mathbf{u}_1\ _{L^2(\Omega_1)}$	$\ P_1^h - p_1\ _{L^2(\Omega_1)}$	$\ \mathbf{D}(\mathbf{U}_1^h - \mathbf{u}_1)\ _{L^2(\Omega_1)}$	$\ P_2^h - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_2^h - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	1.591×10^{-2}	6.346×10^{-2}	1.365×10^{-1}	1.341×10^{-2}	1.637×10^{-1}
$\frac{1}{16}$	9.984×10^{-4}	6.645×10^{-3}	3.390×10^{-2}	8.210×10^{-4}	3.856×10^{-2}
$\frac{1}{64}$	6.227×10^{-5}	7.363×10^{-4}	8.469×10^{-3}	5.150×10^{-5}	9.552×10^{-3}
rate (h)	2.00	1.58	1.00	2.00	1.00

Table 1

Errors and rates for (W_S^h) (FEM/NIPG $\sigma = 1, k_2 = 1$).

We then solve problem $(W_{h,SD})$ on the same sequence of meshes as in Table 1. Tables 2 and 3 show the errors and convergence rates for $(W_{h,SD})$ using the NIPG method with $\sigma = 1$ and the SIPG method with $\sigma = 6$ in Ω_2 respectively. First we note that the errors obtained from the NIPG and SIPG solutions are comparable. For the SIPG method the expected convergence rate is obtained by making the penalty parameter large enough. This is due to the fact that the coercivity of the bilinear form a_D for the SIPG method depends on the choice of the penalty parameter. From [20], we choose $\sigma = k_2(k_2 + 1) \cot(\theta)$, where θ is the smallest angle over all triangles in the mesh. We obtain order one convergence for the errors in the energy norm in the parameter h for both methods. The order of convergence is better than the predicted theoretical convergence rate of $h^{3/4}$. The errors are comparable to those obtained by solving the fully coupled problem (Table 1) for both methods. However the Stokes pressure error is relatively larger for the two-grid problem compared to the fully coupled problem.

Next we increase the degree of approximation in Ω_2 to two. Table 4 and 5 show the errors and convergence rates for the two grid algorithm using the NIPG method with $\sigma = 0$ and SIPG method with $\sigma = 18$ in Ω_2 respectively. Increasing the degree of approximation in Ω_2 improves the quality of the Darcy pressure solution. For example on a mesh of size $\frac{1}{64}$ the error in the Darcy

h	H	$\ U_{h,1} - u_1\ _{L^2(\Omega_1)}$	$\ P_{h,1} - p_1\ _{L^2(\Omega_1)}$	$\ D(U_{h,1} - u_1)\ _{L^2(\Omega_1)}$	$\ P_{h,2} - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_{h,2} - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	$\frac{1}{2}$	1.599×10^{-2}	6.647×10^{-2}	1.364×10^{-1}	1.235×10^{-2}	1.657×10^{-1}
$\frac{1}{16}$	$\frac{1}{4}$	9.984×10^{-4}	1.571×10^{-2}	3.397×10^{-2}	1.018×10^{-3}	3.923×10^{-2}
$\frac{1}{64}$	$\frac{1}{8}$	6.073×10^{-5}	4.349×10^{-3}	8.473×10^{-3}	2.699×10^{-4}	9.638×10^{-3}
rate (h)	-	2.00	0.92	1.00	0.95	1.00

Table 2

Errors and rates for $(W_{h,SD})$ (FEM/NIPG $\sigma = 1, k_2 = 1$).

h	H	$\ U_{h,1} - u_1\ _{L^2(\Omega_1)}$	$\ P_{h,1} - p_1\ _{L^2(\Omega_1)}$	$\ D(U_{h,1} - u_1)\ _{L^2(\Omega_1)}$	$\ P_{h,2} - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_{h,2} - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	$\frac{1}{2}$	1.594×10^{-2}	7.383×10^{-2}	1.363×10^{-1}	8.788×10^{-3}	1.703×10^{-1}
$\frac{1}{16}$	$\frac{1}{4}$	1.002×10^{-3}	1.489×10^{-2}	3.397×10^{-2}	1.471×10^{-3}	4.137×10^{-2}
$\frac{1}{64}$	$\frac{1}{8}$	6.616×10^{-5}	4.166×10^{-3}	8.473×10^{-3}	3.115×10^{-4}	1.011×10^{-2}
rate (h)	-	2.00	0.92	1.00	1.12	1.00

Table 3

Errors and rates for $(W_{h,SD})$ (FEM/SIPG $\sigma = 6, k_2 = 1$).

velocity in Table 4 is about seven times smaller compared to Tables 2 and 3. The Stokes velocity error is unchanged from the case $k_2 = 1$ to $k_2 = 2$. The Stokes pressure error decreases by a factor of four.

h	H	$\ U_{h,1} - u_1\ _{L^2(\Omega_1)}$	$\ P_{h,1} - p_1\ _{L^2(\Omega_1)}$	$\ D(U_{h,1} - u_1)\ _{L^2(\Omega_1)}$	$\ P_{h,2} - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_{h,2} - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	$\frac{1}{2}$	1.599×10^{-2}	6.577×10^{-2}	1.363×10^{-1}	5.134×10^{-3}	2.556×10^{-2}
$\frac{1}{16}$	$\frac{1}{4}$	9.928×10^{-4}	7.708×10^{-3}	3.396×10^{-2}	1.213×10^{-3}	5.654×10^{-3}
$\frac{1}{64}$	$\frac{1}{8}$	6.258×10^{-5}	1.304×10^{-3}	8.470×10^{-3}	2.981×10^{-4}	1.378×10^{-3}
rate (h)	-	2.00	1.28	1.00	1.00	1.00

Table 4

Errors and rates for $(W_{h,SD})$ (FEM/NIPG $\sigma = 0, k_2 = 2$).

h	H	$\ U_{h,1} - u_1\ _{L^2(\Omega_1)}$	$\ P_{h,1} - p_1\ _{L^2(\Omega_1)}$	$\ D(U_{h,1} - u_1)\ _{L^2(\Omega_1)}$	$\ P_{h,2} - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_{h,2} - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	$\frac{1}{2}$	1.599×10^{-2}	6.470×10^{-2}	1.364×10^{-1}	4.965×10^{-3}	2.470×10^{-2}
$\frac{1}{16}$	$\frac{1}{4}$	9.927×10^{-4}	7.253×10^{-3}	3.395×10^{-2}	1.192×10^{-3}	5.585×10^{-3}
$\frac{1}{64}$	$\frac{1}{8}$	6.236×10^{-5}	1.124×10^{-3}	8.473×10^{-3}	2.968×10^{-4}	1.374×10^{-3}
rate (h)	-	2.00	1.34	1.00	1.00	1.00

Table 5

Errors and rates for $(W_{h,SD})$ (FEM/SIPG $\sigma = 18, k_2 = 2$).

Further, we increase the degree of approximation in Ω_2 to three. Table 6 shows the errors and convergence rates for this case. We note that the order of approximation remains one in the Stokes domain and increasing the order to three does not have any further improvement in the accuracy of the solution in both domains. This is expected because the order of approximation in Ω_1 remains one.

It is clear that the two-grid method performs well for the coupled Stokes and Darcy problem. The real advantage of this method will be shown in the case of the coupled Navier-Stokes and Darcy problem because we avoid solving a non-linear problem on the fine mesh.

h	H	$\ \mathbf{U}_{h,1} - \mathbf{u}_1\ _{L^2(\Omega_1)}$	$\ P_{h,1} - p_1\ _{L^2(\Omega_1)}$	$\ \mathbf{D}(\mathbf{U}_{h,1} - \mathbf{u}_1)\ _{L^2(\Omega_1)}$	$\ P_{h,2} - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_{h,2} - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	$\frac{1}{2}$	1.599×10^{-2}	6.417×10^{-2}	1.363×10^{-1}	4.942×10^{-3}	2.328×10^{-2}
$\frac{1}{16}$	$\frac{1}{4}$	9.926×10^{-4}	7.196×10^{-3}	3.395×10^{-2}	1.191×10^{-3}	5.579×10^{-3}
$\frac{1}{64}$	$\frac{1}{8}$	6.233×10^{-5}	1.119×10^{-3}	8.470×10^{-3}	2.967×10^{-4}	1.375×10^{-3}
rate (h)	-	2.00	1.34	1.00	1.00	1.00

Table 6

Errors and rates for $(W_{h,SD})$ (FEM/NIPG $\sigma = 0, k_2 = 3$).

5.2 Convergence Study for Decoupled Navier-Stokes and Darcy problem

First we solve the fully coupled problem (W_{NS}^h) on a sequence of meshes $\mathcal{E}_h$.

Table 7 shows the errors and convergence rates for the solution from the fully

h	$\ \mathbf{U}_1^h - \mathbf{u}_1\ _{L^2(\Omega_1)}$	$\ P_1^h - p_1\ _{L^2(\Omega_1)}$	$\ \mathbf{D}(\mathbf{U}_1^h - \mathbf{u}_1)\ _{L^2(\Omega_1)}$	$\ P_2^h - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_2^h - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	1.598×10^{-2}	6.889×10^{-2}	1.366×10^{-1}	2.598×10^{-2}	1.625×10^{-1}
$\frac{1}{16}$	9.988×10^{-4}	6.754×10^{-3}	3.390×10^{-2}	1.592×10^{-3}	3.841×10^{-2}
$\frac{1}{64}$	6.232×10^{-5}	7.403×10^{-4}	8.467×10^{-3}	9.948×10^{-5}	9.541×10^{-3}
rate (h)	2.00	1.59	1.00	2.00	1.00

Table 7

Errors and rates for (W_{NS}^h) (FEM/NIPG $\sigma = 1, k_2 = 1$).

coupled Navier-Stokes and Darcy problem solved using a Picard iteration with a tolerance set at 10^{-7} . We observe order one convergence rate in the energy norm for the Navier-Stokes velocity and Darcy pressure. We then compare the solution from the fully coupled problem to the decoupled Navier-Stokes and Darcy problems $(W_{h,NSD})$. From Table 8 we note that the errors from

h	H	$\ \mathbf{U}_{h,1} - \mathbf{u}_1\ _{L^2(\Omega_1)}$	$\ P_{h,1} - p_1\ _{L^2(\Omega_1)}$	$\ \mathbf{D}(\mathbf{U}_{h,1} - \mathbf{u}_1)\ _{L^2(\Omega_1)}$	$\ P_{h,2} - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_{h,2} - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	$\frac{1}{2}$	1.594×10^{-2}	9.7277×10^{-2}	1.365×10^{-1}	4.258×10^{-2}	1.708×10^{-1}
$\frac{1}{16}$	$\frac{1}{4}$	9.696×10^{-4}	1.728×10^{-2}	3.399×10^{-2}	7.068×10^{-3}	4.029×10^{-2}
$\frac{1}{64}$	$\frac{1}{8}$	7.137×10^{-5}	3.961×10^{-3}	8.477×10^{-3}	1.567×10^{-3}	9.896×10^{-3}
rate (h)	-	2.00	1.00	1.00	1.20	1.20

Table 8

Errors and rates for $(W_{h,NSD})$ (FEM/NIPG $\sigma = 1, k_2 = 1$).

the two-grid method are comparable to the fully coupled problem for the Navier-Stokes and Darcy velocities in Table 7. The convergence rates are order one for the velocity and pressure in both domains. This is better than the predicted order $h^{3/4}$ for the case $H = \sqrt{h}$. We note that the same improved rate has been reported in [11, 12]. Next, we increase the order of approximation in Ω_2 to $k_2 = 2$. From Table 9 as expected the Darcy solution is more accurate compared to the order one approximation, the convergence rate in the velocity increases to two. In Table 10 we observe that the two grid solution with $k_2 = 2$ is more accurate compared to the one with $k_2 = 1$ in Table 8. The Darcy velocity error decreases by a factor of almost 4, this in-turn results in an improvement in the Navier-Stokes velocity and pressure solutions in Ω_1 . However, due to the fact that the order of approximation in Ω_1 is one only,

h	$\ \mathbf{U}_1^h - \mathbf{u}_1\ _{L^2(\Omega_1)}$	$\ P_1^h - p_1\ _{L^2(\Omega_1)}$	$\ \mathbf{D}(\mathbf{U}_1^h - \mathbf{u}_1)\ _{L^2(\Omega_1)}$	$\ P_2^h - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_2^h - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	1.599×10^{-2}	6.687×10^{-2}	1.366×10^{-1}	7.449×10^{-3}	1.463×10^{-2}
$\frac{1}{16}$	9.994×10^{-4}	6.685×10^{-3}	3.390×10^{-2}	4.953×10^{-4}	9.275×10^{-4}
$\frac{1}{64}$	6.232×10^{-5}	7.379×10^{-4}	8.467×10^{-3}	3.1225×10^{-5}	5.812×10^{-5}
rate (h)	2.00	1.59	1.00	2.00	2.00

Table 9

Errors and rates for (W_{NS}^h) (FEM/NIPG $\sigma = 0, k_2 = 2$).

h	H	$\ \mathbf{U}_{h,1} - \mathbf{u}_1\ _{L^2(\Omega_1)}$	$\ P_{h,1} - p_1\ _{L^2(\Omega_1)}$	$\ \mathbf{D}(\mathbf{U}_{h,1} - \mathbf{u}_1)\ _{L^2(\Omega_1)}$	$\ P_{h,2} - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_{h,2} - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	$\frac{1}{2}$	1.594×10^{-2}	7.758×10^{-2}	1.364×10^{-1}	2.562×10^{-2}	1.708×10^{-1}
$\frac{1}{16}$	$\frac{1}{4}$	9.674×10^{-4}	1.278×10^{-2}	3.397×10^{-2}	6.131×10^{-3}	1.057×10^{-2}
$\frac{1}{64}$	$\frac{1}{8}$	6.698×10^{-5}	2.882×10^{-3}	8.474×10^{-3}	1.511×10^{-5}	2.616×10^{-3}
rate (h)	-	2.00	1.00	1.00	1.20	1.20

Table 10

Errors and rates for $(W_{h,\text{NSD}})$ (FEM/NIPG $\sigma = 0, k_2 = 2$).

the errors in Table 10 are larger compared to the coupled problem in Table 9. The convergence rate for the Navier-Stokes velocity and pressure in Ω_1 is order one, the same holds true for the Darcy velocity in Ω_2 . We also present results for the SIPG method below. As we noted in the case with the Stokes and Darcy coupling the penalty parameter σ has to be chosen to be large enough in order to observe the expected convergence rate. Table 11 shows

h	$\ \mathbf{U}_1^h - \mathbf{u}_1\ _{L^2(\Omega_1)}$	$\ P_1^h - p_1\ _{L^2(\Omega_1)}$	$\ \mathbf{D}(\mathbf{U}_1^h - \mathbf{u}_1)\ _{L^2(\Omega_1)}$	$\ P_2^h - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_2^h - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	1.600×10^{-2}	6.681×10^{-2}	1.366×10^{-1}	5.060×10^{-2}	1.612×10^{-1}
$\frac{1}{16}$	1.000×10^{-3}	6.682×10^{-3}	3.390×10^{-2}	4.000×10^{-4}	3.9791×10^{-2}
$\frac{1}{64}$	6.232×10^{-5}	7.378×10^{-4}	8.467×10^{-3}	2.618×10^{-5}	9.974×10^{-3}
rate (h)	2.00	1.59	1.00	2.00	1.00

Table 11

Errors and rates for (W_{NS}^h) (FEM/SIPG $\sigma = 6, k_2 = 1$).

h	H	$\ \mathbf{U}_{h,1} - \mathbf{u}_1\ _{L^2(\Omega_1)}$	$\ P_{h,1} - p_1\ _{L^2(\Omega_1)}$	$\ \mathbf{D}(\mathbf{U}_{h,1} - \mathbf{u}_1)\ _{L^2(\Omega_1)}$	$\ P_{h,2} - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_{h,2} - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	$\frac{1}{2}$	1.594×10^{-2}	7.931×10^{-2}	1.364×10^{-1}	5.289×10^{-2}	1.661×10^{-1}
$\frac{1}{16}$	$\frac{1}{4}$	9.747×10^{-4}	1.278×10^{-2}	3.397×10^{-2}	5.714×10^{-3}	4.153×10^{-2}
$\frac{1}{64}$	$\frac{1}{8}$	6.350×10^{-5}	2.842×10^{-3}	8.474×10^{-3}	1.482×10^{-3}	1.029×10^{-2}
rate (h)	-	2.00	1.00	1.00	1.20	1.20

Table 12

Errors and rates for $(W_{h,\text{NSD}})$ (FEM/SIPG $\sigma = 6, k_2 = 1$).

the errors and convergence rate for the fully coupled problem with $k_2 = 1$ using the SIPG method in Ω_2 . Table 12 shows the errors and convergence rate from the two grid method on the same meshes. We observe the same trend as in the case with the NIPG method above. The errors for the two-grid method algorithm converge with order one and are comparable to the fully coupled problem.

Tables 13 and 14 show the errors and convergence rates for the fully coupled Navier-Stokes and Darcy model and the two-grid method respectively

h	$\ U_1^h - \mathbf{u}_1\ _{L^2(\Omega_1)}$	$\ P_1^h - p_1\ _{L^2(\Omega_1)}$	$\ D(U_1^h - \mathbf{u}_1)\ _{L^2(\Omega_1)}$	$\ P_2^h - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_2^h - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	1.599×10^{-2}	6.604×10^{-2}	1.366×10^{-1}	6.021×10^{-3}	1.301×10^{-2}
$\frac{1}{16}$	9.993×10^{-4}	6.668×10^{-3}	3.390×10^{-2}	3.759×10^{-4}	8.183×10^{-4}
$\frac{1}{64}$	6.233×10^{-5}	7.373×10^{-4}	8.469×10^{-3}	2.349×10^{-5}	5.123×10^{-5}
rate (h)	2.00	1.59	1.00	2.00	1.00

Table 13

Errors and rates for (W_{NS}^h) (FEM/SIPG $\sigma = 18, k_2 = 2$).

h	H	$\ U_{h,1} - \mathbf{u}_1\ _{L^2(\Omega_1)}$	$\ P_{h,1} - p_1\ _{L^2(\Omega_1)}$	$\ D(U_{h,1} - \mathbf{u}_1)\ _{L^2(\Omega_1)}$	$\ P_{h,2} - p_2\ _{L^2(\Omega_2)}$	$\ \nabla(P_{h,2} - p_2)\ _{L^2(\Omega_2)}$
$\frac{1}{4}$	$\frac{1}{2}$	1.594×10^{-2}	7.722×10^{-2}	1.364×10^{-1}	2.406×10^{-2}	4.206×10^{-2}
$\frac{1}{16}$	$\frac{1}{4}$	9.671×10^{-4}	1.249×10^{-2}	3.397×10^{-2}	6.014×10^{-3}	1.042×10^{-2}
$\frac{1}{64}$	$\frac{1}{8}$	6.764×10^{-5}	2.784×10^{-3}	8.477×10^{-3}	1.503×10^{-3}	2.606×10^{-3}
rate (h)	-	2.00	1.00	1.00	1.20	1.20

Table 14

Errors and rates for $(W_{h,NSD})$ (FEM/SIPG $\sigma = 18, k_2 = 2$).

if quadratic approximation is used in the subsurface. Increasing the order of approximation in Ω_2 to $k_2 = 2$ for the problem $(W_{h,NSD})$ improves the accuracy of the Darcy velocity compared to the $k_2 = 1$. However, we note that the solution to the fully coupled problem with $k_2 = 2$ in the Darcy region is more accurate compared to the two-grid solution for all variables of the model.

One of the applications of this work has been to couple the Navier-Stokes and Darcy solution with a transport equation [21]. Figures 2 and 3 are plots of the

(a) Two Grid

Fig. 2. x- component of velocity $H = 0.5, h = 0.25$, FEM/NIPG, $k_2 = 2$, $\sigma = 0$.

x and y components of the velocity solutions respectively for the fully coupled Navier-Stokes/Darcy scheme and the two-grid schemes. It is clear that even for the coarse mesh $H = 0.5$ the two-grid solution from $h = 0.25$ gives the

(a) Two Grid

Fig. 3. y - component of velocity $H = 0.5, h = 0.25, FEM/NIPG, k_2 = 2, \sigma = 0$.

same information for the velocity as in the case for the fully coupled problem. We note for the application of this work in tracking contaminants the velocity field for the coupled problem is the input we seek to couple with the transport equation [21].

5.3 CPU Times: Two-grid Coupled Navier-Stokes and Darcy Model

The advantage of the two-grid method for the Navier-Stokes and Darcy coupling is that it results in a linearized system. This means that on the fine grid we only solve the linear system resulting from an Oseen problem in Ω_1 once. This results in significant gains in computational time and efficiency of the solver. We first present the CPU times for solving the linear systems.

h	H	Fully Coupled ($t_{W_{NS}^h}$)	Navier-Stokes ($t_{W_{h,NS}}$)	Darcy ($t_{W_{h,D}}$)
$\frac{1}{16}$	$\frac{1}{4}$	1.29×10^0	6.00×10^{-2}	1.20×10^{-1}
$\frac{1}{64}$	$\frac{1}{8}$	$8.731 \times 10^{+1}$	1.69×10^0	7.36×10^0

Table 15

Direct Solver CPU times (s): $W_{NS}^h, W_{h,NS}, W_{h,D}, (FEM/NIPG \ \sigma = 0, k_2 = 2)$.

Table 15 shows the CPU times for the fully coupled Navier-Stokes/Darcy model and the two-grid method. It is very clear that decoupling the problem results in significant reduction in computational time. In most finite element computations the bulk of the time is spent during the assembly of the matrices for the linear system. We also present the total time taken to assemble and solve the system.

h	H	Fully Coupled ($t_{W_{NS}^h}$)	Fully Coupled ($t_{W_{NS}^H}$)	Navier-Stokes ($t_{W_{h,NS}}$)	Darcy ($t_{W_{h,D}}$)
$\frac{1}{16}$	$\frac{1}{4}$	9.4×10^0	3.2×10^{-1}	6.0×10^{-1}	1.6×10^{-1}
$\frac{1}{64}$	$\frac{1}{8}$	$1.0 \times 10^{+3}$	1.1×10^0	2.4×10^2	1.1×10^1

Table 16

Assembly and Solve CPU times (s): W_{NS}^h , $W_{h,NS}$, $W_{h,D}$, ($FEM/NIPG$ $\sigma = 0$, $k_2 = 2$).

From Table 16 it is clear that the decoupling technique cuts down the computational time significantly. In a sequential implementation, the two-grid method is almost 9 times faster than the fully coupled approach for a mesh of size $h = 1/16$. If the mesh is refined and $h = 1/64$, the two-grid method is about 4 times faster than the fully coupled approach. In that case, we note that the linear system for the fully coupled problem is of size (77825×77825) whereas the decoupled problems are of size (28673×28673) and (49152×49152) for the Oseen and Darcy problems respectively.

5.4 Case of Impermeable Inclusion in Subsurface

Next we consider a case for which the exact solution for the coupled flow problem is unknown. The computational domain Ω is divided into $\Omega_1 = (0.0, 1.5) \times (1.0, 2.0)$ and $\Omega_2 = (0.0, 1.5) \times (0.0, 1.5)$. The porous medium Ω_2 has an enclosure of low permeability $(0.5, 1.0) \times (0.25, 0.75)$ with $\mathbf{K} = 10^{-5}$ surrounded by an area with $\mathbf{K} = 1.0$. We consider the following boundary conditions:

$$\mathbf{u}_1(x, y) = \begin{cases} (-(y - 2.0)(y - 1.0), 0.0), & \text{for } x = 0.0 \\ (-0.5(y - 2.0)(y - 1.0), 0.0), & \text{for } x = 1.5 \\ (0.0, 0.0) & \text{otherwise} \end{cases}$$

The Dirichlet and Neumann boundary conditions are chosen to be zero and the data functions $\mathbf{f}_1$ and f_2 are also zero. The expected solution is a parabolic flow from left to right in the free flow domain with some fluid percolating into the porous medium. We compare the computational time taken to solve the fully coupled problem and the two-grid method. First we solve the problem using the two-grid method, this requires solving the problem on a coarse mesh. The choice of a coarse mesh for this problem is $H = \frac{1}{8}$. The coarse mesh consists of 854 triangular elements. The choice of MINI elements in Ω_1 and the NIPG method of order two with zero penalty in Ω_2 results in a fully coupled problem of size 3995×3995 which takes 6.25 seconds to solve. Next we refine the coarse mesh twice to obtain a fine mesh with 13664 triangular elements ($h = \frac{1}{32}$). On the fine mesh we solve a decoupled Navier-Stokes problem of size 26935×26935 and a Darcy problem of size 36000×36000 . It

takes 2.12×10^2 seconds to solve the resulting decoupled problems. In Figure

Fig. 4. Euclidian norm of velocity $H = \frac{1}{8}, h = \frac{1}{32}, FEM/NIPG, k_2 = 2, \sigma = 0$.

4 we compare the solutions obtained from solving the fully coupled problem on the fine mesh. It is clear that both solutions capture the flow well and are comparable. The fully coupled problem for this case takes 1.904×10^3 seconds to solve. The two-grid method is 8 times faster than the fully coupled method in this example and gives an accurate solution.

6 Conclusion

We have presented a two-grid method for the coupled Navier-Stokes and Darcy problem. The method naturally yields decoupled and linear problems for the surface and subsurface regions. We have shown that the two-grid method is computationally efficient: the accuracy of the solution is comparable to fully coupled solution, and the CPU times are significantly smaller. Using the flexibility of discontinuous Galerkin methods in the subsurface, we obtained a more accurate solution by locally increasing the order of approximation.

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