The Rate of Convergence of the
Modified Method of Characteristics for
Linear Advection Equations in One Dimension

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I. Introduction

In a recent paper by Russell and two of the authors [2], it was demonstrated that the error for the modified method of characteristics (MMOC) applied to periodic linear advection problems in n dimensions is at worst $O(h^r + \Delta t)$ in L^2 at each time level. Here $r = \min(s, k)$, where k is the degree of the piecewise polynomial approximating space and s is related to the smoothness of the solution. Thus, a suboptimal rate of convergence in L^2 was derived; the rate is suboptimal in the sense that the power of h is one lower than is possible for a best approximation.

In this paper, we show that in general the rate of convergence is suboptimal, but that it is optimal in at least one case. We consider the simple one-dimensional linear advection equation

$$\begin{cases} u_t + u_x = f, & x \in (0, 1) \equiv I, \\ & t \in (0, T], \\ u(x, 0) = u^0(x), & x \in I, \end{cases} \quad (1.1)$$

with periodic initial and boundary conditions, i.e., $u^0(0) = u^0(1)$ and $u(0, t) = u(1, t)$. In particular, we consider a finite element MMOC approximation to (1.1) with continuous piecewise linear and continuous piecewise quadratic approximating spaces. In the first case (piecewise linears), we show that under certain assumptions on the smoothness of u and assuming $\Delta t = \mathcal{O}(h^2)$, one actually has optimality in L^2 , thus, the L^2 error is $\mathcal{O}(h^2)$. For the second case (piecewise quadratics), assuming $\Delta t = o(h^2)$, we construct an example where the L^2 error is at best $\mathcal{O}(h^2)$, thus, a suboptimal order of convergence holds here.

The nature of these results and the tools used are similar to the results and tools in [3] where the rate of convergence of the continuous-time Galerkin method was studied.

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Our result for piecewise quadratics can be used to see that the rate of convergence of the continuous-time Galerkin method is suboptimal in this case; just take Δt to zero.

The remainder of this paper is divided into five sections. In the next section, we establish notation, discuss the MMOC and its application to (1.1), and state the two major results of this paper. The first major result, an optimal order L^2 error estimate for the case of a continuous piecewise linear approximating space, is derived in Section 3. The second major result is derived in Section 4; we show by construction that for the case of continuous piecewise quadratic approximation the rate in L^2 is suboptimal in space. Technical lemmas used to derive the results of Sections 3 and 4 are proved in Section 5. Finally, in Section 6, we verify these theoretical results numerically by calculating experimental L^2 convergence rates for a smooth test problem.

II. Notation and Preliminaries

Let $L^2(I)$, $L^\infty(I)$, and $H^k(I)$ denote the standard Sobolev spaces defined on the unit interval. Let $\tilde{L}^2$ denote the set of functions defined on $\mathbf{R}$ which are one-periodic and whose restrictions to I are on $L^2(I)$. Adopt a similar definition for $\tilde{H}^k$. For u an element of $\tilde{L}^2$ let $\|u\| = \|u\|_{L^2(I)}$, and for $u \in \tilde{H}^k$, $\|u\|_k = \|u\|_{H^k(I)}$.

If $\mathcal{X}$ is a normed space with norm $\|\cdot\|_{\mathcal{X}}$ and $\phi : [0, T] \rightarrow \mathcal{X}$, let

$$\|\phi\|_{L^\infty(\mathcal{X})} = \sup_{0 \leq t \leq T} \|\phi(t)\|_{\mathcal{X}}.$$

For M_h a finite-dimensional subspace of $\tilde{H}^1$, the finite element MMOC is a map $U : \{0 = t^0, t^1, \dots, t^N = T\} \rightarrow M_h$ defined by

$$\begin{cases} \left(\frac{U^{n+1} - \hat{U}^n}{\Delta t}, v \right) = (f^{n+1}, v), & v \in M_h, \\ U^0 = \tilde{U}^0. \end{cases} \quad (2.1)$$

Here $(\cdot, \cdot)$ denotes the L^2 inner product on I , $f^n(x) = f(x, t^n)$, $U^n(x) = U(x, t^n)$, where $t^n = n\Delta t$, $n = 0, 1, \dots, N$,

$$\hat{U}^n(x) = U^n(x - \Delta t), \quad (2.2)$$

and $\tilde{U}^0$ is the L^2 projection of u^0 into M_h . (For application and analysis of the MMOC to more complicated problems, see [4], [5], [6], [7].)

We note that since the true solution satisfies

$$u(x, t^{n+1}) = u(x - \Delta t, t^n) + \int_0^{\Delta t} f(x - \Delta t + s, t^n + s) ds$$

we have

$$\left(\frac{u^{n+1} - \hat{u}^n}{\Delta t}, v \right) = (f^{n+1} + \rho^n, v), \quad v \in M_h, \quad (2.3)$$

where $\rho_n = \mathcal{O}(\Delta t)$ provided $f_\tau \in L^\infty(L^2)$, where τ is the characteristic direction.

Let J be a positive integer, $h = 1/J$, and for $j = 1, \dots, J$,

$$\begin{aligned} x_j &= jh, \\ X_j &= [x_{j-1}, x_j], \\ x_{j+1/2} &= (j + 1/2)h. \end{aligned}$$

Denote by $\tilde{M}_0(1, h)$ the set of functions $v \in \mathcal{C}^0(I)$ with v linear on each x_j , and $v(0) = v(1)$. Moreover, denote by $\tilde{M}_0(2, h)$ the set of functions $v \in \mathcal{C}^0(I)$, v quadratic on X_j , and $v(0) = v(1)$. We now study convergence of the MMOC with $M_h = \tilde{M}_0(1, h)$ and $M_h = \tilde{M}_0(2, h)$.

The main results of this paper can be stated as follows.

Theorem 1 *Let $\Delta t = \mathcal{O}(h^2)$, and assume u is three times continuously differentiable, then*

$$\max_{0 \leq n \leq N} \|u^n - U^n\|_{L^2} = \mathcal{O}(h^2).$$

Theorem 2 *Let $\Delta t = o(h^2)$ and assume u is four times continuously differentiable, then*

$$\max_{0 \leq n \leq N} \|u^n - U^n\|_{L^2} = \mathcal{O}(h^2). \quad (2.4)$$

In addition, if u_{xxx} is not identically zero, there exists $C_0 > 0$ independent of h and $\mu^n \in \tilde{M}_0(2, h)$ with $\max_n \|\mu^n\| \geq C_0$, and

$$\|u^n - U^n + h^2 \mu^n\|_{L^2} = \mathcal{O}(h^3). \quad (2.5)$$

Consequently, the estimate (2.4) is sharp.

In the following analysis, C represents a generic constant independent of h and Δt .

III. Proof of Theorem 1.

Let $M_h = \tilde{M}_0(1, h)$, and for each $t \in (0, T]$, let $\tilde{U}(t)$ be the $\tilde{L}^2$ projection of $u(t)$ into M_h , i.e.

$$(\tilde{U}(t) - u(t), v) = 0, \quad v \in M_h. \quad (3.1)$$

Then, setting $\zeta = U - \tilde{U}$, $\xi = u - \tilde{U}$, we have by (2.1), (2.3), and (3.1)

$$\left(\frac{\zeta^{n+1} - \zeta^n}{\Delta t}, v \right) = \left(\frac{\xi^n - \hat{\xi}^n}{\Delta t} + \rho^n, v \right), \quad v \in M_h. \quad (3.2)$$

Noting by periodicity that $\|\hat{\zeta}^n\| = \|\zeta^n\|$, and setting $v = \zeta^{n+1}$ in (3.2) we obtain

$$\|\zeta^{n+1}\|^2 - \|\zeta^n\|^2 + \|\zeta^{n+1} - \hat{\zeta}^n\|^2 = 2(\xi^n - \hat{\xi}^n + \Delta t \rho^n, \zeta^{n+1}). \quad (3.3)$$

A suboptimal L^2 estimate for ζ can be obtained as in [2] by noting that

$$(\xi^n - \hat{\xi}^n + \Delta t \rho^n, \zeta^{n+1}) \leq C \Delta t (\|\xi_x^n\|^2 + \|\rho^n\|^2 + \|\zeta^{n+1}\|^2). \quad (3.4)$$

Moreover, by approximation theory

$$\|\xi_x^n\| \leq Ch \|u^n\|_2. \quad (3.5)$$

Thus, substituting (3.4) and (3.5) into (3.3) and summing on $n, n = 0, \dots, N-1$ we find that

$$\|\zeta^N\|^2 \leq C_u (h^2 + (\Delta t)^2) + C \sum_{n=0}^{N-1} \|\zeta^{n+1}\|^2 \Delta t,$$

where C_u depends on derivatives of u through second order. By the discrete Gronwall Lemma,

$$\max_{0 \leq n \leq N} \|\zeta^n\| \leq C(h + \Delta t). \quad (3.6)$$

Combining (3.6) with the well-known estimate

$$\|\xi\|_{L^\infty(L^2)} \leq Ch^2 \|u\|_{L^\infty(H^2)}, \quad (3.7)$$

we have by the triangle inequality

$$\max_{0 \leq n \leq N} \|u^n - U^n\| \leq C(h + \Delta t). \quad (3.8)$$

We can see by the above argument that obtaining a “better” L^2 estimate requires more careful treatment of the term

$$(\xi^n - \hat{\xi}^n, \zeta^{n+1}). \quad (3.9)$$

Thus, we examine this term more closely.

We now state a lemma concerning the term (3.9), whose proof we defer to Section 5.

Lemma 1 *Assume the hypothesis of Theorem 1 holds, then*

$$\left| \left(\frac{\xi^n - \hat{\xi}^n}{\Delta t}, \zeta^{n+1} \right) \right| \leq C(h^4 + \|\zeta^{n+1}\|^2). \quad (3.10)$$

Substituting (3.10) into (3.3), assuming $\Delta t = \mathcal{O}(h^2)$, we find

$$\frac{1}{\Delta t} \|\zeta^{n+1}\|^2 \leq \frac{1}{\Delta t} \|\zeta^n\|^2 + Ch^4 + \|\zeta^{n+1}\|^2.$$

Summing above on n and applying the discrete Gronwall Lemma proves Theorem 1.

IV. Proof of Theorem 2.

As in the proof of Theorem 1 with $\xi = u - \tilde{U}$ and $\zeta = U - \tilde{U}$, we obtain

$$\left(\frac{\zeta^{n+1} - \hat{\zeta}^n}{\Delta t}, \chi \right) = \left(\frac{\xi^n - \hat{\xi}^n}{\Delta t} + \rho^n, \chi \right), \quad \chi \in \tilde{M}_0(2, h).$$

Setting $\chi = \zeta^{n+1}$, we have

$$\frac{1}{\Delta t} \|\zeta^{n+1}\|^2 \leq \frac{1}{\Delta t} \|\zeta^n\|^2 + \left(\frac{\xi^n - \hat{\xi}^n}{\Delta t} + \rho^n, \zeta^{n+1} \right) \quad (4.1)$$

Just as in the derivation of the optimal rate in the proof of Theorem 1 we get that

$$\frac{1}{\Delta t} \|\zeta^{n+1}\|^2 \leq \frac{1}{\Delta t} \|\zeta^n\|^2 + C(h^4 + \|\zeta^{n+1}\|^2). \quad (4.2)$$

Here we have also used the fact that $\Delta t = o(h^2)$. Summing (4.2) on n , $n = 0, 1, \dots, N$ and applying Gronwall's Lemma we obtain (2.5).

In Lemma 4 below we construct $\mu^n \in \tilde{M}_0(2, h)$ satisfying $\max_n \|\mu^n\| \geq C_0 > 0$ and

$$\|\zeta^n + h^2 \mu^n\| = \mathcal{O}(h^3).$$

Inequality (2.4) follows immediately using approximation theory.

V. Lemmas.

V.1 Proof of Lemma 1.

Let $\mathcal{V} = \{v_0(x), v_1(x), \dots, v_{J-1}(x)\}$ be the set of 1-periodic “hat” functions defined over the mesh $\{x_j\}$. That is, $v_j \in \tilde{M}_0(1, h)$ and

$$v_j(x_k) = \begin{cases} 1, & j = k, \\ 0, & 0 \leq k < J, \quad k \neq j. \end{cases} \quad (5.1)$$

Then $\mathcal{V}$ is a basis for $\tilde{M}_0(1, h)$ and

$$\zeta^n(x) = \sum_{j=0}^{J-1} \zeta_j^n v_j(x), \quad (5.2)$$

where

$$\zeta_j^n = \zeta^n(x_j).$$

It is easily seen that if

$$\chi(x) = \sum_{j=0}^{J-1} \chi_j v_j(x)$$

and

$$\|\chi\|_h^2 = \sum_{j=0}^{J-1} \chi_j^2 h, \quad (5.3)$$

then

$$\frac{1}{3} \|\chi\|_h^2 \leq \|\chi\|^2 \leq \|\chi\|_h^2. \quad (5.4)$$

Thus $\|\cdot\|_h$ and $\|\cdot\|$ define equivalent norms on $\tilde{M}_0(1, h)$.

Suppressing time-dependence momentarily, we express ξ as

$$\xi(x) = \psi(x) + g(x) \quad (5.5)$$

where on each X_{j+1}

$$\psi(x) = \frac{1}{2} u_{xx}(x_{j+\frac{1}{2}})(x - y_{j1})(x - y_{j2}) \quad (5.6)$$

with y_{j1} and y_{j2} being the two Gauss points on X_{j+1} . It then follows [3] that on each X_{j+1}

$$|g(x)| + h|g'(x)| = \mathcal{O}(h^3); \quad (5.7)$$

i.e., ψ is the leading term in the error in the L^2 projection.

This expression is used to show that (ξ_x, v_j) is smaller than is apparent at first glance. Specifically, integration by parts shows that

$$\begin{aligned} (\psi', v_j) &= \frac{1}{2} (x_j - y_{j1})(x_j - y_{j2})(u_{xx}(x_{j-\frac{1}{2}}) - u_{xx}(x_{j+\frac{1}{2}})) \\ &\quad + (\psi, v'_j). \end{aligned} \quad (5.8)$$

The last integral is zero since ψ vanishes at the Gauss points. Combining this with (5.7) we see that

$$(\xi_x, v_j) = \mathcal{O}(h^3). \quad (5.9)$$

Next, let

$$\xi(x) - \xi(x - \Delta t) = \Delta t \xi_x(x) + \eta(x). \quad (5.10)$$

Then for each j

$$\eta(x) = \begin{cases} \mathcal{O}(\Delta t^2), & \text{on } (x_j + \Delta t, x_{j+1}), \\ \mathcal{O}(h\Delta t), & \text{on } (x_j, x_j + \Delta t). \end{cases} \quad (5.11)$$

The first bound holds because $\xi_{xx} = u_{xx}$ on (x_j, x_{j+1}) ; the second holds since $\xi_x = \mathcal{O}(h)$ and

$$\xi(x) - \xi(x - \Delta t) = \int_{x-\Delta t}^x \xi_x(s) ds.$$

From (5.11) we see that

$$(\eta, v_j) = \mathcal{O}(h\Delta t^2). \quad (5.12)$$

Thus from (5.9), (5.10) and (5.12) we see that

$$\begin{aligned} (\xi - \hat{\xi}, v_j) &= \mathcal{O}(h^3 \Delta t + h \Delta t^2) \\ &= \mathcal{O}(h^3 \Delta t). \end{aligned}$$

Hence,

$$\begin{aligned} (\xi^n - \hat{\xi}^n, \zeta^{n+1}) &= \sum_j (\xi^n - \hat{\xi}^n, v_j) \zeta_j^{n+1} \\ &\leq C \Delta t \sum_j h^2 |\zeta_j^{n+1}| h \\ &\leq C \Delta t (h^4 + \|\zeta^{n+1}\|^2). \end{aligned} \quad (5.13)$$

Here we used (5.4). This completes the proof of Lemma 1.

V.2 Quadratic Case Truncation Error

Let v_j , $j = 0, \dots, J-1$ be the “hat” functions defined by (5.1) and let b_j , $j = 0, \dots, J-1$ be the “bump” functions given by

$$b_j(x) = \begin{cases} \frac{(x-x_j)(x_{j+1}-x)}{h^2}, & x_j \leq x \leq x_{j+1}, \\ 0, & \text{otherwise.} \end{cases} \quad (5.14)$$

Then, $\{v_j\} \cup \{b_j\}$ is a basis for $\tilde{M}_0(2, h)$. We now show that

$$\left(\frac{\xi^n - \hat{\xi}^n}{\Delta t}, v_j \right) = \mathcal{O}(h^4) \quad (5.15)$$

and

$$\left(\frac{\xi^n - \hat{\xi}^n}{\Delta t}, b_j \right) = -\frac{1}{360} u_{xxx}(x_j, t^n) h^3 + \text{higher order terms.} \quad (5.16)$$

Again suppressing time-dependence momentarily, we have

$$\frac{\xi(x) - \hat{\xi}(x)}{\Delta t} = \xi_x(x) + e(x),$$

where $e(x) = \bar{\xi}_x(x) - \xi_x(x)$ and $\bar{\xi}_x$ is the average value of ξ_x over the interval $[x - \Delta t, x]$. Using Lemmas 2 and 3, the results (5.15) and (5.16) follow.

Lemma 2 *Let u be C^4 and 1-periodic. Then*

$$(\xi_x, v_j) = \mathcal{O}(h^4)$$

and

$$(\xi_x, b_j) = -\frac{1}{360} u_{xxx}(x_j) h^3 + \mathcal{O}(h^4),$$

where v_j and b_j are defined by (5.1) and (5.14).

Proof: Let Iu denote the $\mathcal{C}^0$ piecewise quadratic interpolant of u on X_{j+1} defined by interpolation of u at x_j , $x_{j+\frac{1}{2}}$ and x_{j+1} . Using a Hermite-Birkhoff cubic interpolant which in addition to the conditions for Iu matches u_{xxx} at $x_{j+\frac{1}{2}}$, and applying the Bramble-Hilbert Lemma [1], we have on X_{j+1}

$$\begin{aligned}(u - Iu)(x) &= \frac{1}{6}u_{xxx}(x_{j+\frac{1}{2}})(x - x_j)(x - x_{j+\frac{1}{2}})(x - x_{j+1}) + g(x) \\ &\equiv \psi(x) + g(x),\end{aligned}\tag{5.17}$$

where $|g(x)| + h|g'(x)| = \mathcal{O}(h^4)$. Note that

$$(\psi, v_j) = \mathcal{O}(h^5).$$

and

$$(\psi, b_j) = 0.$$

Now $\xi(x) = (u - Iu) + (Iu - \tilde{U})$. Thus, it follows easily that

$$\xi = \psi + \mathcal{O}(h^4)\tag{5.18}$$

and

$$\xi_x = \psi' + \mathcal{O}(h^3).$$

Thus, the leading order term in the expression for $\xi_x(x)$ is $\psi'(x)$. The lemma now follows since

$$(\psi', b_j) = -\frac{1}{360}u_{xxx}(x_{j+\frac{1}{2}})h^3.$$

Lemma 3 *Let $\bar{\xi}_x(x)$ denote the average value of ξ_x over the interval $[x - \Delta t, x]$. Assume $\Delta t = o(h^2)$. Then $e = \bar{\xi}_x - \xi_x$ is $\mathcal{O}(h^3)$.*

Proof. In the proof of Lemma 2, Equation (5.18), we showed that

$$\xi(x) = \psi(x) + \text{higher order terms}$$

on X_j , where ψ is given by (5.17). Now

$$|\psi'(x_j + 0) - \psi'(x_j - 0)| \leq Ch^3,$$

which implies that

$$|\xi_x(x_j + 0) - \xi_x(x_j - 0)| \leq Ch^3.\tag{5.19}$$

Moreover

$$\bar{\xi}_x(x) = \frac{1}{\Delta t} \int_{x-\Delta t}^x \xi_x(s) ds = \xi_x(x) + \frac{1}{\Delta t} \int_{x-\Delta t}^x (\xi_x(s) - \xi_x(x)) ds.\tag{5.20}$$

By Taylor expansion and (5.19)

$$\begin{aligned}|\xi_x(s) - \xi_x(x)| &\leq Ch|x - s| + Ch^3 \\ &\leq Ch\Delta t + Ch^3;\end{aligned}\tag{5.21}$$

Hence, by (5.20), (5.21), and the assumption on Δt

$$\begin{aligned} |e(x)| &= |\tilde{\xi}_x(x) - \xi_x(x)| \\ &\leq Ch\Delta t \\ &\leq Ch^3. \end{aligned}$$

V.3 Lemma 4

Lemma 4 *Assume that u_{xxx} is not identically zero. Let*

$$\mu^n(x) = \sum_j r(x_j, t^n) w_j(x)$$

where $w_j = -\frac{1}{3}v_j + b_j + b_{j-1}$. Assume $\Delta t = o(h^2)$ and that the coefficients $r(x, t)$ satisfy the first order equation

$$r_t - 10r_x = -\frac{1}{4}u_{xxx},$$

with $r(x, 0) \equiv 0$. Then, there exists $C_0 > 0$, independent of h , such that

$$\max_{0 \leq n \leq N} \|\mu^n\| \geq C_0$$

and

$$\|\zeta^n + h^2 \mu^n\| = \mathcal{O}(h^3).$$

Proof: It is clear that r is a function that is not identically zero and does not depend on h . The size of μ is determined by the size of r . In fact,

$$\|\mu^n\|^2 \geq \frac{1}{45} \sum_j |r(x_j, t^n)|^2 h.$$

Thus, the existence of $C_0 > 0$ independent of h is established.

Let $\lambda = \frac{\Delta t}{h}$. A calculation yields

$$\begin{aligned} &\frac{6}{h} (\mu^{n+1} - \hat{\mu}^n, v_j) \\ &= -\frac{1}{3} [r_{j-1}^{n+1} + 4r_j^{n+1} + r_{j+1}^{n+1} - ((1+3\lambda)r_{j-1}^n + 4r_j^n + (1-3\lambda)r_{j+1}^n)] \\ &\quad + \frac{1}{2} [r_{j-1}^{n+1} + 2r_j^{n+1} + r_{j+1}^{n+1} - ((1+2\lambda)(r_{j-1}^n + r_j^n) + (1-2\lambda)(r_j^n + r_{j+1}^n))] \\ &\quad + o(\Delta t) \\ &= o(h^2). \end{aligned} \tag{5.22}$$

and

$$\begin{aligned}
& \frac{60}{h} (\mu^{n+1} - \hat{\mu}^n, b_j) \\
&= -\frac{5}{3} [r_j^{n+1} + r_{j+1}^{n+1} - (r_j^n + r_{j+1}^n) + 2\lambda(r_{j+1}^n - r_j^n)] \\
&\quad + 2[r_j^{n+1} + r_{j+1}^{n+1} - (r_j^n + r_{j+1}^n)] + o(\Delta t) \\
&= \Delta t \left[\frac{1}{3} r_t(x_j, t^n) - \frac{10}{3} r_x(x_j, t^n) \right] + \mathcal{O}(h\Delta t) + o(\Delta t).
\end{aligned}$$

Thus,

$$\left(\frac{\mu^{n+1} - \hat{\mu}^n}{\Delta t}, b_j \right) = -\frac{h}{360} u_{xxx}(x_j, t^n) + o(h). \quad (5.23)$$

Letting $\phi^n = \zeta^n + h^2 \mu^n$ and using Lemmas 2 and 3, (5.22) and (5.23), we note that

$$\left(\frac{\phi^{n+1} - \hat{\phi}^n}{\Delta t}, \phi^{n+1} \right) = \mathcal{O}(h^3 \|\phi^{n+1}\|).$$

Thus, by Gronwall's Lemma

$$\|\phi^n\| = \mathcal{O}(h^3)$$

and the proof is complete.

VI. Experimental results

In this section, we verify experimentally the theoretical results of the previous sections.

Experimental rates of convergence for the MMOC with piecewise linear and piecewise quadratic approximating spaces were computed for the test problem

$$\begin{cases} u_t + u_x = 0, & x \in (0, 1) \equiv I, \\ u^0(x) = \sin(2\pi x), & x \in I, \end{cases} \quad (6.24)$$

which has solution $u(x, t) = \sin(2\pi(x - t))$. Discrete L^2 errors were calculated at $t = 1/16$ for $h^{-1} = 2^k$, $k = 3, \dots, 6$. For the case $M_h = \tilde{M}_0(1, h)$, $\Delta t = h^2$ and the error was calculated by the trapezoidal rule of integration. For the case $M_h = \tilde{M}_0(2, h)$, $\Delta t = h^3$ and Simpson's rule was used. A least squares fit was used to calculate the experimental rate of convergence. Table 1 below verifies that in both the linear and quadratic cases, h^2 convergence was observed.

We would also like to report that, in the linear case with $f \equiv 0$, one obtains superconvergence of h^4 at the nodes, provided that the interpolant is used as the initial condition. This result can be verified theoretically, however, since it is very specialized, the details are not included here.

Moreover, it is clear from the proof of Theorem 2 that in the $\tilde{M}_0(2, h)$ case the convergence at the Gauss points y_{j1} and y_{j2} is $\mathcal{O}(h^3)$ since μ vanishes to the next higher order at these points. This was observed in our experiments.

Approximating space	Rate
$\tilde{M}_0(1, h)$	2.00
$\tilde{M}_0(2, h)$	1.94

Table 1: Experimental rates

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